ATTRACTIVE PROPERTIES VIA GENERALIZED DERIVATIVES OF CONTINUOUS FUNCTIONS

BY

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ABSTRACT. Let I be a closed interval in $\mathbf{R}$ and let $x_0 \in I$. We obtain a condition on a set E having x_0 as a point of accumulation that guarantees that any continuous function $f : I \to I$ that has x_0 as a fixed point and posesses a derivative $f'_E(x_0)$ relative to E at x_0 with $|f'_E(x_0)| < 1$ has some point $x \neq x_0$ attracted to x_0. We also obtain conditions that guarantee that the point x can be chosen in E.

1. Introduction. Let f be a continuous function mapping a closed interval I into itself and let x_0 be a fixed point of f. An elementary theorem asserts that if $|f'(x_0)| < 1$, then there is a neighborhood U of x_0 such that for each $x \in U$, the sequence $\{f^n(x)\}$ of iterates of x under f converges to x_0. Here $f^0(x) = x$ and $f^n(x) = f^{n-1}(x)$ $n = 1, 2, \ldots$.

A variant of this theorem can be obtained by replacing the condition on the derivative by a condition on the difference- quotient: if there is a number $d \in (0, 1)$ such that

$$-d < \frac{f(x) - f(x_0)}{x - x_0} < d \tag{1}$$

for all $x \in U$, then $\lim_{n\to\infty} f^n(x) = x_0$ for all $x \in U$.

Suppose, now, that we replace the full neighborhood U of x_0 with some other set E. We can ask whether the condition $|f'_E(x_0)| < 1$ where $f'_E(x_0)$ denotes the derivative of f with respect to E at x_0, or the condition (1) holding for all $x \in E$, still implies some sort of attraction to the fixed point x_0. The strength of the conclusion will depend, of course, on the set E. It is, perhaps surprising that porosity conditions, rather than density conditions, are the relevant ones.

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2. Main Results.

Suppose x_0 is a fixed point of f and $0 < f'(x_0) < 1$. It is then clear that for x sufficiently close to x_0 the sequence $\{f^n(x)\}$ approaches x_0 monotonically. Suppose we weaken the assumption to $0 < f'_{ap}(x_0) < 1$. This means that there is a set E having x_0 as a point of density such that

$$0 < \lim_{x \to x_0; x \in E} \frac{f(x) - f(x_0)}{x - x_0} < 1.$$

If $x \in E$ and $f^n(x) \in E$ for all $n = 1, 2, ...$, then $f^n(x) \to x_0$. Perhaps the fact that E has x_0 as a density point suggests there must be an x attracted to x_0 whose orbit lies entirely in E. We shall see that this is not necesseraly the case, although there will be a point not necesseraly in E, whose orbit approaches x_0. Our goal in this section is to obtain two theorems. The first provides a condition on a set E that guarantees that any continuous function such that $0 < \lim_{x \to x_0; x \in E} \frac{f(x) - f(x_0)}{x - x_0} = d < 1$ has some point x (not necessarily in E) whose orbit is attracted to the fixed point x_0 without landing on x_0. The second theorem provides a condition that guarantees that if $0 < \lim_{x \to x_0; x \in E} \frac{f(x) - f(x_0)}{x - x_0} < 1$ then there is a point $x \in E$ whose orbit is attracted to x_0 without landing on x_0. Actually each theorem reveals a bit more.

We begin with an example that may be instructive.

Example 2.1. Assume that $0 \in \text{int}(I)$, where $\text{int}(I)$ denotes the interior of the interval I. Then there exists a continuous function $f : I \to I$ and a set E such that $f(0) = 0$, 0 is a right density point of E, $\lim_{x \searrow 0; x \in E} \frac{f(x)}{x} = 1/2$, and $|f^n(x)| > \delta_x > 0$ $n = 1, 2, ...$, for every $x \in E$, $x > 0$. The notation $x \searrow y$ means that x converges to y from the right.

Verification of Example 2.1. Before turning to the details we want to sketch the idea of our example. We split each interval $[2^{-n}, 2^{-n+1}]$ into 2^n subintervals $I_{n,m}$ $n = 1, 2, ..., m = 1, 2, ..., 2^n$ and we choose small subintervals $J_{n,m}$ about the endpoints of $I_{n,m}$. The set E will be $(0, 1] \setminus \cup_{n,m} J_{n,m}$ and the function f will be constant on each set

$E \cap I_{n,m}$. These constants will be chosen so that we can obtain $\lim_{x\searrow 0; x\in E} \frac{f(x)}{x} = 1/2$ and each constant will equal a fixed point of f. Hence for any $x \in E$ we have $f^k(x) = f(x)$, $k = 1, 2, ...$, that is, no point of E will be attracted to 0.

