CHARGES, BV FUNCTIONS, AND MULTIPLIERS FOR GENERALIZED RIEMANN INTEGRALS

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Abstract

For an invariant generalized Riemann integral in $\mathbb{R}^m$, we obtain the following results. (1) A function is a multiplier for the space of locally integrable functions if and only if it is locally bounded and locally BV. (2) The dual of the space of all functions integrable in a bounded BV set A is linearly isomorphic to the space of all bounded BV functions vanishing outside A, and each element of the dual has the usual integral representation. (3) On Lipschitz domains an integration by parts formula holds for any continuous function that is pointwise Lipschitz everywhere except on a set of σ-finite $(m-1)$-dimensional Hausdorff measure.

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1 Introduction

In dimension one, the problem of recovering derivative by integration has been solved a long time ago by Denjoy and Perron [20, Chapters 6 and 8]. In higher dimensions, however, a substantial progress was made only in the eighties. Following the ideas of Henstock, Kurzweil, and McShane, several authors suggested various Riemann type extensions of the Lebesgue integral which provide the Gauss-Green theorem for any differentiable (not necessarily continuously) vector field. Among these generalized Riemann integrals, the most viable appear those introduced in [15, Chapters 12 and 13]: they are defined over bounded BV sets, are invariant with respect to lipeomorphic changes of coordinates, and integrate functions with large, topologically unrestricted, sets of singularities. They have a common extension, referred to as the R_*-integral in the present paper, and as the continuous integral in [13, Section 9].

Our main result is a complete solution of the multiplier problem for the R_*-integral. Specifically, we show the family of all functions, called *multipliers*, whose product with every locally R_*-integrable function is again locally R_*-integrable is the algebra of all locally bounded locally BV functions. Using this, we obtain an integral representation of continuous linear functionals on the space of all functions that are R_*-integrable in a bounded BV set. In the remainder of this section, we outline briefly the main points of our work for a fixed bounded BV set $A \subset \mathbb{R}^m$.

In the linear space $BV^\infty(A)$ of all bounded BV functions vanishing outside A, we consider a sequentially complete locally convex topology τ induced by the following convergence: $f_i \to 0$ if and only if $\{f_i\}$ converges to zero in $L^1(A)$, and the numerical sequences formed by the L^∞ norms and the total variations of the functions f_i are bounded. The family $\boldsymbol{\mathcal{BV}}(A)$ of all BV subsets of A is topologized as a subspace of $\big(BV^\infty(A), \tau\big)$.

A charge is an additive τ-continuous function defined on $\boldsymbol{\mathcal{BV}}(A)$. The linear space of all charges, topologized by the uniform convergence on τ-compact subfamilies of $\boldsymbol{\mathcal{BV}}(A)$, is a Banach space denoted by $CH(A)$ (Proposition 3.4). Following [12] and [16], with each charge F, we associate a function $F \times \mathcal{L}^1$ defined by the formula

$$(F \times \mathcal{L}^1)(B) = \int_{\mathbb{R}} F(B^t)\, dt\,,$$

where B is a bounded BV subset of $A \times \mathbb{R}$ and $B^t = \big\{x \in A : (x,t) \in B\big\}$. Observing the subgraph $\Sigma_g = \big\{(x,t) \in A \times \mathbb{R} : 0 < t < g(x)\big\}$ of a nonnegative function $g \in BV^\infty(A)$ is a bounded BV subset of $A \times \mathbb{R}$, we obtain a nondegenerate, bilinear, and separately continuous pairing

$$\langle F, g\rangle = (F \times \mathcal{L}^1)(\Sigma_{g^+}) - (F \times \mathcal{L}^1)(\Sigma_{g^-})\,,$$

which puts the Banach space $CH(A)$ and the space $\big(BV^\infty(A), \tau\big)$ in duality. The map Θ that sends a charge F to the functional

$$\Theta(F) : BV^\infty(A) \to \mathbb{R} : g \mapsto \langle F, g\rangle$$

is a linear homeomorphism from $CH(A)$ onto the dual of $\big(BV^{\infty}(A), \tau\big)$ equipped with the topology of uniform convergence on τ-compact subfamilies of $BV^{\infty}(A)$ (Theorem 4.3 and Remark 4.4).

It is well known each absolutely continuous (abbreviated as AC) charge F is derivable almost everywhere, and F is the indefinite Lebesgue integral of the derivate F'. Given a charge F, there is a Borel regular measure V_*F, which is absolutely continuous and finite if and only if F is AC [2, Proposition 5.11]. If V_*F is merely absolutely continuous (not necessarily finite), we say F is an AC$_*$ charge. It was proved in [3, Corollaries 4.3 and 4.7] each AC$_*$ charge F is derivable almost everywhere, and F is uniquely determined by the derivate F'. Thus, in analogy with the Lebesgue integral, we say a function f defined on A is R-integrable whenever there is an AC$_*$ charge F such that $F' = f$ almost everywhere. If A is a compact interval, this definition produces the generalized Riemann integral of [15, Definition 12.2.1]. We denote by $R(A)$ the linear space of all R-integrable functions. Since the R-integral extends the Lebesgue integral, writing $\int_A f \, d\mathcal{L}^m$ for $f \in R(A)$ leads to no confusion.

If F is an AC$_*$ charge, $g \in BV^{\infty}(A)$, and χ_B is the indicator function of $B \in \boldsymbol{\mathcal{BV}}(A)$, we let

$$G(B) = \langle F, g\chi_B \rangle ,$$

and show G is an AC$_*$ charge with $G' = F'g$ almost everywhere. Rewriting this in terms of the R-integral solves the difficult part of the multiplier problem (Theorem 6.4): if $f \in R(A)$ and $g \in BV^{\infty}(A)$, then $fg \in R(A)$ and

$$\int_A fg \, d\mathcal{L}^m = \langle F, g \rangle .$$

Denote by $CH_{AC}(A)$ and $CH_{AC_*}(A)$ the linear spaces of all AC and AC$_*$ charges, respectively, and topologize $R(A)$ so that the linear bijection

$$F \mapsto F' : CH_{AC_*}(A) \to R(A)$$

becomes a homeomorphism. We show $CH_{AC}(A)$ is dense in $CH(A)$ (Corollary 4.9 — a different proof presented in [4, Corollary 4.1] is incomplete) and infer $\big(BV^{\infty}(A), \tau\big)$ is dual to each linear space $X \subset CH(A)$ containing $CH_{AC}(A)$. In particular, every $g \in BV^{\infty}(A)$ defines a continuous linear functional

$$T_g : R(A) \to \mathbb{R} : f \mapsto \int_A fg \, d\mathcal{L}^m ,$$

and the map $g \mapsto T_g$ is a continuous linear bijection from $\big(BV^{\infty}(A), \tau\big)$ onto the dual of $R(A)$ equipped with the w^*-topology (Proposition 7.6).

Since the R$_*$-integral is obtained by a transfinite iteration of improper R-integrals (Definition 8.1), the extension of the previous results to the R$_*$-integral is easy. We prove a very general Gauss-Green theorem (Theorem 8.6) and, in case A is a Lipschitz domain, we establish the integration by parts formula (Theorem 7.17) for continuous vector fields that are pointwise Lipschitz except on a set of σ-finite $(m-1)$-dimensional Hausdorff measure.

2 Preliminaries

The set of all real numbers is denoted by $\mathbb{R}$, and the ambient space of this paper is $\mathbb{R}^m$ where $m \geq 1$ is a fixed integer. In $\mathbb{R}^m$ we use exclusively the metric induced by the maximum norm $|\cdot|$. Specifically, the diameter of a set $E \subset \mathbb{R}^m$ is defined as

$$d(E) = \sup\{|x-y| : x, y \in E\}.$$

For $r > 0$, we let $B(r) = (-r, r)^m$ and $B[r] = [-r, r]^m$, and if $x \in \mathbb{R}^m$, then

$$B(x, r) = \{y \in \mathbb{R}^m : |x-y| < r\} \quad \text{and} \quad B[x, r] = \{y \in \mathbb{R}^m : |x-y| \leq r\}.$$

By $x \cdot y$ we denote the usual inner product of $x, y \in \mathbb{R}^m$, and let $\|x\| = \sqrt{x \cdot x}$.

A *cell* is a compact nondegenerate subinterval of $\mathbb{R}^m$, and a *figure* is a finite (possibly empty) union of cells. The family of all figures is denoted by $\mathcal{F}$. The closure, interior, and boundary of a set $E \subset \mathbb{R}^m$ are denoted by $\operatorname{cl} E$, $\operatorname{int} E$, and ∂E, respectively. If A and B are sets, then $A \triangle B$ denotes their symmetric difference.

When no attributes are added, a function is always real-valued. If f is a function defined on a set $E \subset \mathbb{R}^m$, the sets

$$\operatorname{null} f = \{x \in E : f(x) = 0\} \quad \text{and} \quad \operatorname{supp} f = \operatorname{cl}\{x \in E : f(x) \neq 0\}$$

are called, respectively, the *null set* and *support* of f. By χ_E we denote the *indicator function* of a set $E \subset \mathbb{R}^m$.

The Lebesgue measure in $\mathbb{R}^m$ is denoted by $\mathcal{L}^m$; however, for $E \subset \mathbb{R}^m$, we write $|E|$ instead of $\mathcal{L}^m(E)$. A set $E \subset \mathbb{R}^m$ with $|E| = 0$ is called *negligible*. Functions f and g defined on a set $E \subset \mathbb{R}^m$ are called *equivalent* if they differ only on a negligible subset of E. Subsets A and B of $\mathbb{R}^m$ are called *equivalent* whenever their indicator functions are equivalent, i.e., whenever $|A \triangle B| = 0$. As usual, we shall tacitly identify a set $E \subset \mathbb{R}^m$, or a function f defined on E, with the equivalence class determined by E, or f, respectively. Unless specified otherwise, the words "measure" and "measurable" as well as the expressions "almost all," "almost everywhere," and "absolutely continuous" always refer to the Lebesgue measure $\mathcal{L}^m$.

For a measurable set $E \subset \mathbb{R}^m$ and $0 \leq \alpha \leq 1$, let

$$E(\alpha) = \left\{ x \in \mathbb{R}^m : \lim_{r \to 0+} \frac{|E \cap B[x, r]|}{|B[x, r]|} = \alpha \right\}.$$

The sets $\operatorname{int}_* E = E(1)$, $\operatorname{cl}_* E = \mathbb{R}^m - E(0)$, and $\partial_* E = \mathbb{R}^m - [E(0) \cup E(1)]$ are called the *essential interior, essential closure*, and *essential boundary* of E, respectively. We say E is *essentially closed* whenever $\operatorname{cl}_* E$ is closed.

If $s \geq 0$ and $k > s$ is an integer, we denote by $\mathcal{H}^s$ the s-dimensional Hausdorff measure in $\mathbb{R}^k$; since k is always determined by the context, this notation leads to no confusion. A set $S \subset \mathbb{R}^m$ of σ-finite measure $\mathcal{H}^{m-1}$ is called *thin*. Clearly, each thin set is negligible but not vice versa.

In our exposition we shall use extensively the *functions of bounded variation*, called BV functions, and the *sets of finite perimeter*, called BV sets. For their definitions and properties we refer to [6, Chapter 5].

The linear space of all bounded BV functions in $\mathbb{R}^m$ with compact support is denoted by BV_c^∞, and the linear space of all locally bounded locally BV functions in $\mathbb{R}^m$ is denoted by BV_{loc}^∞, i.e.,

$$BV_c^\infty = BV(\mathbb{R}^m) \cap L_c^\infty(\mathbb{R}^m) \quad \text{and} \quad BV_{\mathrm{loc}}^\infty = BV_{\mathrm{loc}}(\mathbb{R}^m) \cap L_{\mathrm{loc}}^\infty(\mathbb{R}^m).$$

If $f \in BV(\mathbb{R}^m)$, we denote by Df and $\|Df\|$ the *distributional gradient* and the *variational measure* of f, respectively; the number $\|f\| = \|Df\|(\mathbb{R}^m)$ is called the *total variation* of f. For $n = 1, 2, \ldots$, topologize

$$BV_n = \bigl\{ f \in BV_c^\infty : \operatorname{supp} f \subset B[n] \text{ and } \|f\| + \|f\|_\infty \le n+1 \bigr\}$$

by the $L^1(\mathbb{R}^m)$ norm $\|\cdot\|_1$. According to [6, Section 5.2, Theorems 1 and 4], each BV_n is a compact set. The largest topology τ in BV_c^∞ for which all inclusion maps $BV_n \hookrightarrow BV_c^\infty$ are continuous is a nonmetrizable locally convex Hausdorff topology that is sequential [5, Section 1.7] and sequentially complete. A sequence $\{f_i\}$ in $\bigl(BV_c^\infty, \tau\bigr)$ converges to zero if and only if $\lim \|f_i\|_1 = 0$, $\sup\bigl(\|f_i\| + \|f_i\|_\infty\bigr) < \infty$, and $\operatorname{supp} f_i \subset K$ for a compact set $K \subset \mathbb{R}^m$ and $i = 1, 2, \ldots$.

The family of all bounded BV subsets of $\mathbb{R}^m$ is denoted by $\boldsymbol{\mathcal{BV}}$. If $E \in \boldsymbol{\mathcal{BV}}$, we write $\|\partial E\|$ instead of $\|D\chi_E\|$, and call $\|E\| = \|\partial E\|(\mathbb{R}^m)$ the *perimeter* of E. The *regularity* of a set $E \in \boldsymbol{\mathcal{BV}}$ is the number

$$r(E) = \begin{cases} \frac{|E|}{d(E)\|E\|} & \text{if } d(E)\|E\| > 0, \\ 0 & \text{otherwise,} \end{cases}$$

which refines the usual concept of regularity employed in connection with Vitali's covering theorem [20, Chapter 4, Section 2].

According to [6, Section 5.11, Theorem 1], a bounded measurable set E is a BV set if and only if $\mathcal{H}^{m-1}(\partial_* E) < \infty$, in which case $\|E\| = \mathcal{H}^{m-1}(\partial_* E)$. If E is a BV set, then $\partial_* E$ is $\mathcal{H}^{m-1}$-measurable, and there is an $\mathcal{H}^{m-1}$-measurable unit vector field ν_E defined $\mathcal{H}^{m-1}$-almost everywhere on $\partial_* E$ such that the Gauss-Green formula

$$\int_E \operatorname{div} v \, d\mathcal{L}^m = \int_{\partial_* E} v \cdot \nu_E \, d\mathcal{H}^{m-1}$$

holds for each $v \in C_c^1(\mathbb{R}^m; \mathbb{R}^m)$ [6, Section 5.8, Theorem 1].

Identifying each set $E \in \boldsymbol{\mathcal{BV}}$ with its indicator function χ_E, the family $\boldsymbol{\mathcal{BV}}$ is a closed subspace of (BV_c^∞, τ). It follows the topology τ restricted to $\boldsymbol{\mathcal{BV}}$, still denoted by τ, is a sequential topology induced by a sequentially complete uniformity. For $n = 1, 2, \ldots$, the topology τ restricted to

$$\boldsymbol{\mathcal{BV}}_n = \boldsymbol{\mathcal{BV}} \cap BV_n = \bigl\{ A \in \boldsymbol{\mathcal{BV}} : A \subset B[n] \text{ and } \|A\| \le n \bigr\}$$

is compact and induced by the metric $(B,C) \mapsto |B \triangle C|$. A sequence $\{A_i\}$ in $(\mathcal{BV}, \tau)$ converges to a set $A \in \mathcal{BV}$ if and only if $\lim |A_i \triangle A| = 0$, $\sup \|A_i\| < \infty$, and $A_i \subset K$ for a compact set $K \subset \mathbb{R}^m$ and $i = 1, 2, \ldots$. For a set $A \in \mathcal{BV}$, let

$$BV^\infty(A) = \{f \in BV_c^\infty : \mathbb{R}^m - A \subset \operatorname{null} f\},$$
$$\mathcal{BV}(A) = \mathcal{BV} \cap BV^\infty(A) = \{B \in \mathcal{BV} : B \subset A\}.$$

3 Charges

A *charge* is an additive τ-continuous function defined on $\mathcal{BV}$. Explicitly, a function $F : \mathcal{BV} \to \mathbb{R}$ is a charge if it satisfies the following two conditions.

Additivity: $F(A \cup B) = F(A) + F(B)$ for each pair of disjoint bounded BV sets A and B.

Continuity: Given $\varepsilon > 0$, there is an $\eta > 0$ such that $|F(C)| < \varepsilon$ for each BV set $C \subset B(1/\varepsilon)$ with $\|C\| < 1/\varepsilon$ and $|C| < \eta$.

Observe an additive function $F : \mathcal{BV} \to \mathbb{R}$ is a charge whenever $\lim F(A_i) = 0$ for each sequence $\{A_i\}$ in $\mathcal{BV}$ that τ-converges to the empty set. By continuity, a charge vanishes on all bounded negligible sets; in particular, the value of a charge on a set $E \in \mathcal{BV}$ depends only on the equivalence class of E. If F is a charge and A is a locally BV set, define a charge $F \llcorner A$ by letting

$$(F \llcorner A)(B) = F(A \cap B)$$

for each $B \in \mathcal{BV}$. We say F is a *charge in* A whenever $F = F \llcorner A$. The linear spaces of all charges and of all charges in a locally BV set A are denoted by CH and $CH(A)$, respectively. The following are canonical examples of charges.

(1) If $f \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$ and $F(A) = \int_A f \, d\mathcal{L}^m$ for each bounded BV set A, then F is a charge by the absolute continuity of the Lebesgue integral. It is called the *indefinite Lebesgue integral* of f, denoted by $\int f \, d\mathcal{L}^m$.

(2) If $v \in C(\mathbb{R}^m; \mathbb{R}^m)$ and $F(A) = \int_{\partial_* A} v \cdot \nu_A \, d\mathcal{H}^{m-1}$ for each bounded BV set A, then F is a charge by [15, Proposition 11.2.8]. It is called the *flux* of v, denoted by $\int v \cdot \nu \, d\mathcal{H}^{m-1}$.

According to DeGiorgi's approximation theorem [2, Proposition 1.1], the family $\mathcal{F}$ of all figures is sequentially dense in $(\mathcal{BV}, \tau)$, and hence $(\mathcal{BV}, \tau)$ is the sequential completion of $(\mathcal{F}, \tau)$. As each additive τ-continuous function on $\mathcal{F}$ is uniformly continuous, it has a unique extension to a charge. The next proposition, proved in [12, Proposition 1], provides a useful characterization of charges.

Proposition 3.1. *An additive function F on $\mathcal{BV}$ is a charge if and only if given $\varepsilon > 0$, there is a $\theta > 0$ such that*

$$|F(B)| < \theta |B| + \varepsilon \big(\|B\| + 1\big)$$

for each BV set $B \subset B(1/\varepsilon)$.

For a charge F and $n = 1, 2, \dots$, let

$$\lvert F \rvert_n = \sup\bigl\{|F(B)| : B \in \boldsymbol{\mathcal{BV}}_n\bigr\}.$$

As $\boldsymbol{\mathcal{BV}}$ is the union of an increasing sequence $\{\boldsymbol{\mathcal{BV}}_n\}$ of compact sets, $\bigl\{\lvert \cdot \rvert_n\bigr\}$ is an increasing sequence of seminorms in CH which defines a metrizable locally convex topology σ in CH. Since σ is the topology of uniform convergence on the families $\boldsymbol{\mathcal{BV}}_n$, it is easy to verify (CH, σ) is a Fréchet space $\bigl($cf. the *natural topology* of [4, Definition 4.2]$\bigr)$.

Lemma 3.2. *Each $C \in \boldsymbol{\mathcal{BV}}$ is the union of disjoint BV sets $C_1, \dots, C_p$ such that $p \le \max\bigl\{1, 2\|C\|\bigr\}$ and $\|C_i\| < 1 + 4d(C)^{m-1}$ for $i = 1, \dots, p$.*

Proof. We employ an idea used in the proof of [12, Lemma 1]. Avoiding a triviality, assume $\gamma = 2d(C)^{m-1} > 0$, and for every $t \in \mathbb{R}$, let

$$C_t^- = C \cap \bigl((-\infty, t) \times \mathbb{R}^{m-1}\bigr) \quad \text{and} \quad C_t^+ = C \cap \bigl([t, +\infty) \times \mathbb{R}^{m-1}\bigr).$$

Then C is the disjoint union of the sets $C_t^\pm$, and

$$\|C_t^-\| + \|C_t^+\| \le \|C\| + \gamma.$$

The function $t \mapsto \|C_t^-\|$ increases from 0 to $\|C\|$, and

$$\lim_{r \to t+} \|C_r^-\| - \lim_{r \to t-} \|C_r^-\| \le \frac{\gamma}{2}.$$

Thus there is an $s \in \mathbb{R}$ with

$$\frac{1}{2}\|C\| \le \|C_s^-\| \le \frac{1}{2}\|C\| + \gamma,$$

which implies $\|C_s^\pm\| \le \frac{1}{2}\|C\| + \gamma$. Now let $n \ge 0$ be the least integer with $\|C\| \le 2^n$, and observe $2^n \le \max\bigl\{1, 2\|C\|\bigr\}$. Proceeding inductively, construct disjoint BV sets $C_1, \dots, C_{2^n}$ so that $C = \bigcup_{k=1}^{2^n} C_k$ and

$$\|C_i\| \le 2^{-n}\|C\| + \gamma \sum_{k=0}^{n-1} 2^{-k} < 1 + 2\gamma$$

for $i = 1, \dots, 2^n$. ☐

Corollary 3.3. *Let $C \in \boldsymbol{\mathcal{BV}}\bigl(B[k]\bigr)$ for an integer $k \ge 1 + 4d(C)^{m-1}$, and assume $\|C\| \ge k$. If F is a charge, then $\bigl|F(C)\bigr| \le 2\|C\| \cdot \lvert F \rvert_k$.*

Proof. By Lemma 3.2, the set C is the union of disjoint BV sets $C_1, \dots, C_p$ such that $p \le 2\|C\|$ and $\|C_i\| < k$ for $i = 1, \dots, p$. Thus each C_i belongs to $\boldsymbol{\mathcal{BV}}_k$, and we have

$$\bigl|F(C)\bigr| \le \sum_{i=1}^{p} \bigl|F(C_i)\bigr| \le 2\|C\| \cdot \lvert F \rvert_k.$$

☐

Proposition 3.4. *Let $A \in \mathcal{BV}(B[k])$ for an integer $k \geq 1 + 4d(A)^{m-1}$. If $n \geq k$ is an integer and F is a charge in A, then*

$$\lVert F \rVert_n \leq 2(\|A\| + n)\lVert F \rVert_k .$$

The seminorm $\lVert \cdot \rVert_k$ is a norm in $CH(A)$ that induces the subspace topology in $CH(A) \subset (CH, \sigma)$. In particular, $(CH(A), \lVert \cdot \rVert_k)$ is a Banach space.

Proof. Given a set $C \in \mathcal{BV}_n$, we distinguish two cases. Either $\|A \cap C\| \leq k$ which yields $|F(C)| = |F(A \cap C)| \leq \lVert F \rVert_k$, or $\|A \cap C\| > k$ and then

$$|F(C)| = |F(A \cap C)| \leq 2\|A \cap C\| \cdot \lVert F \rVert_k \leq 2(\|A\| + n)\lVert F \rVert_k$$

by Corollary 3.3. In either case $\lVert F \rVert_n \leq 2(\|A\| + n)\lVert F \rVert_k$, and we conclude $\lVert \cdot \rVert_k$ and $\lVert \cdot \rVert_n$ are equivalent norms in $CH(A)$. □

If E is a bounded BV subset of $\mathbb{R}^{m+1}$, it follows from [10, Theorems 20 and 33] the set

$$E^t = \{x \in \mathbb{R}^m : (x, t) \in E\}$$

belongs to $\mathcal{BV}$ for $\mathcal{L}^1$-almost all $t \in \mathbb{R}$.

Proposition 3.5. *Let F be a charge, and let E be a bounded BV subset of $\mathbb{R}^{m+1}$. Then the function $t \mapsto F(E^t)$, defined for $\mathcal{L}^1$-almost all $t \in \mathbb{R}$, belongs to $L^1(\mathbb{R})$. Letting*

$$(F \times \mathcal{L}^1)(E) = \int_{\mathbb{R}} F(E^t)\, dt$$

for each bounded BV set $E \subset \mathbb{R}^{m+1}$ defines a charge $F \times \mathcal{L}^1$ in $\mathbb{R}^{m+1}$. The map $F \mapsto F \times \mathcal{L}^1$ is a continuous linear map from $(CH(\mathbb{R}^m), \sigma)$ to $(CH(\mathbb{R}^{m+1}), \sigma)$.

Proof. If $A \subset \mathbb{R}^{m+1}$ is an $(m+1)$-dimensional figure, then

$$\widehat{F}(A) = \int_{\mathbb{R}} F(A^t)\, dt$$

is well defined, since $t \mapsto F(A^t)$ is an $\mathcal{L}^1$-measurable simple function vanishing outside a compact set. Choose an $\varepsilon > 0$ and $a < b$. By Proposition 3.1, there is a $\theta > 0$ such that

$$|F(B)| < \theta|B| + \varepsilon(\|B\| + 1)$$

for each BV set $B \subset [a, b]^m$. Thus if A is an $(m+1)$-dimensional subfigure of $[a, b]^{m+1}$, then

$$\begin{aligned} |\widehat{F}(A)| &\leq \int_a^b |F(A^t)|\, dt \leq \theta \int_a^b |A^t|\, dt + \varepsilon \int_a^b (\|A^t\| + 1)\, dt \\ &\leq \theta\mathcal{L}^{m+1}(A) + \varepsilon[\mathcal{H}^m(\partial A) + (b - a)] . \end{aligned} \tag{3.1}$$

We infer $\widehat{F}$ has a unique extension to a charge in $\mathbb{R}^{m+1}$, denoted by $F \times \mathcal{L}^1$.