Put $I_{n,m} = [2^{-n} + (m-1)4^{-n}, 2^{-n} + m4^{-n}]$ and $J_{n,m} = (2^{-n} + (m - 4^{-n})4^{-n}, 2^{-n} + (m + 4^{-n})4^{-n})$ for $n = 1, 2, ...,$ $m = 1, 2, ..., 2^n$. Put $E = (0, 1] \setminus \cup_{n,m} J_{n,m}$ and $f(x) = \frac{1}{2}(2^{-n} + m4^{-n})$ for $x \in I_{n,m} \cap E$, $n = 1, 2, ...,$ $m = 1, 2, ..., 2^n$. We also put $f(2^{-n} + m4^{-n}) = 2^{-n} + m4^{-n}$ for $n = 1, 2, ...,$ $m = 1, 2, ..., 2^n$. Otherwise define f arbitrarily so as to be continuous. Then for every $x \in E$ there exist n and m such that $x \in I_{n,m}$. Thus $f(x) = \frac{1}{2}(2^{-n} + m4^{-n}) = 2^{-n-1} + 2m4^{-n-1}$ and hence $f^k(f(x)) = f(x)$ $k = 1, 2, ...$ As the reader easily can verify 0 is a right density point of E and $\lim_{x\searrow 0; x\in E} \frac{f(x)}{x} = 1/2$.

We remark that it will follow from Theorem 2.1 below that there is one point x not in E whose orbit is attracted to 0.

Lemma 2.1. Suppose that $f(0) = 0 \in \mathrm{int}(I)$, $0 < f(x) \leq x$ for $x > 0$, $f : I \to I$ is continuous, there exists $a_n \searrow 0$ such that $\frac{f(a_n)}{a_n} < d \in (0, 1)$, and $a_{n+1}/a_n > d$. We also assume that an interval $[c_0, b_0]$ is given such that $0 < c_0 < d \cdot b_0 \leq a_1$, $f([c_0, b_0]) \cap [c_0, b_0] \neq \emptyset$. Then $0 \in cl(\cup\{f^n([c_0, b_0]) : n = 1, 2, ...\})$.

Proof. Since f is continuous $f^n([c_0, b_0])$ is an interval denoted by $[c_n, b_n]$. From $0 < f(x) \leq x$ it follows that $0 \leq ... \leq c_n \leq c_{n-1} \leq ... < c_0$, and $0 \leq ... \leq b_n \leq b_{n-1} \leq ... \leq b_0$. From $f([c_0, b_0]) \cap [c_0, b_0] \neq \emptyset$ it follows that there exist $z, w \in [c_0, b_0]$ such that $f(w) = z$. Then $f^{n+1}(w) = f^n(z) \in [c_{n+1}, b_{n+1}] \cap [c_n, b_n]$ for every $n \geq 1$. Thus $\cup_{k=0}^{n} f^k([c_0, b_0]) = [c_n, b_0]$.

To obtain a contradiction suppose that $\lim_{n\to\infty} c_n = \gamma > 0$. Then $\cup_{k=0}^{\infty}[c_k, b_k] = \cup_{k=0}^{\infty} f^k([c_0, b_0]) \supset (\gamma, b_0]$. We also have $f(\gamma) = \gamma$ since otherwise $f(\gamma) < \gamma$ and by the continuity of f there would exist a c_n such that $f(c_n) < \gamma$ which is clearly impossible. If there is $x \in (\gamma, b_0]$ such that $f(x) < \gamma$ then from $(\gamma, b_0] \subset \cup_{k=0}^{\infty}[c_k, b_k]$ it follows that $x \in [c_k, b_k]$ for a $k \in \mathbf{N}$. This would imply that $c_{k+1} \leq f(x) < \gamma$ which again contradicts the definition of γ. Therefore $f(x) \geq \gamma$ for $x \in (\gamma, b_0]$. Since $a_{n+1}/a_n > d$ and $b_0 < a_1$ there exists an n such that $\gamma < a_n \leq \gamma/d$. By the assumption of this lemma $f(a_n) < d \cdot a_n$. On the other hand also by the assumption of this lemma $\gamma \leq c_0 < d \cdot b_0$ and hence $\gamma/d \in (a_n, b_0]$. Thus $f(a_n) \geq \gamma = d \cdot \gamma/d \geq d \cdot a_n$, a contradiction proving $\lim_{n\to\infty} c_n = 0$, which implies $0 \in cl(\cup\{f^n([c_0, b_0]) : n = 1, 2, ...\})$.