Select a BV set E in $\mathbb{R}^{m+1}$ with $\operatorname{cl} E \subset (a,b)^{m+1}$. By DeGiorgi's approximation theorem [2, Proposition 1.1], there is a sequence $\{A_i\}$ of $(m+1)$-dimensional figures contained in $[a,b]^{m+1}$ such that

$$\lim \mathcal{L}^{m+1}(A_i \triangle E) = 0 \quad \text{and} \quad \sup \mathcal{H}^m(\partial A_i) < \infty \,. \tag{3.2}$$

If $f_i : t \mapsto F(A_i^t)$, then

$$\left|f_i(t) - f_j(t)\right| \leq \left|F(A_i^t - A_j^t)\right| + \left|F(A_j^t - A_i^t)\right|$$

for all $t \in \mathbb{R}$, and inequality (3.1) implies

$$\begin{aligned} \int_{\mathbb{R}} \left|f_i(t) - f_j(t)\right| dt = \int_a^b \left|f_i(t) - f_j(t)\right| dt \leq \\ \theta\mathcal{L}^{m+1}(A_i \triangle A_j) + 2\varepsilon\big[\mathcal{H}^{m+1}(\partial A_i) + \mathcal{H}^{m+1}(\partial A_j) + (b-a)\big] \,. \end{aligned} \tag{3.3}$$

As ε is arbitrary, relations (3.2) and (3.3) imply $\{f_i\}$ is a Cauchy sequence in $L^1(\mathbb{R})$, and hence it converges in $L^1(\mathbb{R})$ to an $f \in L^1(\mathbb{R})$. It follows $\{f_i\}$ has a subsequence, still denoted by $\{f_i\}$, such that $\lim f_i(t) = f(t)$ for all $t \in \mathbb{R} - N$ where $\mathcal{L}^1(N) = 0$. In view of the Fubini and dominated convergence theorems, we may assume $\lim |A_i^t \triangle E^t| = 0$ for each $t \in \mathbb{R} - N$. Fatou's lemma yields

$$\int_{\mathbb{R}} \liminf \|A_i^t\| \, dt \leq \liminf \int_{\mathbb{R}} \|A_i^t\| \, dt \leq \sup \mathcal{H}^{m+1}(\partial A_i) < \infty \,,$$

so we may further assume $\liminf \|A_i^t\| < \infty$ for every $t \in \mathbb{R} - N$. Select a $t \in \mathbb{R} - N$, and find a subsequence $\{A_{i_k}\}$ of $\{A_i\}$ with $\sup \|A_{i_k}^t\| < \infty$. Since F is a charge,

$$f(t) = \lim f_{i_k}(t) = \lim F(A_{i_k}^t) = F(E^t) \,,$$

and we conclude $f(t) = F(E^t)$ for all $t \in \mathbb{R} - N$. As $\{f_i\}$ converges to f in $L^1(\mathbb{R})$, we obtain

$$(F \times \mathcal{L}^1)(E) = \lim \widehat{F}(A_i) = \lim \int_{\mathbb{R}} F(A_i^t) \, dt = \int_{\mathbb{R}} F(E^t) \, dt \,.$$

Now given an integer $n \geq 1$, let $k = \max\{n, 1 + 4(2n)^{m-1}\}$. If $E \subset [-n,n]^{m+1}$ is a BV set in $\mathbb{R}^{m+1}$ and $\|E\| \leq n$, let

$$J_- = \big\{t \in [-n,n] : \|E^t\| \leq k\big\} \quad \text{and} \quad J_+ = \big\{t \in [-n,n] : \|E^t\| > k\big\} \,.$$

The first part of the proof and Corollary 3.3 imply

$$\begin{aligned} \left|(F \times \mathcal{L}^1)(E)\right| &\leq \int_{J_-} \left|F(E^t)\right| dt + \int_{J_+} \left|F(E^t)\right| dt \\ &\leq \mathcal{L}^1(J_-)|F|_k + 2|F|_k \int_{J_+} \|E^t\| \, dt \\ &\leq \big(\mathcal{L}^1(J_-) + 2\|E\|\big)|F|_k \leq 4n|F|_k \,, \end{aligned}$$

and hence $|F \times \mathcal{L}^1|_n \leq 4n|F|_k$. As the map $F \mapsto F \times \mathcal{L}^1$ is clearly linear, it is also continuous. $\square$

4 Duality

Denote by CH^* and $(BV_c^\infty)^*$ the dual spaces of (CH, σ) and (BV_c^∞, τ), respectively. We endow $(BV_c^\infty)^*$ with a Fréchet topology ν induced by seminorms

$$|T|_n = \sup\{|T(f)| : f \in BV_n\}$$

where $T \in (BV_c^\infty)^*$ and $n = 1, 2 \dots$ (cf. the topology σ in CH defined in Section 3).

The *subgraph* of a nonnegative function g defined on $\mathbb{R}^m$ is the set

$$\Sigma_g = \{(x,t) \in \mathbb{R}^m \times \mathbb{R} : 0 < t < g(x)\}.$$

It follows from the coarea theorem [6, Section 5.5, Theorem 1] that the subgraph Σ_g of a nonnegative function $g \in BV_c^\infty$ is a bounded BV subset of $\mathbb{R}^{m+1}$. In view of Proposition 3.5, the number

$$\langle F, g \rangle = (F \times \mathcal{L}^1)(\Sigma_{g^+}) - (F \times \mathcal{L}^1)(\Sigma_{g^-})$$

is defined for each $F \in CH$ and each $g \in BV_c^\infty$, and we introduce the following functionals:

$$\begin{aligned} \Theta(F) &: BV_c^\infty \to \mathbb{R} : g \mapsto \langle F, g \rangle, \\ \Phi(g) &: CH \to \mathbb{R} : F \mapsto \langle F, g \rangle. \end{aligned}$$

Proposition 4.1. *The pairing $(F, g) \mapsto \langle F, g \rangle$ is a nondegenerate separately continuous bilinear functional on $(CH \times BV_c^\infty, \sigma \times \tau)$. In particular, for each $F \in CH$ and each $g \in BV_c^\infty$, the functionals $\Theta(F)$ and $\Phi(g)$ belong to $(BV_c^\infty)^*$ and CH^*, respectively.*

Proof. If $\langle F, g \rangle = 0$ for all $g \in BV_c^\infty$, then $F = 0$. On the other hand, if g is a nonzero element of BV_c^∞, we may assume by symmetry that $\mathcal{L}^{m+1}(\Sigma_{g^+}) > 0$. Let $E_+ = \{x \in \mathbb{R}^m : g(x) > 0\}$ and observe $F = \mathcal{L}^m \llcorner E_+$ is a charge with $\langle F, g \rangle = \mathcal{L}^{m+1}(\Sigma_{g^+})$. It follows the pairing $(F, g) \mapsto \langle F, g \rangle$ is nondegenerate. According to Proposition 3.5, the linear functional $\Phi(g)$ belongs to CH^*.

Let $\{g_i\}$ be a sequence in BV_c^∞ that τ-converges to a $g \in BV_c^\infty(\mathbb{R}^m)$. Then

$$\lim \mathcal{L}^{m+1}(\Sigma_{g_i^\pm} \triangle \Sigma_{g^\pm}) \le \lim \|g_i - g\|_1 = 0.$$

Moreover, $\sup(\|g_i\| + \|g_i\|_\infty) = c < \infty$, and there is a bounded open set $U \subset \mathbb{R}^m$ such that $\operatorname{supp} g_i \subset U$ for $i = 1, 2, \dots$. Thus for each i,

$$\|\Sigma_{g_i^\pm}\| \le 2|U| + \|g_i\| \le 2|U| + c$$

and $\Sigma_{g_i^\pm} \subset U \times [0, c]$. We infer the sequences $\{\Sigma_{g_i^\pm}\}$ τ-converge to $\Sigma_{g^\pm}$. Consequently $\lim \langle F, g_i \rangle = \langle F, g \rangle$ for each charge F, and as τ is a sequential topology, $\Theta(F)$ is a continuous functional on (BV_c^∞, τ). It is a consequence of the coarea theorem [6, Section 5.5, Theorem 1] the linear space S_{BV} of all simple functions in BV_c^∞ is dense in (BV_c^∞, τ). A standard argument reveals if $g \in S_{BV}$ has the form $g = \sum_{i=1}^p c_i E_i$ where $c_1, \dots, c_p$ are real numbers and $E_1, \dots, E_p$ are bounded BV sets, then

$$\Theta(F)(g) = \langle F, g \rangle = \sum_{i=1}^p c_i F(E_i)\,.$$

It follows $\Theta(F)$ is a linear functional on S_{BV}, and hence on BV_c^∞. Alternatively, by [16, Theorem 3.4], the functional $\Theta(F)$ is linear on the vector space of all Lipschitz functions $g \in BV_c^\infty$, which is also dense in (BV_c^∞, τ). We conclude $\Theta(F)$ belongs to $(BV_c^\infty)^*$. □

Remark 4.2. As (CH, σ) is a Fréchet space, an application of the Banach-Steinhaus theorem [19, Theorem 2.17] improves the separate continuity established in Proposition 4.1 as follows: $\lim \langle F_i, g_i \rangle = \langle F, g \rangle$ whenever $\{(F_i, g_i)\}$ is a sequence in $CH \times BV_c^\infty$ that $(\sigma \times \tau)$-converges to (F, g).

Theorem 4.3. *For $T \in (BV_c^\infty)^*$ and $E \in \boldsymbol{\mathcal{BV}}$, let $\Lambda(T)(E) = T(\chi_E)$. Then $\Lambda(T)$ is a charge, and the map*

$$\Lambda : \big((BV_c^\infty)^*, \nu\big) \to (CH, \sigma) : T \mapsto \Lambda(T)$$

is a linear homeomorphism with $\Lambda^{-1} : F \mapsto \Theta(F)$.

Proof. Since $\Lambda(T)$ is the composition of T and the inclusion map $\boldsymbol{\mathcal{BV}} \hookrightarrow BV_c^\infty$, it is easy to see that $\Lambda(T)$ is a charge, and $\|\Lambda(T)\|_n \leq \|T\|_n$ for $n = 1, 2, \dots$. Thus Λ, which is clearly linear, is also continuous and injective. Let

$$\Theta : CH \to (BV_c^\infty)^* : F \mapsto \Theta(F)\,,$$

and observe that for each charge F and each bounded BV set E,

$$(\Lambda \circ \Theta)(F)(E) = \Theta(F)(\chi_E) = (F \times \mathcal{L}^1)(\Sigma_{\chi_E}) = F(E)\,.$$

Hence $\Lambda \circ \Theta$ is the identity map on CH, which implies Λ is surjective. By the closed graph theorem [19, Theorem 2.15], Λ is a linear homeomorphism and $\Lambda^{-1} = \Theta$. □

Remark 4.4. Let A be a bounded BV set. Considering $BV^\infty(A)$ and $CH(A)$ instead of BV_c^∞ and CH, respectively, we obtain results completely analogous to Proposition 4.1 and Theorem 4.3. In particular, it follows from Proposition 3.4 the topology in $BV^\infty(A)^*$ is induced by a Banach norm.

Remark 4.5. If $\mathcal{D}(\mathbb{R}^m)$ is the space of Schwartz's test functions equipped with its usual direct limit topology [19, Definition 6.3], then the inclusion map $\mathcal{D}(\mathbb{R}^m) \hookrightarrow BV_c^\infty$ is continuous. Thus given a charge F, the restriction T of $\Theta(F)$ to $\mathcal{D}(\mathbb{R}^m)$ is a distribution, and it is easy to see the order of T is one.

A charge F is *derivable* at $x \in \mathbb{R}^m$ if a finite limit

$$\lim \frac{F(B_n)}{|B_n|}$$

exists for each sequence $\{B_n\}$ of figures such that

$$\lim d\big(B_n \cup \{x\}\big) = 0 \quad \text{and} \quad \inf r\big(B_n \cup \{x\}\big) > 0\,. \tag{4.1}$$

If all these limits exist, they have the same value called the *derivate* of F at x, denoted by $F'(x)$. Convenient necessary and sufficient conditions for a charge to be derivable almost everywhere in a set $E \subset \mathbb{R}^m$ can be found in [2, Theorem 3.3 and Corollary 4.9].

Remark 4.6. It follows from [2, Lemma 3.1] that if a charge F is derivable at x, then $\lim\big[F(B_n)/|B_n|\big] = F'(x)$ exists for each sequence $\{B_n\}$ of bounded BV sets satisfying condition (4.1). This is an important fact we shall often use.