Lemma 2.2. If $f(0) = 0$, $0 < f(x) \leq x$ for $x > 0$, f is continuous, and there exist $0 < c_0 < b_0$ such that $0 \in cl(\cup\{f^n([c_0, b_0]) : n = 1, 2, ...\})$ then there exists $y \in [c_0, b_0]$ such that $\lim_{n\to\infty} f^n(y) = 0$ and $f^n(y) > 0$ for $n = 1, 2, ...$.

Proof. From the continuity of f^n it follows that $f^n([c_0, b_0])$ is an interval $[c_n, b_n]$. Since $f(x) \leq x$ we have $c_0 \geq c_1 \geq ... \geq c_n \geq ... > 0$, $b_0 \geq b_1 \geq ... \geq b_n \geq ... > 0$. The assumption of this lemma implies that $\lim_{n\to\infty} c_n = 0$. If there exists c_n such that $c_n = c_{n+1}$ then $f([c_n, b_n]) = [c_{n+1}, b_{n+1}] = [c_n, b_{n+1}] \subset [c_n, b_n]$ and hence $f^m([c_n, b_n]) \subset f^{m-1}([c_n, b_n]) \subset f([c_n, b_n]) \subset [b_n, c_n]$ contradicting the fact that $\lim_{n\to\infty} c_n = 0$. Thus $c_0 > c_1 > ... > c_n > ... > 0$. Put $F_n = f^{-n}([c_n, c_{n-1}]) \cap [c_0, b_0]$. If we have finitely many F_n's say $F_{n_1}, F_{n_2}, ..., F_{n_m}$, $n_1 > n_2 > ... > n_m$ then we show that their intersection is nonempty. Choose a $z \in [c_{n_m}, c_{n_m - 1}) \cap [c_{n_m}, b_{n_m}]$. Then there exists a $u \in F_{n_m}$ such that $f^{n_m}(u) = z$. Suppose that there exists a j such that $u \notin F_{n_j}$, that is, $f^{n_j}(u) \notin [c_{n_j}, c_{n_j - 1}]$. Thus $f^{n_j}(u) \in (c_{n_j - 1}, b_{n_j}] \subset [c_{n_j - 1}, b_{n_j - 1}]$ and hence there exists a $w \in [c_0, b_0]$ such that $f^{n_j}(u) = f^{n_j - 1}(w)$ and hence $z = f^{n_j + (n_m - n_j)}(u) = f^{n_j - 1 + (n_m - n_j)}(w) = f^{n_m - 1}(w) \in [c_{n_m - 1}, b_{n_m - 1}]$ contradicting to $z \in [c_{n_m}, c_{n_m - 1})$. Thus by compactness there exists a $y \in \cap_{n=1}^{\infty} F_n$. Since $f^n(y) \in [c_n, c_{n-1}]$ we obtain $\lim_{n\to\infty} f^n(y) = 0$.

Lemma 2.3. Suppose that $f(0) = 0$, f is continuous, and there exist $q_n, r_n \to 0$, $q_1 > r_1 > q_2 > r_2 > ... > q_k > r_k > ... > 0$, such that $f(q_k) > 0$, and $f(r_k) = 0$, $k = 1, 2, ...$ then there exists a nonempty perfect set P such that $\lim_{n\to\infty} f^n(y) = 0$, $f^n(y) > 0$, for every $y \in P$, $n = 1, 2, ...$.

Proof. Put $J_\emptyset = [r_1, q_1]$. Suppose that s is a finite zero-one sequence of length $m \in \{0, 1, 2, ...\}$ (when $m = 0$ then $s = \emptyset$) and $J_s = [r_{n(s)}, q_{n(s)}]$. Since $f(q_{n(s)}) > 0$

we can choose $n(s0) > n(s1) > n(s)$ such that $J_{s0} = [r_{n(s0)}, q_{n(s0)}] \subset [0, f(q_{n(s)})]$, and $J_{s1} = [r_{n(s1)}, q_{n(s1)}] \subset [0, f(q_{n(s)})]$.