A charge F is called *absolutely continuous* (abbreviated as AC) if for each $\varepsilon > 0$ there is a $\delta > 0$ such that $\big|F(E)\big| < \varepsilon$ whenever $E \subset B(1/\varepsilon)$ is a BV set with $|E| < \delta$. The linear spaces of all AC charges and all AC charges in a locally BV set A are denoted by CH_{AC} and $CH_{AC}(A)$, respectively. It follows from [15, Theorem 9.3.7] that each $F \in CH_{AC}$ is derivable almost everywhere with $F' \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$ and $\int F'\,d\mathcal{L}^m = F$.

Lemma 4.7. *Let B be a BV set, $y \in \mathbb{R}^m$, and $B + y = \{x + y : x \in B\}$. Then*

$$\big|(B + y) \triangle B\big| \le \|y\| \cdot \|B\|\,.$$

Proof. If $g \in BV(\mathbb{R}^m) \cap C^1(\mathbb{R}^m)$, then

$$g(x - y) - g(x) = \int_0^1 \frac{d}{dt} g(x - ty)\,dt = -\int_0^1 y \cdot \nabla g(x - ty)\,dt$$

for all $x \in \mathbb{R}^m$, and Schwartz's inequality yields

$$\begin{aligned}\int_{\mathbb{R}^m} \big|g(x - y) - g(x)\big|\,dx &\le \int_{\mathbb{R}^m} \left(\int_0^1 \|y\| \cdot \big\|\nabla g(x - ty)\big\|\,dt\right) dx \\ &= \|y\| \int_0^1 \left(\int_{\mathbb{R}^m} \big\|\nabla g(x - ty)\big\|\,dx\right) dt = \|y\| \cdot \|g\|\,.\end{aligned}$$

By the approximation theorem [6, Section 5.2.2, Theorem 2], there is a sequence $\{g_i\}$ in $BV(\mathbb{R}^m) \cap C^1(\mathbb{R}^m)$ with $\lim \|g_i - \chi_B\|_1 = 0$ and $\lim \|g_i\| = \|B\|$. Since

$$\begin{aligned}&\big|(B + y) \triangle B\big| = \int_{\mathbb{R}^m} \big|\chi_B(x - y) - \chi_B(x)\big|\,dx \le \\ &\quad \int_{\mathbb{R}^m} \big|\chi_B(x - y) - g_i(x - y)\big|\,dx + \int_{\mathbb{R}^m} \big|g_i(x - y) - g_i(x)\big|\,dx + \\ &\quad \int_{\mathbb{R}^m} \big|g_i(x) - \chi_B(x)\big|\,dx \le 2\|g_i - \chi_B\|_1 + \|y\| \cdot \|g_i\|\end{aligned}$$

for $i = 1, 2, \dots$, the lemma follows. □

Proposition 4.8. *The linear space CH_{AC} is dense in (CH,σ).*

Proof. Let F be a charge, and let $\varphi \in C_c^1(\mathbb{R}^m)$ be nonnegative. For a bounded BV set B, let

$$F_\varphi(B) = \Theta(F)(\varphi * \chi_B) = \langle F, \varphi * \chi_B \rangle$$

where $*$ denotes the convolution operation. We shall need the following inequalities obtained by standard manipulations of convolutions:

$$\begin{aligned} &\|\varphi * \chi_B\|_\infty \le \|\varphi\|_1 \,, \\ &\|\varphi * \chi_B\|_1 \le |B| \cdot \|\varphi\|_1 \,, \\ &\|\varphi * \chi_B\| \le \min\bigl\{\|B\| \cdot \|\varphi\|_1, \|\varphi\| \cdot |B|\bigr\} \,. \end{aligned} \tag{4.2}$$

Given $\varepsilon > 0$, find a $\theta > 0$ according to Proposition 3.1, and select a BV set $B \subset B(1/\varepsilon)$. Letting $b = \|\varphi * \chi_B\|_\infty$, Proposition 3.1, Fubini's theorem, coarea theorem, and inequalities (4.2) yield

$$\begin{aligned} \bigl|F_\varphi(B)\bigr| &= \bigl|(F \times \mathcal{L}^1)(\Sigma_{\varphi * \chi_B})\bigr| \le \int_0^b \bigl|F(\Sigma^t_{\varphi*\chi_B})\bigr|\,dt \\ &\le \theta \int_0^b |\Sigma^t_{\varphi*\chi_B}|\,dt + \varepsilon \int_0^b \bigl(\|\Sigma^t_{\varphi*\chi_B}\| + 1\bigr)\,dt \\ &= \theta \|\varphi * \chi_B\|_1 + \varepsilon\bigl(\|\varphi * \chi_B\| + b\bigr) \\ &\le \theta |B| \cdot \|\varphi\|_1 + \varepsilon\bigl(\|\varphi\| \cdot |B| + \|\varphi\|_1\bigr) \,, \end{aligned}$$

and we conclude F_φ is an AC charge.

For $i = 1, 2, \dots$, let $\varphi_i = \eta_{1/i}$ be the standard mollifier [6, Section 4.2.1], and write F_i instead of F_{φ_i}. We establish the proposition by showing the sequence $\{F_i\}$ of AC charges converges to F uniformly on each $\boldsymbol{\mathcal{BV}}_n$. To this end, select an integer $n \ge 1$ and a $B \in \boldsymbol{\mathcal{BV}}_n$. Clearly $\operatorname{supp}(\chi_B - \varphi_i * \chi_B) \subset B[n+1]$, and since $\|\varphi_i\|_1 = 1$, the inequalities (4.2) imply

$$\|\varphi * \chi_B\|_\infty \le 1 \quad \text{and} \quad \|\varphi_i * \chi_B\| \le \|B\| \le n \,.$$

Moreover, by Fubini's theorem and Lemma 4.7,

$$\begin{aligned} \|\chi_B - \varphi_i * \chi_B\|_1 &= \int_{\mathbb{R}^m} \left| \chi_B(x) \int_{\mathbb{R}^m} \varphi_i(y)\,dy - \int_{\mathbb{R}^m} \chi_B(x-y)\varphi_i(y)\,dy \right| dx \\ &\le \int_{\mathbb{R}^m} \varphi_i(y) \left(\int_{\mathbb{R}^m} \bigl|\chi_B(x) - \chi_B(x-y)\bigr|\,dx \right) dy \\ &= \int_{\|y\| \le \frac{1}{i}} \varphi_i(y) \bigl|(B+y) \triangle B\bigr|\,dy \\ &\le \|B\| \int_{\|y\| \le \frac{1}{i}} \|y\| \varphi_i(y)\,dy \le \frac{1}{i}\|B\| \le \frac{n}{i} \,. \end{aligned}$$

Thus $\{\varphi_i * \chi_B\}$ is a sequence in BV_c^∞ which τ-converges to χ_B uniformly with respect to $B \in \boldsymbol{\mathcal{BV}}_n$. Since the linear functional $\Theta(F)$ defined on BV_c^∞ is

τ-continuous (Proposition 4.1), we obtain

$$F(B) = \Theta(F)(\chi_B) = \lim \Theta(F)(\varphi_i * \chi_B) = \lim F_i(B)$$

where the convergence is uniform with respect to $B \in \mathcal{BV}_n$. $\square$

Corollary 4.9. *For each locally BV set A, the linear space $CH_{AC}(A)$ is dense in $\bigl(CH(A), \sigma\bigr)$.*

Proof. Let F be a charge in A. By Proposition 4.8, there is a sequence $\{F_i\}$ in CH_{AC} that σ-converges to F. Clearly, $\{F_i \llcorner A\}$ is a sequence in $CH_{AC}(A)$ that also σ-converges to F. $\square$

Endow $L^1_{\mathrm{loc}}(\mathbb{R}^m)$ with the Fréchet topology ρ induced by seminorms

$$\|f\|_{1,n} = \int_{B[n]} \bigl|f(x)\bigr|\, dx$$

defined for each $f \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$ and $n = 1, 2, \ldots$. If F is an AC charge, then

$$|F|_n = \sup\left\{\left|\int_B F'(x)\, dx\right| : B \in \mathcal{BV}_n\right\} \le \int_{B[n]} \bigl|F'(x)\bigr|\, dx = \|F'\|_{1,n}$$

and consequently, the linear bijection

$$\Upsilon : \bigl(L^1_{\mathrm{loc}}(\mathbb{R}^m), \rho\bigr) \to (CH_{AC}, \sigma) : f \mapsto \int f\, d\mathcal{L}^m$$

is continuous. Note the inverse map $\Upsilon^{-1} : F \mapsto F'$ is not continuous.

Lemma 4.10. *If T is a continuous linear functional on $\bigl(L^1_{\mathrm{loc}}(\mathbb{R}^m), \rho\bigr)$, then there is a $g \in L^\infty_c(\mathbb{R}^m)$ such that for each $f \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$,*

$$fg \in L^1(\mathbb{R}^m) \quad \textit{and} \quad T(f) = \int_{\mathbb{R}^m} f(x)g(x)\, dx\,.$$

Proof. For $n = 1, 2, \ldots$, the subspace $L_n = \bigl\{f\chi_{B[n]} : f \in L^1_{\mathrm{loc}}(\mathbb{R}^m)\bigr\}$ of $\bigl(L^1_{\mathrm{loc}}(\mathbb{R}^m), \rho\bigr)$ is isomorphic to $L^1\bigl(B[n]\bigr)$. According to [18, Theorem 6.16], there is a $g_n \in L^\infty\bigl(B[n]\bigr)$ such that

$$T(f) = \int_{B[n]} f(x)g_n(x)\, dx$$

for each $f \in L_n$. Since

$$\int_B g_n(x)\, dx = T(\chi_B) = \int_B g_{n+1}(x)\, dx$$

for every measurable set $B \subset B[n]$, the restriction of g_{n+1} to $B[n]$ equals g_n almost everywhere. We infer there is a function g defined on $\mathbb{R}^m$ whose restriction to each $B[n]$ equals g_n almost everywhere; in particular g is measurable.

Select an $f \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$ and observe the sequence $\{f\chi_{B[n]}\}$ ρ-converges to f. As $|f| \operatorname{sign} g$ belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^m)$, we obtain

$$\begin{aligned} T\big(|f| \operatorname{sign} g\big) &= \lim T\Big[\big(|f| \operatorname{sign} g\big)\chi_{B[n]}\Big] \\ &= \lim \int_{B[n]} \big|f(x)g(x)\big|\,dx = \int_{\mathbb{R}^m} \big|f(x)g(x)\big|\,dx\,. \end{aligned}$$

Hence fg belongs to $L^1(\mathbb{R}^m)$, which implies

$$T(f) = \lim T(f\chi_{B[n]}) = \lim \int_{B[n]} f(x)g(x)\,dx = \int_{\mathbb{R}^m} f(x)g(x)\,dx\,.$$

Now it is easy to show by contradiction the equivalence class of g contains a bounded function with compact support. ☐

Theorem 4.11. *Let X be a linear subspace of CH, and let X^* denote the dual space of (X,σ) equipped with the w^* topology. If $CH_{AC} \subset X$, then the map*

$$\Phi : (BV_c^\infty, \tau) \to X^* : g \mapsto \Phi(g)$$

is a continuous linear bijection.

Proof. In view of Proposition 4.1 and the definition of the w^*-topology in X^* [19, Section 3.14], we only need to show Φ is surjective. To this end, select a $T \in X^*$, and observe $T \circ \Upsilon$ is a continuous linear functional on $L^1_{\mathrm{loc}}(\mathbb{R}^m)$. By Lemma 4.10, there is a $g \in L_c^\infty(\mathbb{R}^m)$ such that $F'g$ belongs to $L^1(\mathbb{R}^m)$ and

$$T(F) = (T \circ \Upsilon)(F') = \int_{\mathbb{R}^m} F'(x)g(x)\,dx$$

for every AC charge F. Employing the σ-continuity of T, find positive integers n and k so that $\big|T(F)\big| \le k|F|_n$ for all $F \in X$. If $h \in C_c^1(\mathbb{R}^m;\mathbb{R}^m)$, then $H = \int \operatorname{div} h\,d\mathcal{L}^m$ is an AC charge. By the Gauss-Green theorem,

$$\big|H(B)\big| = \left|\int_{\partial_* B} h \cdot \nu_B \,d\mathcal{H}^{m-1}\right| \le \|B\| \cdot \|h\|_\infty$$

for each BV set B, and hence $|H|_n \le n\|h\|_\infty$. Thus

$$\int_{\mathbb{R}^m} g(x) \operatorname{div} h(x)\,dx = T(H) \le k|H|_n \le kn\|h\|_\infty\,,$$

which implies $g \in BV^\infty(A)$. Now Fubini's theorem yields

$$\begin{aligned} T(F) &= \int_{\mathbb{R}^m} \left(F'(x)\int_0^{g^+(x)} dt\right) dx - \int_{\mathbb{R}^m} \left(F'(x)\int_0^{g^-(x)} dt\right) dx \\ &= \int_{\mathbb{R}} \left(\int_{\{g^+(x)>t\}} F'(x)\,dx\right) dt - \int_{\mathbb{R}} \left(\int_{\{g^-(x)>t\}} F'(x)\,dx\right) dt \\ &= \int_{\mathbb{R}} F(\Sigma^t_{g^+})\,dt - \int_{\mathbb{R}} F(\Sigma^t_{g^-})\,dt \\ &= (F \times \mathcal{L}^1)(\Sigma_{g^+}) - (F \times \mathcal{L}^1)(\Sigma_{g^-}) = \langle F, g\rangle = \Phi(g)(F) \end{aligned}$$

for each $F \in CH_{AC}$. According to Corollary 4.9 and Proposition 4.1, the equality $T(F) = \Phi(g)(F)$ extends to all $F \in X$, and the surjectivity of Φ is established. $\square$

Question 4.12. *Is there a simple description of the topology σ^* in CH^* for which $\Phi : (BV_c^\infty, \tau) \to (CH^*, \sigma^*)$ is a homeomorphism?*

Remark 4.13. Let A be a bounded BV set. In view of Corollary 4.9, a result completely analogous to Theorem 4.11 holds when the spaces CH and BV_c^∞ are replaced by $CH(A)$ and $BV^\infty(A)$, respectively (cf. Remark 4.2).