Put $F_\emptyset = J_\emptyset$. Suppose that s is of length $n-1$, the closed set F_s is defined, and $f^{n-1}(F_s) = J_s$. Then put $F_{s0} = f^{-n}(J_{s0}) \cap F_s$ and $F_{s1} = f^{-n}(J_{s1}) \cap F_s$. From $J_{s0} \subset [0, f(q_{n(s)})]$ it follows that for every $x \in J_{s0}$ there exists a $z \in J_s$ such that $f(z) = x$. Since $f^{n-1}(F_s) = J_s$ we can find a $w \in F_s$ such that $f^{n-1}(w) = z$. Thus $f^n(w) = x$ and $w \in F_{s0}$. This implies that $f^n(F_{s0}) = J_{s0}$. A similar argument shows that $f^n(F_{s1}) = J_{s1}$.

Put $H_m = \cup\{F_s : s$ is a zero-one sequence of length $m\}$ and $H = \cap_{m=1}^{\infty} H_m$. Since the sets F_s are closed and there are 2^m zero-one sequences of length m the set H_m is closed. Therefore H is also closed.

Suppose that $y \in H$. Then $f^m(y) \in [r_{n(s)}, q_{n(s)}]$, where s is of length m. It is easy to see that $n(s) > m$. Thus $f^m(y) \to 0$ and $f^m(y) > 0$.

If ϕ is an infinite zero-one sequence then denote by $\phi|m$ the first m terms of ϕ.

For arbitrary zero-one sequence ϕ choose a $y(\phi) \in \cap_{m=1}^{\infty} F_{\phi|m} \in H$. We remark that $y(\phi)$ is well defined since $F_{\phi|m}$ $m = 1, 2, ...$ is a nested sequence of closed sets. If ϕ, ψ are different zero- one sequences then there exists an m such that $\phi|m = \psi|m = s$ and $\phi|m+1 \neq \psi|m+1$. By symmetry we may assume that $\phi|m+1 = s0$ and $\psi|m+1 = s1$. Then $y(\phi) \in F_{s0}$ and $y(\psi) \in F_{s1}$. Since $f^{m+1}(y(\phi)) \in J_{s0}$, $f^{m+1}(y(\psi)) \in J_{s1}$ and $J_{s0} \cap J_{s1} = \emptyset$ we obtain $y(\phi) \neq y(\psi)$. Therefore the cardinality of H is that of the continuum and there exists a nonempty perfect subset $P \subset H$. This proves Lemma 2.3.

It will be convenient to state our results in the language of porosity. Let S be a set, x_0 a point and $h > 0$. Let $\ell(h)$ denote the length of the longest interval in $[x_0, x_0 + h] \setminus S$. The right porosity of S at x_0 is defined as

$$p_+(x_0, S) = \limsup_{h \to 0} \frac{\ell(h)}{h}.$$

If $p_+(x_0, S) = 0$, we say S is *nonporous from the right* at x_0. If $p_+(x_0, S) = 1$ we say S is *strongly porous from the right at* x_0. The notions of left porosity and (bilateral) porosity are now defined in the obvious way.

Theorem 2.1. *If x_0 is a fixed point of the continuous function f and*

$$0 < \frac{f(x) - f(x_0)}{x - x_0} < d < 1$$

for all x in set E such that $p_+(E, x_0) < 1-d$, then there exists y such that $\lim_{n\to\infty} f^n(y) = x_0$ and for every n, $f^n(y) \neq x_0$.

Proof. Replacing f by $f(x + x_0) - x_0$ we may assume that $x_0 = 0$. The porosity condition now implies the existence of a strictly decreasing sequence $\{a_n\}$ such that $a_n \to 0$, $0 < f(a_n)/a_n < d$, and $a_{n+1}/a_n > d$.

If there exist $x_n \searrow 0$ such that $f(x_n) \leq 0$ then using $0 < f(a_n)$ one can choose $q_n, r_n \to 0$, $q_1 > r_1 > ... > q_k > r_k > ... > 0$, such that $f(q_k) > 0$, $f(r_k) = 0$ for $k = 1, 2, ...$ In this case applying Lemma 2.3 we can complete the proof of Theorem 2.1.

Thus we may assume that there exists $\delta > 0$ such that $0 < f(x)$ for $x \in (0, \delta]$. Put

$$f_1(x) = \begin{cases} f(x), & \text{if } f(x) \leq x, \text{ and } 0 \leq x < \delta; \\ x, & \text{if } f(x) > x \text{ and } 0 \leq x < \delta; \\ f(x), & \text{if } x < 0; \\ \min(\delta, f(\delta)), & \text{if } \delta \leq x. \end{cases}$$

Since $a_n \searrow 0$ there exists $m \in \mathbf{N}$ such that $a_m < \delta$. Put $a'_n = a_{n+m}$ for $n = 1, 2, ...$.

Obviously $0 < f_1(x) \leq x$ for $x > 0$. Put $b_0 = a'_1$ and choose a $c_0 \in (0, f_1(b_0)]$. Then $f_1([c_0, b_0]) \cap [c_0, b_0] \neq \emptyset$. Applying Lemma 2.1 for f_1, a'_n and $[c_0, b_0]$ we obtain that $0 \in cl(\cup\{f_1^n([c_0, b_0]) : n = 1, 2, ...\})$. Therefore we can apply Lemma 2.2 for f_1 and hence there exists $y \in [c_0, b_0]$ such that $\lim_{n\to\infty} f_1^n(y) = 0$ and $f_1^n(y) > 0$ for $n = 1, 2, ...$. From $0 < f_1^n(x) \leq x$ it follows that $0 < ... \leq f_1^{n+1}(y) \leq f_1^n(y) \leq ... \leq y$. If $f_1^{n+1}(y) = f_1^n(y)$ for an $n \in \mathbf{N}$ then $f_1^{\ell+1}(y) = f_1^{n+1+(\ell-n)}(y) = f_1^{n+(\ell-n)}(y) = f_1^{\ell}(y)$ for every $\ell \geq n$ and this would contradict the fact that $\lim_{n\to\infty} f_1^n(y) = 0$. Therefore $0 < f_1(f_1^n(y)) < f_1^n(y)$ for $n = 0, 1, ...$. Since $0 < ... < f_1^n(y) < ... < y < b_0 = a_{m+1} \leq a_m < \delta$ we obtain that $0 < f_1^n(y) < \delta$ and $f_1(f_1^n)(y) < f_1(y)$ $n = 1, 2, ...$. From the definition of f_1 it follows that $f^n(y) = f_1^n(y)$ for $n = 1, 2, ...$. Thus $\lim_{n\to\infty} f^n(y) = 0$ and this completes the proof of Theorem 2.1.

Example 2.2 below shows that the porosity condition in Theorem 2.1 cannot be weakened.

Example 2.2. Suppose that $0 < \epsilon < 1/2$. Put $f(x) = 2^{-n}$ for $x \in [2^{-n}, \frac{2^{-n+1}}{1+\epsilon}]$, $n = 1, 2, \dots$. If $\frac{2^{-n+1}}{1+\epsilon} < x < 2^{-n+1}$, $n = 1, 2, \dots$ choose $2^{-n} \le f(x) \le 2^{-n+1}$ so that it continuously connects $f(\frac{2^{-n+1}}{1+\epsilon}) = 2^{-n}$ and $f(2^{-n+1}) = 2^{-n+1}$. If $x \le 0$ or $x > \frac{2}{1+\epsilon}$ choose f so that $f(0) = 0$, $f(\frac{2}{1+\epsilon}) = 1$ and f is continuous. Then the porosity of $E = \{x : f(x) < (\frac{1}{2} + \epsilon)x\}$ at 0 is no greater than 1/2 since $f(\frac{2^{-n+1}}{1+\epsilon}) = \frac{1+\epsilon}{2}(\frac{1}{1+\epsilon} \cdot 2^{-n+1}) < (\frac{1}{2} + \epsilon)\frac{2^{-n+1}}{1+\epsilon}$ $n = 1, 2, \dots$. It is also clear that each interval $[2^{-n}, 2^{-n+1}]$ is mapped into itself and hence no point is attracted from the right to 0.

We saw in Example 2.1 that even when E is nonporous at x_0 and $f'_E(x_0) = 1/2$ there may be no point in E whose orbit is attracted from the right to x_0. Observe that $\mathbf{R} \setminus E$ was also nonporous from the right at x_0 in that example. One can rectify this flaw by requiring $\mathbf{R} \setminus E$ to be strongly porous from the right at x_0.