5 AC$_*$ charges

We show that a weaker notion of absolute continuity leads to a generalized Riemann integral akin to that of [15, Chapter 12].

A *partition* is a collection (possibly empty) $P = \{(A_1, x_1), \dots, (A_p, x_p)\}$ where $A_1, \dots, A_p$ are disjoint bounded BV sets, and $x_i \in \mathbb{R}^m$ for $i = 1, \dots, p$. Given $\varepsilon > 0$ and a nonnegative function δ on $E \subset \mathbb{R}^m$, the partition P is called

$\mathcal{F}$-partition if $A_1, \dots, A_p$ are figures,

ε-regular if $r(A_i \cup \{x_i\}) > \varepsilon$ for $i = 1, \dots, p$,

δ-fine if $x_i \in E$ and $d(A_i \cup \{x_i\}) < \delta(x_i)$ for $i = 1, \dots, p$.

Note if P is δ-fine, then each x_i belongs to $E - \operatorname{null} \delta$. A nonnegative function δ defined on a set $E \subset \mathbb{R}^m$ is called a *gage* on E whenever $\operatorname{null} \delta$ is a thin set.

The *variation* of a charge F on a set $E \subset \mathbb{R}^m$ is the number

$$V_* F(E) = \sup_{\varepsilon > 0} \inf_{\delta} \sup_{P} \sum_{i=1}^{p} \left| F(A_i) \right|$$

where δ is a gage on E, and $P = \{(A_1, x_1), \dots, (A_p, x_p)\}$ is an ε-regular δ-fine $\mathcal{F}$-partition. According to [2, Lemmas 4.6 and 4.4], the extended real-valued function $V_* F : E \mapsto V_* F(E)$ is a Borel regular measure in $\mathbb{R}^m$, which coincides with the classical variation of F on every figure. In particular, $V_* F$ is Radon [6, Section 1.1] whenever it is locally finite. Our first proposition, proved in [3, Proposition 3.3], characterizes AC charges in terms of variations.

Proposition 5.1. *A charge F is AC if and only if $V_* F$ is absolutely continuous and locally finite.*

It turns out many important properties of AC charges remain valid when $V_* F$ is merely absolutely continuous. Specifically, we have a proposition, which follows from [9, Lemma 6] and [3, Corollary 4.3 and Theorem 4.4].

Proposition 5.2. *Let F be a charge. If $V_* F$ is absolutely continuous, then the following is true.*

(1) *F is derivable almost everywhere, and $F \geq 0$ whenever $F' \geq 0$ almost everywhere.*

(2) *There are closed sets $C_i \subset \mathbb{R}^m$ with $V_*F(C_i) < \infty$ for $i = 1, 2, \ldots$ and such that $\mathbb{R}^m - \bigcup_{i=1}^{\infty} C_i$ is a thin set. In particular, V_*F is σ-finite.*

Definition 5.3. A charge F is called AC_* whenever the measure V_*F is absolutely continuous.

The linear space of all AC_* charges or all AC_* charges in a bounded BV set A is denoted by CH_{AC_*} or $CH_{AC_*}(A)$, respectively. We wish to alert the reader our concept of AC_* differs substantially from that of [20, Chapter 8, Section 8].

Remark 5.4. While neither of the spaces (CH_{AC_*}, σ) and $\big(CH_{AC_*}(A), \sigma\big)$ is complete, an argument similar to that employed in [4, Theorem 4.1] shows the space $\big(CH_{AC_*}(A), \sigma\big)$ is barrelled.

In view of Proposition 5.2,(1), we can present a descriptive definition of the *generalized Riemann integral* (abbreviated as R-integral), which is completely analogous to the usual descriptive definition of the Lebesgue integral.

Definition 5.5. A function f defined on $\mathbb{R}^m$ is *locally R-integrable* whenever there is an AC_* charge F such that $F'(x) = f(x)$ for almost all $x \in \mathbb{R}^m$. The charge F, uniquely determined by f, is called the *indefinite R-integral* of f. If A is a bounded BV set, the number $F(A)$ is called the *R-integral* of f over A.

The linear space of all locally R-integrable functions is denoted by $R_{\mathrm{loc}}(\mathbb{R}^m)$. Clearly $L^1_{\mathrm{loc}}(\mathbb{R}^m) \subset R_{\mathrm{loc}}(\mathbb{R}^m)$, and the indefinite R-integral F of $f \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$ coincides with the indefinite Lebesgue integral of f. Thus without danger of confusion, we can denote F by $\int f \, d\mathcal{L}^m$, and $F(A)$ by $\int_A f \, d\mathcal{L}^m$ for each bounded BV set A. An argument similar to [20, Chapter 4, Theorem 4.2] shows each $f \in R_{\mathrm{loc}}(\mathbb{R}^m)$ is measurable. It follows a *nonnegative* function f on $\mathbb{R}^m$ belongs to $R_{\mathrm{loc}}(\mathbb{R}^m)$ if and only if it belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^m)$ [13, Lemma 5.13]. The name "generalized Riemann integral" is justified by the next theorem, proved in [3, Theorem 4.6].

Theorem 5.6. *A function f defined on $\mathbb{R}^m$ is locally R-integrable if and only if there is a charge F satisfying the following condition: given $\varepsilon > 0$, we can find a gage δ on $B(1/\varepsilon)$ so that*

$$\sum_{i=1}^{p} \Big| f(x_i)|A_i| - F(A_i) \Big| < \varepsilon$$

for each ε-regular δ-fine $\boldsymbol{\mathcal{F}}$-partition $\big\{(A_1, x_1), \ldots, (A_p, x_p)\big\}$. In this case F is the indefinite R-integral of f.

To stress further the proximity of the R-integral and Lebesgue integral, we state a theorem due to McShane [11, Chapter 4, Definition 1-1 and Corollary 6-5]. It shows the regularity of partitions and use of gages are critical components of Theorem 5.6.

Theorem 5.7. *A function f defined on $\mathbb{R}^m$ is locally Lebesgue integrable if and only if there is a charge F satisfying the following condition: given $\varepsilon > 0$, we can find a positive function δ on $B(1/\varepsilon)$ so that*

$$\sum_{i=1}^{p} \Big| f(x_i)|A_i| - F(A_i) \Big| < \varepsilon$$

for each δ-fine $\boldsymbol{\mathcal{F}}$-partition $\big\{(A_1, x_1), \ldots, (A_p, x_p)\big\}$. In this case F is the indefinite Lebesgue integral of f.

The following proposition, which simplifies many technical manipulations, is a consequence of [17, Theorem 3.3], [13, Proposition 7.7], and Remark 4.6.

Proposition 5.8. *If F is a charge and $E \subset \mathbb{R}^m$, then*

$$V_* F(E) = \sup_{\varepsilon > 0} \inf_{\delta} \sup_{P} \sum_{i=1}^{p} \big| F(A_i) \big|$$

where δ is a gage on E, and $P = \big\{(A_1, x_1), \ldots, (A_p, x_p)\big\}$ is any ε-regular δ-fine partition. Similarly, Theorems 5.6 and 5.7 remain valid when $\boldsymbol{\mathcal{F}}$-partitions are replaced by partitions.

One of the principal reasons for introducing the R-integral is the divergence theorem for discontinuously differentiable vector fields. Let $E \subset \mathbb{R}^m$ and let $v : E \to \mathbb{R}^m$ be a vector field on E. We say v is *Lipschitz at $x \in E$* whenever there are positive numbers c and r such that

$$\big| v(y) - v(x) \big| \leq c|y - x|$$

for each $y \in E \cap B(x, r)$. If E is an *open set* and v is Lipschitz at each point of a set $C \subset E$, then v is differentiable almost everywhere in C by Stepanoff's theorem [7, Theorem 3.1.9].

Lemma 5.9. *Let F be the flux of a $v \in C(\mathbb{R}^m; \mathbb{R}^m)$. For each $x \in \mathbb{R}^m$, the following is true.*

(1) *If v is Lipschitz at x, then given $\varepsilon > 0$, we can find positive numbers c and δ so that $\big|F(B)\big| \leq c|B|$ for each BV set B with $r\big(B \cup \{x\}\big) > \varepsilon$ and $d\big(B \cup \{x\}\big) < \delta$.*

(2) *If v is differentiable at x, then F is derivable at x and $F'(x) = \operatorname{div} v(x)$.*

The proof of Lemma 5.9 is not difficult. The reader can find it in [15, Lemma 11.7.4].

Lemma 5.10. *Let F be the flux of a $v \in C(\mathbb{R}^m; \mathbb{R}^m)$. If v is Lipschitz at each point of a set $E \subset \mathbb{R}^m$, then $(V_* F) \llcorner E$ is absolutely continuous.*

Proof. Choose a negligible set $N \subset E$ and an $\varepsilon > 0$. Given $x \in N$, use Lemma 5.9,(1) to find positive numbers c_x and δ_x so that $\left|F(B)\right| \leq c_x|B|$ for each BV set B with $r\big(B \cup \{x\}\big) > \varepsilon$ and $d\big(B \cup \{x\}\big) < \delta_x$. For $k = 1, 2, \dots,$ let

$$N_k = \{x \in N : k - 1 \leq c_x < k\}\,,$$

and let $U_k \subset \mathbb{R}^m$ be an open set with $N_k \subset U_k$ and $|U_k| < \varepsilon 2^{-k}/k$. Making δ_x smaller, we may assume $B[x, \delta_x] \subset U_k$ whenever $x \in N_k$. Observe $\delta : x \mapsto \delta_x$ is a positive function on N, and select an ε-regular δ-fine partition $\big\{(A_1, x_1), \dots, (A_p, x_p)\big\}$. Then

$$\begin{aligned}\sum_{i=1}^{p}\left|F(A_i)\right| &\leq \sum_{i=1}^{p} c_{x_i}|A_i| < \sum_{k=1}^{\infty}\sum_{x_i \in N_k} k|A_i| \\ &\leq \sum_{k=1}^{\infty} k|U_k| < \sum_{k=1}^{\infty} \varepsilon 2^{-k} = \varepsilon\,,\end{aligned}$$

and from the arbitrariness of ε, we conclude $V_*F(N) = 0$. □

Theorem 5.11. *Let $v \in C(\mathbb{R}^m; \mathbb{R}^m)$, and suppose there is a thin set S such that v is Lipschitz at each point of $\mathbb{R}^m - S$. Then* $\operatorname{div} v$ *is locally R-integrable, and*

$$\int \operatorname{div} v\, d\mathcal{L}^m = \int v \cdot \nu\, d\mathcal{H}^{m-1}\,.$$

Proof. Let F be the flux of v. Since $V_*F(S) = 0$, Lemma 5.10 implies F is AC_*. An application of Stepanoff's theorem and Lemma 5.9,(2) completes the argument. □

6 Multipliers

A *multiplier* for a family X of functions defined on a set E is a function g defined on E such that $fg \in X$ for each $f \in X$. The collection of all multipliers for X is denoted by MX. If X is a linear space containing constant functions, then MX is a subalgebra of X — for instance, $ML^1_{\mathrm{loc}}(\mathbb{R}^m) = L^\infty(\mathbb{R}^m)$. In this section we show $MR_{\mathrm{loc}}(\mathbb{R}^m) = BV^\infty_{\mathrm{loc}}$.

Lemma 6.1. $MR_{\mathrm{loc}}(\mathbb{R}^m) \subset BV^\infty_{\mathrm{loc}}$.

Proof. Since the one-dimensional R-integral is a restriction of the Denjoy-Perron integral [8, Chapter 11], for $m = 1$ the lemma follows from [21]. Thus we assume $m \geq 2$, and select an $f \in MR_{\mathrm{loc}}(\mathbb{R}^m)$. According to the definition of a multiplier, f and f^2 belong to $R_{\mathrm{loc}}(\mathbb{R}^m)$. Thus f is measurable and f^2 is in $L^1_{\mathrm{loc}}(\mathbb{R}^m)$. In particular, $f \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$.