Theorem 2.2. *Suppose E is nonporous from the right at x_0 and $\mathbf{R} \setminus E$ is strongly porous from the right at x_0. If f is a continuous function having x_0 as a fixed point and satisfies an inequality*

$$0 < d_1 < \frac{f(x) - f(x_0)}{x - x_0} < d_2 < 1$$

for all x in E then there exists $y \in E$ such that $\lim_{n\to\infty} f^n(y) = x_0$ and $f^n(y) > 0$ for $n = 1, 2, \dots$. In particular, if $f'_E(x_0)$ exists and $0 < f'_E(x_0) < 1$, the conclusion follows.

Proof. Replacing f by $f(x + x_0) - x_0$ we can reduce the proof to the case that $x_0 = 0$. The assumptions of the theorem imply that there exist $c_0 < b_0$, and a sequence $\{a_n\}$ converging monotone decreasingly to 0 such that for each n, $a_n \in E$, $a_{n+1}/a_n > d_2$, $[c_0, b_0] \subset E \cap [0, a_1]$ and $\frac{c_0}{b_0} < d_1$. Put

$$f_1(x) = \begin{cases} f(x), & \text{if } f(x) \le x \text{ and } 0 \le x; \\ x, & \text{if } f(x) > x \text{ and } 0 \le x; \\ f(x), & \text{if } x < 0. \end{cases}$$

Observe that $f_1(a_n)/a_n < d_2$, $c_0 < d_1 b_0 < d_2 b_0 < b_0 \le a_1$ and finally $f_1(b_0)/b_0 > d_1 > c_0/b_0$. Therefore $f_1(b_0) > c_0$, and hence $f_1([c_0, b_0]) \cap [c_0, b_0] \ne \emptyset$. Thus we can apply Lemma 2.1 and Lemma 2.2 for f_1 with $\{a_n\}, d_2, c_0, b_0$ and find a point $y \in [c_0, b_0] \subset E$ such that

$\lim_{n\to\infty} f_1^n(y) = 0$, $f_1^n(y) > 0$, $n = 0, 1, \ldots$. If there exists an $n \geq 1$ such that $f_1^n(y) \neq f^n(y)$ then $f_1(f_1^{n-1}(y)) = f_1^{n-1}(y)$ and hence $f_1^m(y) = f_1^{m-n+1}(f_1^{n-1}(y)) = f_1^{n-1}(y)$ which contradicts to $\lim_{n\to\infty} f_1^n(y) = 0$. Thus $f_1^n(y) = f^n(y)$ and this proves Theorem 2.2.

Example 2.3. In this example we want to show that in Theorem 2.2 the condition $0 < d_1 < \frac{f(x)-f(x_0)}{x-x_0} < d_2 < 1$ can not be replaced by $0 < \frac{f(x)-f(x_0)}{x-x_0} < 1$. Put

$$[c_n, b_n] = [\frac{1}{n}\frac{1}{(n+1)!}, \frac{1}{n}\frac{1}{n!}]$$

and $E = \cup_{n=1}^{\infty}[c_n, b_n]$. Then E is non porous from the right at 0 and $\mathbf{R} \setminus E$ is strongly porous from the right at 0. Put $f(\frac{b_{n+1}+c_n}{2}) = \frac{b_{n+1}+c_n}{2}$, $n = 1, 2, \ldots$, and $f(x) = \frac{b_{n+1}+c_n}{2}$ for $x \in [c_n, b_n]$, $n = 1, 2, \ldots$. Otherwise choose f so that it is continuous on $\mathbf{R}$. Then, obviously no point from E is attracted to 0.

3. Results for "typical" continuous functions.

The previous results show that adequate behavior of the difference quotient $\frac{f(x)-f(x_0)}{x-x_0}$ on sets satisfying certain porosity conditions with respect to a fixed point x_0 will guarantee certain attractive behavior to x_0. But "most " continuous functions will not exhibit this behavior with respect to any fixed point [BH]. In this section, we make this statement precise and show that, nonetheless, most continuous functions have the property that to each fixed point x_0 corresponds a nonempty perfect set $P(x_0)$ such that for each $y \in P(x_0)$, $\lim_{n\to\infty} f^n(y) = x_0$ with $f^n(y) \neq x_0$ for each $n = 1, 2, \ldots$.

Let $\mathcal{C}$ denote the class of continuous functions mapping I into I. When $\mathcal{C}$ is furnished with the sup norm it becomes a complete metric space.