Suppose $f \notin L^\infty_{\mathrm{loc}}(\mathbb{R}^m)$. By symmetry, we may assume there is a cell C with $f^+ \notin L^\infty(C)$. Each $C_n = \{x \in C : f(x) \geq 2^n\}$ is a measurable set of finite positive measure, and

$$g = \sum_{n=1}^{\infty} \frac{1}{2^n |C_n|} \chi_{C_n}$$

is a function from $L^1(\mathbb{R}^m)$. If $E = \{x \in \mathbb{R}^m : f(x) \geq 0\}$, then $g\chi_E$ belongs to $L^1(\mathbb{R}^m)$, and hence to $R_{\mathrm{loc}}(\mathbb{R}^m)$. Since $f \in MR_{\mathrm{loc}}(\mathbb{R}^m)$, the function $f^+ g = (f\chi_E)g = f(g\chi_E)$ is in $R_{\mathrm{loc}}(\mathbb{R}^m)$ and, being nonnegative, in $L^1_{\mathrm{loc}}(\mathbb{R}^m)$. On the other hand, as

$$|C_n| \leq 2^{-n} \int_{\mathbb{R}^m} |f| \, d\mathcal{L}^m \qquad \text{and} \qquad \int_{C_n} f^+ g \geq 1$$

for $n = 1, 2, \ldots$, the function $f^+ g$ is not in $L^1(C)$, a contradiction.

Next suppose $f \notin BV_{\mathrm{loc}}(\mathbb{R}^m)$, and select an open cube $U = (a, b)^m$ so that $f \notin BV(U)$. For $a < t < b$, let

$$U_{t-} = (a, t) \times (a, b)^{m-1} \quad \text{and} \quad U_{t+} = (t, b) \times (a, b)^{m-1}.$$

There is a $v_1 \in C^1(\mathbb{R}^m, \mathbb{R}^m)$ with $\|v_1\|_\infty \leq 1$ and $\int_U f \operatorname{div} v_1 > 2 \cdot 2^2$. Find a $t \in (a, b)$ so that

$$\int_{U_{t-}} f \operatorname{div} v_1 = \int_{U_{t+}} f \operatorname{div} v_1 = \frac{1}{2} \int_U f \operatorname{div} v_1 > 2^2,$$

and denote by V_1 the element of $\{U_{t-}, U_{t+}\}$ in which f is not BV; such a V_1 exists by the extension property of BV functions [6, Section 5.4]. The other element of $\{U_{t-}, U_{t+}\}$ is denoted by $U_1 = (r_1, s_1) \times (a, b)^{m-1}$. Applying the previous argument to V_1, we obtain disjoint subsets $U_2 = (r_2, s_2) \times (a, b)^{m-1}$ and $V_2 = (t, t') \times (a, b)^{m-1}$ of V_1 such that f is not BV in V_2 and

$$\int_{U_2} f \operatorname{div} v_2 > 2^4$$

for a $v_2 \in C^1(\mathbb{R}^m, \mathbb{R}^m)$ with $\|v_2\|_\infty \leq 1$. Proceeding inductively, we construct disjoint open intervals $U_k = (r_k, s_k) \times (a, b)^{m-1}$ and $v_k \in C^1(\mathbb{R}^m, \mathbb{R}^m)$ with

$$\|v_k\|_\infty \leq 1 \quad \text{and} \quad \int_{U_k} f \operatorname{div} v_k > 2^{2k} \quad \text{for} \quad k = 1, 2, \ldots.$$

From the construction it is clear the set $\{r_k, s_k : k = 1, 2, \ldots\}$ has at most two cluster points α and β. As $m \geq 2$, we can define disjoint open intervals

$$U_{k,i} = (r_k, s_k) \times \left(a + (i-1)\frac{b-a}{2^k}, a + i\frac{b-a}{2^k} \right) \times (a, b)^{m-2},$$

$i = 1, \ldots, 2^k$, which cover U_k almost entirely. Thus for each k there is an i_k with $\int_{U_{k,i_k}} f \operatorname{div} v_k > 2^k$. Now U_{k,i_k} contains a cell A_k such that

$$\int_{A_k} f \operatorname{div} v_k > 2^k,$$

and by mollification, we may assume v_k vanishes outside U_{k,i_k}. The vector field $v = \sum_{k=1}^{\infty} 2^{-k} v_k$ is continuous everywhere, and it is differentiable at each $x \in \mathbb{R}^m$ that is not in the thin set $\{\alpha, \beta\} \times (a,b)^{m-1}$. According to Theorem 5.11, $\operatorname{div} v$ belongs to $R_{\mathrm{loc}}(\mathbb{R}^m)$ and, as $f \in MR_{\mathrm{loc}}(\mathbb{R}^m)$, so does $f \operatorname{div} v$. Denote by F the indefinite R-integral of $f \operatorname{div} v$. Since

$$\sum_{k=1}^{\infty} \|A_k\| \leq \sum_{k=1}^{\infty} \|U_{k,i_k}\| \leq \|U\|,$$

the union $A = \bigcup_{k=1}^{\infty} A_k$ is a bounded BV set. A contradiction follows:

$$\begin{aligned} F(A) = \lim F\left(\bigcup_{k=1}^{n} A_k\right) &= \sum_{k=1}^{\infty} F(A_k) \\ = \sum_{k=1}^{\infty} \int_{A_k} f \operatorname{div} v &= \sum_{k=1}^{\infty} 2^{-k} \int_{A_k} f \operatorname{div} v_k = \infty . \end{aligned}$$

□

Let F be a charge, and let $g \in BV_{\mathrm{loc}}^{\infty}$. For each bounded BV set B, the function $g\chi_B$ belongs to BV_c^{∞}, and we let

$$G : B \mapsto \langle F, g\chi_B \rangle$$

where $\langle \cdot, \cdot \rangle$ is the pairing introduced at the beginning of Section 4. Using Proposition 4.1, it is easy to see G is a charge; we say G is *induced by F via g*.

Lemma 6.2. *Let F be an AC_* charge, and let $g \in BV_{\mathrm{loc}}^{\infty}$ satisfy $0 \leq g \leq 1$. If G is a charge induced by F via g, then G is AC_*.*

Proof. Denote by I the interval $[0,1]$. For a bounded BV set B and $t \in I$, let

$$B^t = \{x \in B : g(x) > t\},$$

and observe that $G(B) = \int_I F(B^t)\, dt$. It suffices to show $V_* G$ vanishes on each bounded negligible set. Choose such a set N, and an $\varepsilon > 0$ so that $N \subset B(1/\varepsilon)$. By Proposition 3.1, there is a $\theta > 0$ such that

$$|F(B)| < \theta |B| + \varepsilon(\|B\| + 1) \tag{6.1}$$

for every BV set $B \subset B(1/\varepsilon)$. As F is AC_*, there is a gage δ on N such that

$$\sum_{i=1}^{p} |F(A_i)| < \varepsilon$$

for each $(\varepsilon/6)$-regular δ-fine partition $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ (Proposition 5.8). Find an open set $U \subset B(1/\varepsilon)$ with $N \subset U$ and $|U| < \varepsilon/\theta$. Making δ smaller, we may assume $\delta(x) \leq 1$ and $B[x, \delta(x)] \subset U$ for every $x \in N$.

Select an ε-regular δ-fine $\mathcal{F}$-partition $\{(B_1, x_1), \dots, (B_q, x_q)\}$, and note each $B_i \subset U$ by the choice of δ. If $U_i = \operatorname{int} B_i$, then

$$\|\partial B_i^t\|(U_i) \le \|B_i^t\| \le \|\partial B_i^t\|(U_i) + \|B_i\| \tag{6.2}$$

for $\mathcal{L}^1$-almost all $t \in I$ and $i = 1, \dots, q$. Now let $S_i = \{t \in I : \|B_i^t\| \le 2\|B_i\|\}$, $T_i = I - S_i$, and distinguish two cases.

(1) Let $S_i' = \{t \in S_i : |B_i| \le 2|B_i^t|\}$, $S_i'' = S_i - S_i'$, and

$$A_{i,t} = \begin{cases} B_i^t & \text{if } t \in S_i', \\ B_i - B_i^t & \text{if } t \in S_i''. \end{cases}$$

As $\|A_{i,t}\| \le \|B_i\| + \|B_i^t\| \le 3\|B_i\|$ and $|B_i| \le 2|A_{i,t}|$, we obtain

$$r(A_{i,t}, x_i) \ge \frac{1}{2} \cdot \frac{|B_i|}{d\big(B_i \cup \{x_i\}\big)\|B_i\|} \cdot \frac{\|B_i\|}{\|A_{i,t}\|} > \frac{\varepsilon}{6}$$

for each $t \in S_i$.

(2) By inequality (6.2),

$$\|B_i^t\| \le \|\partial B_i^t\|(U_i) + \|B_i\| < \|\partial B_i^t\|(U_i) + \frac{1}{2}\|B_i^t\|,$$

and hence $\|B_i^t\| < 2\|\partial B_i^t\|(U_i)$ for almost all $t \in T_i$.

Using cases (1) and (2), we estimate

$$\sum_{i=1}^q \big|G(B_i)\big| = \sum_{i=1}^q \left|\int_I F(B_i^t)\,dt\right| \le \sum_{i=1}^q \int_{S_i} \big|F(B_i^t)\big|\,dt + \sum_{i=1}^q \left|\int_{T_i} F(B_i^t)\,dt\right|$$

by estimating separately each sum on the right. According to case (1), the collection $P_t = \{(A_{i,t}, x_i) : t \in S_i\}$ is an $(\varepsilon/6)$-regular δ-fine partition for each $t \in I$. Thus

$$\begin{aligned}
\sum_{i=1}^q \int_{S_i} \big|F(B_i^t)\big|\,dt &= \sum_{i=1}^q \left(\int_{S_i'} \big|F(A_{i,t})\big|\,dt + \int_{S_i''} \big|F(B_i - A_{i,t})\big|\,dt\right) \\
&\le \sum_{i=1}^q \left(\int_{S_i} \big|F(A_{i,t})\big|\,dt + \int_{S_i''} \big|F(B_i)\big|\,dt\right) \\
&\le \int_I \left(\sum_{i=1}^q \chi_{S_i}(t)\big|F(A_{i,t})\big|\right) dt + \int_I \left(\sum_{i=1}^q \big|F(B_i)\big|\right) dt \\
&< \int_I \left(\sum_{t \in S_i} \big|F(A_{i,t})\big|\right) dt + \varepsilon < 2\varepsilon\,.
\end{aligned}$$

After a suitable reordering, we may assume there is a nonnegative integer $p \leq q$ such that $\int_{T_i} F(B_i^t)\,dt \geq 0$ for $i \leq p$, and $\int_{T_i} F(B_i^t)\,dt < 0$ for $i > p$. Letting

$$B_+ = \bigcup \{B_i^t : t \in T_i \text{ and } i \leq p\} \quad \text{and} \quad B_- = \bigcup \{B_i^t : t \in T_i \text{ and } i > p\},$$

case (2), inequality (6.1), and the coarea theorem imply

$$\begin{aligned}
\sum_{i=1}^q \left| \int_{T_i} F(B_i^t)\,dt \right| &= \sum_{i=1}^p \int_{T_i} F(B_i^t)\,dt - \sum_{i=p+1}^q \int_{T_i} F(B_i^t)\,dt \\
&= \int_I \left(\sum_{i=1}^p \chi_{T_i}(t) F(B_i^t) - \sum_{i=p+1}^q \chi_{T_i}(t) F(B_i^t) \right) dt \\
&= \int_I \left(\sum_{\{i \leq p : t \in T_i\}} F(B_i^t) - \sum_{\{i > p : t \in T_i\}} F(B_i^t) \right) dt \\
&= \int_I \Big[F(B_+) - F(B_-) \Big] dt \leq \int_I \Big(\big|F(B_+)\big| + \big|F(B_-)\big| \Big) dt \\
&\leq \int_I \Big[\theta\big(|B_+| + |B_-|\big) + \varepsilon\big(\|B_+\| + \|B_-\|\big) + 2\varepsilon \Big] dt \\
&\leq \int_I \theta |U|\,dt + \varepsilon \int_I \left(\sum_{t \in T_i} \|B_i^t\| \right) dt + 2\varepsilon \\
&< 3\varepsilon + 2\varepsilon \int_I \left(\sum_{t \in T_i} \|\partial B_i^t\|(U_i) \right) dt \\
&\leq 3\varepsilon + 2\varepsilon \sum_{i=1}^q \int_I \|\partial B_i^t\|(U_i)\,dt \\
&= 3\varepsilon + 2\varepsilon \sum_{i=1}^p \|Dg\|(U_i) \leq 3\varepsilon + 2\varepsilon \|Dg\|(U).
\end{aligned}$$

Combining the previous inequalities yields

$$\sum_{i=1}^q \big|G(B_i)\big| < \varepsilon \Big[5 + 2\|Dg\|(U) \Big],$$

and the lemma follows. ☐

Lemma 6.3. *Let F be an AC_* charge, let $g \in BV_{\mathrm{loc}}^\infty$ satisfy $0 \leq g \leq 1$, and let G be the charge induced by F via g. There is a set $E \subset \mathbb{R}^m$ with $|\mathbb{R}^m - E| = 0$ and sequences $\{r_{x,n}\}$ of positive reals such that the following conditions are met:*

(1) *$F'(x)$ exists and $\lim r_{x,n} = 0$ for every $x \in E$;*

(2) *given $\varepsilon > 0$, we can find a gage δ on $E \cap B(1/\varepsilon)$ such that*

$$\sum_{i=1}^{p} \Big| g(x_i) F'(x_i) |A_i| - G(A_i) \Big| < \varepsilon$$

whenever $A_i = B[x_i, r_{x_i,n_i}]$ and $r_{x_i,n_i} < \delta(x_i)$ for $i = 1, \ldots, p$.

Proof. The absolutely continuous part of the distributional gradient Dg of g has a density vector $\nabla g \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$ [6, Section 5.1]. Let E consist of all $x \in \mathbb{R}^m$ at which F is derivable, and

$$\frac{1}{|B(x,s)|} \int_{B(x,s)} \big| g(y) - g(x) - \nabla g(x) \cdot (y - x) \big| \, dy = o(s)$$

as $s \to 0$. It follows from Proposition 5.2,(1), [6, Section 6.1, Theorem 1], and Hölder's inequality that $|\mathbb{R}^m - E| = 0$. Fix an $x \in E$ and let $B_s = B[x, s]$. For all sufficiently small $s > 0$,

$$\begin{aligned} \int_{B_s} \big| g(y) - g(x) \big| \, dy &\le o(s^{m+1}) + \int_{B_s} \big| \nabla g(x) \cdot (y - x) \big| \, dy \\ &\le o(s^{m+1}) + s\sqrt{m} \big\| \nabla g(x) \big\| \cdot |B_s| \\ &\le \Big[1 + \sqrt{m} \big\| \nabla g(x) \big\| \Big] 2^m s^{m+1} \end{aligned}$$

and hence, by the Fubini or coarea theorem,

$$\int_0^s \left(\int_{\partial B_r} \big| g(y) - g(x) \big| \, d\mathcal{H}^{m-1}(y) \right) dr \le \\ (m+1) \Big[1 + \sqrt{m} \big\| \nabla g(x) \big\| \Big] \int_0^s (2r)^m \, dr \, .$$

Thus there is a sequence $\{r_n\}$ of positive reals such that $\lim r_n = 0$ and

$$\begin{aligned} \int_{B_{r_n}} \big| g(y) - g(x) \big| \, dy &\le r_n \Big[1 + \sqrt{m} \big\| \nabla g(x) \big\| \Big] \cdot |B_{r_n}| \, , \\ \int_{\partial B_{r_n}} \big| g(y) - g(x) \big| \, d\mathcal{H}^{m-1}(y) &\le (m+1) \Big[1 + \sqrt{m} \big\| \nabla g(x) \big\| \Big] \cdot |B_{r_n}| \end{aligned} \tag{6.3}$$

for $n = 1, 2, \ldots$. Let $r_{x,n} = r_n$ and choose a positive $\varepsilon < 1/(2m)$. To simplify the notation, let $U = B(1/\varepsilon)$ and $\widehat{F} = F \times \mathcal{L}^1$.

According to Propositions 3.5 and 3.1, there is a $\theta > 0$ such that

$$\big| \widehat{F}(C) \big| < \theta |C| + \varepsilon \big(\|C\| + 1 \big) \tag{6.4}$$

for every BV set $C \subset U \times [0, 1]$. Extend F' to a function f defined on $\mathbb{R}^m$. Since both f and $\|\nabla g\|$ belong to $R_{\mathrm{loc}}(\mathbb{R}^m)$, Theorem 5.6 implies there is a gage Δ

on U such that

$$\sum_{i=1}^{p} \Big| f(x_i)|A_i| - F(A_i) \Big| < \varepsilon \,,$$

$$\sum_{i=1}^{p} \left| \big\| \nabla g(x_i) \big\| \cdot |A_i| - \int_{A_i} \| \nabla g(x) \| \, dx \right| < 1$$

for each ε-regular Δ-fine $\boldsymbol{\mathcal{F}}$-partition $\big\{(A_1, x_1), \ldots, (A_p, x_p)\big\}$. Let δ be the restriction of $\Delta/2$ to $E \cap U$. Making δ smaller, we may assume $\delta < \varepsilon/\theta$ and $B\big[x, \delta(x)\big] \subset U$ for each x in $E \cap U - \operatorname{null} \delta$. Thus for every ε-regular 2δ-fine $\boldsymbol{\mathcal{F}}$-partition $\big\{(A_1, x_1), \ldots, (A_p, x_p)\big\}$, the following inequalities hold:

$$\sum_{i=1}^{p} \Big| F'(x_i)|A_i| - F(A_i) \Big| < \varepsilon \,, \tag{6.5}$$

$$\begin{aligned} \sum_{i=1}^{p} \big\| \nabla g(x_i) \big\| \cdot |A_i| &< 1 + \sum_{i=1}^{p} \int_{A_i} \| \nabla g(x) \| \, dx \\ &\leq 1 + \sum_{i=1}^{p} \| Dg \| (A_i) \leq 1 + \| Dg \| (U) \,. \end{aligned} \tag{6.6}$$

If $A_i = B[x_i, r_{x_i, n_i}]$ where $x_i \in E \cap U$ and $r_{x_i, n_i} < \delta(x_i)$, then $\big\{(A_1, x_1), \ldots, (A_p, x_p)\big\}$ is an ε-regular 2δ-fine $\boldsymbol{\mathcal{F}}$-partition. The sets $C_i^1 = A_i \times [0, g(x)]$ and $C_i^2 = \Sigma_{g\chi_{A_i}}$ are contained in $U \times [0,1]$ by the choice of δ. For $j, k = 1, 2$ and $S \subset \{1, \ldots, p\}$, inequalities 6.3 and 6.6 yield

$$\begin{aligned} \left| \bigcup_{i \in S} (C_i^j - C_i^k) \right| &= \sum_{i \in S} |C_i^j - C_i^k| \leq \sum_{i=1}^{p} \int_{A_i} \big| g(y) - g(x_i) \big| \, dy \\ &\leq \sum_{i=1}^{p} r_{x_i, n_i} \Big[1 + \sqrt{m} \big\| \nabla g(x_i) \big\| \Big] \cdot |A_i| \\ &< \frac{\varepsilon}{\theta} \Big(|U| + \sqrt{m} \big[1 + \| Dg \| (U) \big] \Big) \,, \end{aligned}$$

$$\begin{aligned} \left\| \bigcup_{i \in S} (C_i^j - C_i^k) \right\| &\leq \sum_{i \in S} \| C_i^j - C_i^k \| \\ &\leq \sum_{i=1}^{p} \left[2|A_i| + \| Dg \| (A_i) + \int_{\partial A_i} \big| g(y) - g(x_i) \big| \, d\mathcal{H}^{m-1}(y) \right] \\ &< 2|U| + \| Dg \| (U) + (m+1) \sum_{i=1}^{p} \Big[1 + \sqrt{m} \big\| \nabla g(x_i) \big\| \Big] \cdot |A_i| \\ &< (m+3) \Big(|U| + \sqrt{m} \big[1 + \| Dg \| (U) \big] \Big) \,. \end{aligned}$$

From these inequalities and (6.4), we obtain

$$\sum_{i\in S} \widehat{F}(C_i^j - C_i^k) = \widehat{F}\left[\bigcup_{i\in S}(C_i^j - C_i^k)\right] < \varepsilon\alpha\,,$$

where

$$\alpha = 1 + (m+4)\Big(|U| + \sqrt{m}\big[1 + \|Dg\|(U)\big]\Big)$$

does not depend on ε. As S is arbitrary,

$$\sum_{i=1}^{p}\big|\widehat{F}(C_i^1) - \widehat{F}(C_i^2)\big| \le \sum_{i=1}^{p}\Big[\big|\widehat{F}(C_i^1 - C_i^2)\big| + \big|\widehat{F}(C_i^2 - C_i^1)\big|\Big] < 4\alpha\varepsilon\,.$$

Since $\widehat{F}(C_i^1) = g(x_i)F(A_i)$ and $\widehat{F}(C_i^2) = G(A_i)$, the previous inequality together with (6.5) implies

$$\sum_{i=1}^{p}\Big|g(x_i)F'(x_i)|A_i| - G(A_i)\Big| \le$$

$$\sum_{i=1}^{p} g(x_i)\Big|F'(x_i)|A_i| - F(A_i)\Big| + \sum_{i=1}^{p}\Big|g(x_i)F(A_i) - G(A_i)\Big| < \varepsilon(1+4\alpha)\,,$$

and the lemma is proved. □

Theorem 6.4. *Let F be the indefinite R-integral of an $f \in R_{\mathrm{loc}}(\mathbb{R}^m)$, and let $g \in BV_{\mathrm{loc}}^{\infty}$. Then $fg \in R_{\mathrm{loc}}(\mathbb{R}^m)$ and*

$$\int_B fg\,d\mathcal{L}^m = \langle F, g\chi_B\rangle$$

for each bounded BV set B. In particular, $MR_{\mathrm{loc}}(\mathbb{R}^m) = BV_{\mathrm{loc}}^{\infty}$.

Proof. With no loss of generality, suppose $0 \le g \le 1$. Let E and the sequences $\{r_{x,n}\}$ be as in Lemma 6.3, and let G be the charge induced by F via g. In view of Lemma 6.2 and Proposition 5.2,(1), making E smaller, we may assume $G'(x)$ exists and $F'(x) = f(x)$ for each $x \in E$. We complete the argument by showing $G'(x) = f(x)g(x)$ for almost all $x \in E$.

For $k = 1, 2, \ldots$, let N_k be the set of all $x \in E \cap B(k)$ such that

$$\frac{1}{k} < \left|\lim \frac{G\big(B[x, r_{x,n}]\big)}{\big|B[x, r_{x,n}]\big|} - f(x)g(x)\right|. \tag{6.7}$$

Fix an integer $k \ge 1$, choose a positive $\varepsilon < 1/k$, and use Lemma 6.3 to find a gage δ on $E \cap B(1/\varepsilon)$ so that

$$\sum_{i=1}^{p}\Big|g(x_i)f(x_i)|A_i| - G(A_i)\Big| < \frac{\varepsilon}{k}$$

whenever $A_i = B[x_i, r_{x_i,n_i}]$ and $r_{x_i,n_i} < \delta(x_i)$ for $i = 1, \dots, p$. In view of (6.7), the collection $\mathcal{B}$ of all cells $B[x, r_{x,n}]$ such that $x \in N_k - \operatorname{null}\delta$, $r_{x,n} < \delta(x)$, and

$$\frac{1}{k}\big|B[x, r_{x,n}]\big| < \Big|G\big(B[x, r_{x,n}]\big) - f(x)g(x)\big|B[x, r_{x,n}]\big|\Big|$$

covers $N_k - \operatorname{null}\delta$. According to [15, Lemma 9.2.1], there are disjoint cells $B[x_1, r_1], B[x_2, r_2], \dots$ in $\mathcal{B}$ such that $N_k - \operatorname{null}\delta \subset \bigcup_i B[x_i, 5r_i]$. Since

$$\sum_i \big|B[x_i, r_i]\big| < k \sum_i \Big|G\big(B[x_i, r_i]\big) - f(x_i)g(x_i)\big|B[x_i, r_i]\big|\Big| \le \varepsilon\,,$$

we have $|N_k| = |N_k - \operatorname{null}\delta| < 5^m\varepsilon$. By the arbitrariness of ε, the set N_k is negligible, and so is $N = \bigcup_{k=1}^{\infty} N_k$. As $G'(x) = f(x)g(x)$ for each $x \in E - N$, our assertion is proved. $\square$

The following natural question remains unanswered.

Question 6.5. *If F is an AC_* charge, is the charge $F \times \mathcal{L}^1$ also AC_*?*

7 The R-integral

If f is a function defined on a set $E \subset \mathbb{R}^m$, let

$$\overline{f}(x) = \begin{cases} f(x) & \text{if } x \in E, \\ 0 & \text{if } x \in \mathbb{R}^m - E. \end{cases}$$

Definition 7.1. A function f defined on a bounded BV set A is called *R-integrable* in A whenever $\overline{f}$ is locally R-integrable. The indefinite R-integral F of $\overline{f}$ is called the *indefinite R-integral* of f, and the number $F(A)$ is called the *R-integral* of f over A.

The linear space of all R-integrable functions in a bounded BV set A is denoted by $R(A)$. If $f \in R(A)$, we denote by $\int_A f\, d\mathcal{L}^m$ the R-integral of f over A.