Theorem 3.1. *a) Let A consist of those functions f in $\mathcal{C}$ such that for each $x_0 \in I$ the set*

$$S(x_0) = \{x : |\frac{f(x) - f(x_0)}{x - x_0}| < 1\}$$

is bilaterally strongly porous at x_0. Then A is a residual subset of $\mathcal{C}$.

b) Let B consist of those functions f in $\mathcal{C}$ such that to every fixed point x_0 of f corresponds a nonempty perfect set $P(x_0)$ such that $\lim_{n\to\infty} f^n(y) = x_0$ for $y \in P(x_0)$ and $f^n(y) \neq x_0$ for all n. Then B is a residual subset of $\mathcal{C}$.

Proof. a) In [BH] one finds that for f in some residual subset D of $\mathcal{C}$, the set $S = \{x : f(x) = g(x)\}$ is bilaterally strongly porous at each point of S whenever f satisfies a Lipschitz condition. If for some $x_0 \in I$ the set $S(x_0)$ in part a) of the statement fails to be bilaterally strongly porous at x_0, we can argue as in the proof of [BH] Thm. 3.2, that is, we can find a Lipschitz function g that agrees with f on a subset of $S(x_0)$ that is also not strongly porous at x_0. It follows that $D \subset A$ so A is residual in $\mathcal{C}$.

To prove b) assume that $I = [a_0, b_0]$ and put $\mathcal{C}_1 = \{f \in \mathcal{C} : f(a_0) \neq a_0 \text{ and } f(b_0) \neq b_0\}$. Obviously $\mathcal{C}_1$ is a residual subset of $\mathcal{C}$.

Assume that $J = [a, b] \subset I$ is given. Denote by $M(J)$ the set of functions $f \in \mathcal{C}_1$ such that the maximum of f on J is achieved at a fixed point of f and this fixed point is different from b. First we show that for a given J the set $M(J)$ is nowhere dense in $\mathcal{C}_1$. Let $N(f, \epsilon)$ be the sphere in $\mathcal{C}_1$ that is centered at f and has radius $\epsilon > 0$. We find a sphere $N(g, \delta)$ inside $N(f, \epsilon)$ whose intersection with $M(J)$ is empty. Without loss of generality, suppose $f \in M(J)$ with $f(c) = c = \max_{x\in J} f(x)$, $a \leq c < b$. We may also assume $\epsilon < b - c$ and choose $d \in (c, b)$ with $d - c < \epsilon/2$. Let $g \in N(f, \epsilon/2)$ satisfy $d < g(c) < f(c) + \epsilon/2$, $g(x) \leq c$ for $x \geq d$. This is possible since $f \leq c$ on J. Let u be the maximum fixed point of g in J. Then $c < u < d$. Choose $\delta \in (0, \epsilon/2)$ such that $g(c) - \delta > d$ and $g(x) + \delta < d$ for $x \geq u$. If $h \in N(g, \delta)$ then $h(c) > d$ and h has no fixed points greater than $d \in J$. Thus $N(g, \delta) \cap M(J)$ is empty and $N(g, \delta) \subset N(f, \epsilon)$. It follows that $M(J)$ is nowhere dense in $\mathcal{C}_1$.

By applying the above result to each interval with rational endpoints and by applying a similar argument for "minimum" instead of "maximum", we find that the set of functions E in $\mathcal{C}_1$ that achieve a local extremum at a fixed point is first category in $\mathcal{C}_1$ and hence these functions form a first category subset of $\mathcal{C}$.

Now according to a theorem of Jarnik [B:p.213], each function in some residual subset S of $\mathcal{C}$ has every real number as a derived number at every point. This means that for $f \in S$, $x_0 \in I$ and $t \in \mathbf{R}$, there exists a sequence x_n converging to x_0 such that $(f(x_n) - f(x_0))/(x_n - x_0)$ converges to t. Let f be in $(\mathcal{C}_1 \cap S) \setminus E$. Applying Lemma 2.3 to f (with respect to x_0 and $f(x_0)$ instead of 0 and $f(0)$) we find f in the required set B. Thus the residual set $(\mathcal{C}_1 \cap S) \setminus E$ is contained in B so B is residual in $\mathcal{C}$. This concludes our proof.

Finally we remark that using similar techniques to those in Chapter VIII of [B] one can show that for the typical $f : I \to I$ the fixed point set is a nowhere dense non-empty perfect set.

References

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