Proposition 7.2. *A function f defined on a bounded BV set A is R-integrable in A whenever there is a $g \in R_{\mathrm{loc}}(\mathbb{R}^m)$ such that $g \restriction A = f$. If G is the indefinite integral of g, then $G \llcorner A$ is the indefinite integral of f. In particular,*

$$\int_A f\, d\mathcal{L}^m = \int_A g\, d\mathcal{L}^m\,.$$

Proof. By Theorem 6.4, the function $\overline{f} = g\chi_A$ is locally R-integrable and

$$\int_B \overline{f}\, d\mathcal{L}^m = \int_B g\chi_A\, d\mathcal{L}^m = \langle G, \chi_{A\cap B}\rangle = (G \llcorner A)(B)$$

for each bounded BV set B. $\square$

According to Proposition 7.2, the indefinite integral F of a function f in $R(A)$ is a charge in A. Since the map

$$F \mapsto F' \restriction A : CH_{AC_*}(A) \to R(A)$$

is a linear bijection, Proposition 3.4 turns $R(A)$ into a normed space, which is barrelled according to Remark 5.4.

Remark 7.3. As the boundaries of figures are thin sets, it follows from Theorem 5.6, [17, Lemma 3.2], and [15, Theorem 12.2.2], that on figures the R-integral coincides with the $\mathcal{F}$-integral of [15, Definition 12.2.1].

The term "locally R-integrable," introduced in Definition 5.5, is justified by the next proposition.

Proposition 7.4. *A function f defined on $\mathbb{R}^m$ is locally R-integrable if and only if $f \restriction A$ is R-integrable in A for each bounded BV set A.*

Proof. As the converse follows from Proposition 7.2, assume $f \restriction A$ is R-integrable in A for each bounded BV set A. For $n = 1, 2, \ldots$, denote by F_n the indefinite R-integral of $f \restriction B[n]$, and observe $F_{n+1} \llcorner B[n] = F_n$ by Proposition 7.2. Thus there is a charge F such that $F \llcorner B[n] = F_n$ for $n = 1, 2, \ldots$, and it is easy to verify $F'(x) = f(x)$ for almost all $x \in \mathbb{R}^m$. □

It follows from Proposition 7.2 the R-integral has the usual properties (such as linearity, nonnegativity, and additivity) associated with the word "integral." Moreover, the common convergence theorems are also valid for the R-integral [13, Lemma 5.13]. By [1, Theorem 5.1], the R-integral is invariant with respect to lipeomorphisms (i.e., bi-Lipschitzian maps).

Theorem 7.5. *Let f be an R-integrable function in a bounded BV set $B \subset \mathbb{R}^m$, and let $\Psi : A \to B$ be a lipeomorphism from a set $A \subset \mathbb{R}^m$ onto B. Then A is a bounded BV set, and if J_Ψ is the Jacobian of Ψ, then $(f \circ \Psi)J_\Psi$ is R-integrable in A with*

$$\int_A (f \circ \Psi) J_\Psi \, d\mathcal{L}^m = \int_B f \, d\mathcal{L}^m .$$

Throughout the remainder of this paper, given $A \in \boldsymbol{\mathcal{BV}}$, we shall identify the linear spaces $BV^\infty(A)$ and $\{g \restriction A : g \in BV^\infty(A)\}$. In other words, we shall view $BV^\infty(A)$ as the linear space of all functions g defined on A for which $\overline{g} \in BV_c^\infty$.

The next two propositions follow easily from Theorems 6.4 and 4.11, and Remarks 4.13 and 4.2.

Proposition 7.6. *Let A be a bounded BV set. Then $MR(A) = BV^\infty(A)$, and for each $g \in BV^\infty(A)$, the map*

$$T_g : f \mapsto \int_A fg \, d\mathcal{L}^m$$

is a continuous linear functional on $R(A)$. The map $g \mapsto T_g$ is a continuous linear bijection from $\big(BV^\infty(A), \tau\big)$ onto the dual space of $R(A)$ equipped with the w^*-topology.

Proposition 7.7. *Let A be a bounded BV set, and let $k \geq 1 + 4d(A)^{m-1}$ be an integer for which $A \subset B[k]$. Let $f \in R(A)$ and $g \in BV^{\infty}(A)$, and let $\{f_i\}$ and $\{g_i\}$ be sequences in $R(A)$ and $BV^{\infty}(A)$, respectively, such that*

(i) $\lim\Big(\sup\Big\{\big|\int_{A\cap B}(f_i - f)\,d\mathcal{L}^m\big| : B \in \boldsymbol{\mathcal{BV}}_k\Big\}\Big) = 0$,

(ii) $\lim \|g_i - g\|_1 = 0$ *and* $\sup\big(\|g_i\| + \|g_i\|_\infty\big) < \infty$.

Then $\lim \int_A f_i g_i \, d\mathcal{L}^m = \int_A fg\, d\mathcal{L}^m$.

Proposition 7.6 inspires a question related to Question 4.12.

Question 7.8. *What is the meaning of the topology σ^* in the dual $R(A)^*$ of $R(A)$ for which*

$$g \mapsto T_g : \big(BV^{\infty}(A), \tau\big) \to \big(R(A)^*, \sigma^*\big)$$

is a homeomorphism?

Observation 7.9. *Let $E \subset \mathbb{R}^m$ be a measurable set, $x \in \operatorname{int}_* E$, and let $B_j = B[x, d_j]$ where $\{d_j\}$ is a sequence of positive numbers converging to zero. If $A_j \subset B_j$ are measurable sets and $|A_j|/|B_j| \geq \alpha > 0$ for $j = 1, 2, \ldots$, then*

$$\lim \frac{|E \cap A_j|}{|A_j|} = 1\,.$$

Proof.

$$\begin{aligned}
1 &\geq \limsup \frac{|E\cap A_j|}{|A_j|} \geq \liminf \frac{|E\cap A_j|}{|A_j|} \geq \liminf \frac{|E\cap B_j| - |B_j - A_j|}{|A_j|}\\
&= \liminf \left[1 - \frac{|B_j|}{|A_j|}\cdot\left(1 - \frac{|E\cap B_j|}{|B_j|}\right)\right] \geq 1 - \frac{1}{\alpha}\lim\left(1 - \frac{|E\cap B_j|}{|B_j|}\right) = 1\,.
\end{aligned}$$

□

Lemma 7.10. *Let $A \in \boldsymbol{\mathcal{BV}}$ and $\varepsilon > \eta > 0$. There is a gage Δ on $\operatorname{cl}_* A$ such that given ε-regular Δ-fine partition $\big\{(A_1, x_1), \ldots, (A_p, x_p)\big\}$, the partition*

$$\big\{(A\cap A_1, x_1), \ldots, (A \cap A_p, x_p)\big\}$$

is η-regular.

Proof. Since the lemma is obvious for $m = 1$, assume $m \geq 2$. By [6, Section 2.3, Theorem 1], there is an $\mathcal{H}^{m-1}$-negligible set $S \subset \operatorname{int}_* A$ such that

$$\lim_{r\to 0+} \frac{\mathcal{H}^{m-1}\big(B[x,r]\cap \partial_* A\big)}{r^{m-1}} = 0$$

for each $x \in \operatorname{int}_* A - S$. Fix an $x \in \operatorname{int}_* A - S$, and select BV sets C_j so that $\lim d\big(C_j \cup \{x\}\big) = 0$ and $r\big(C_j \cup \{x\}\big) \geq \varepsilon$ for $j = 1, 2, \ldots$. Letting $d_j = d\big(C_j \cup \{x\}\big)$ and $B_j = B[x, d_j]$, observe $C_j \subset B_j$ and

$$\|A \cap C_j\| \leq \|C_j\| + \mathcal{H}^{m-1}(B_j \cap \partial_* A)\,.$$

Since $r(C_j \cup \{x\}) \geq \varepsilon$, the isoperimetric inequality yields $|C_j|/d_j^m \geq \kappa\varepsilon^m$ where $\kappa > 0$ depends only on the dimension m [6, Section 5.6.2, Theorem 2]. Hence $|C_j|/|B_j| \geq \kappa(\varepsilon/2)^m$, and $\lim(|A \cap C_j|/|C_j|) = 1$ by Observation 7.9. Now

$$\frac{d_j\|A \cap C_j\|}{|C_j|} \leq \frac{d_j\|C_j\|}{|C_j|} + \frac{d_j^m}{|C_j|} \cdot \frac{\mathcal{H}^{m-1}(B_j \cap \partial_* A)}{d_j^{m-1}}$$
$$\leq \frac{1}{\varepsilon} + \frac{1}{\kappa\varepsilon^m} \cdot \frac{\mathcal{H}^{m-1}(B_j \cap \partial_* A)}{d_j^{m-1}},$$

and we obtain

$$\liminf r\big[(A \cap C_j) \cup \{x\}\big] \geq \liminf \left(\frac{|A \cap C_j|}{|C_j|} \cdot \frac{|C_j|}{d_j\|A \cap C_j\|}\right) \geq \varepsilon .$$

It follows there is a $\Delta_x > 0$ such that $r\big[(A \cap C) \cup \{x\}\big] > \eta$ for each BV set C with $d(C \cup \{x\}) < \Delta_x$ and $r(C \cup \{x\}) \geq \varepsilon$. As $S \cup \partial_* A$ is a thin set, extending the function $x \mapsto \Delta_x$ to $\mathrm{cl}_* A$ by zero produces the desired gage Δ. ☐

Lemma 7.11. *Let F be a charge, and let A be a bounded BV set. Then*

$$\big[V_*(F \llcorner A)\big] \llcorner \mathrm{cl}_* A \leq (V_* F) \llcorner \mathrm{cl}_* A ,$$

and $(F \llcorner A)'(x) = F'(x)$ at almost all $x \in A$ at which $F'(x)$ exists.

Proof. Proceeding towards a contradiction, suppose there is an $E \subset \mathrm{cl}_* A$ with $V_*(F \llcorner A)(E) > V_* F(E)$, and find an $\varepsilon > 0$ so that

$$\inf_{\delta} \sup_{P} \sum_{i=1}^{p} \big|(F \llcorner A)(A_i)\big| > \inf_{\sigma} \sup_{Q} \sum_{j=1}^{q} \big|F(B_j)\big|$$

where δ and σ are gages on E, $P = \big\{(A_1, x_1), \dots, (A_p, x_p)\big\}$ is an ε-regular δ-fine partition and $Q = \big\{(B_1, y_1), \dots, (B_q, y_q)\big\}$ is an $(\varepsilon/2)$-regular δ-fine partition. Let Δ be a gage on $\mathrm{cl}_* A$ associated with $\eta = \varepsilon/2$ according to Lemma 7.10. There is a gage σ on E with $\sigma \leq (\Delta \restriction E)$ and an ε-regular σ-fine partition $\big\{(A_1, x_1), \dots, (A_p, x_p)\big\}$ such that

$$\sum_{i=1}^{p} \big|F(A \cap A_i)\big| = \sum_{i=1}^{p} \big|(F \llcorner A)(A_i)\big| > \sup_{Q} \sum_{j=1}^{q} \big|F(B_j)\big|$$

where $Q = \big\{(B_1, y_1), \dots, (B_q, y_q)\big\}$ is an $(\varepsilon/2)$-regular σ-fine partition. In view of Lemma 7.10, this is a contradiction.

Suppose F is derivable at an $x \in A \cap (\mathrm{int}_* A - \mathrm{null}\,\Delta)$, and select a sequence $\{B_j\}$ of BV sets such that $\lim d(B_j \cup \{x\}) = 0$ and $\inf r(B_i \cup \{x\}) > \varepsilon$. Lemma 7.10 implies $r\big[(A \cap B_j) \cup \{x\}\big] > \varepsilon/2$ for all sufficiently large j. Arguing as in the proof of Lemma 7.10, we show $\lim(|A \cap B_j|/|B_j|) = 1$, and hence

$$\lim \frac{(F \llcorner A)(B_j)}{|B_j|} = \lim \left(\frac{F(A \cap B_j)}{|A \cap B_j|} \cdot \frac{|A \cap B_j|}{|B_j|}\right) = F'(x) .$$

Thus $(F \llcorner A)'(x) = F'(x)$, and it suffices to observe the sets $A \cap (\mathrm{int}_* A - \mathrm{null}\,\Delta)$ and A are equivalent. $\square$

Corollary 7.12. *Let f be a function defined on an* **essentially closed** *set $A \in \boldsymbol{\mathcal{BV}}$. Then $f \in R(A)$ whenever there is a charge F such that $(V_* F) \llcorner \mathrm{cl}_* A$ is absolutely continuous, and $F'(x) = f(x)$ for almost all $x \in A$. In this case $F \llcorner A$ is the indefinite R-integral of f.*

Proof. Since $\mathrm{cl}_* A$ is closed, Lemma 7.11 implies the charge $F \llcorner A$ is AC_*, and $(F \llcorner A)'(x) = \overline{f}(x)$ for almost all $x \in \mathbb{R}^m$. $\square$

We omit the proof of the next lemma, which is similar to that of Whitney's extension theorem presented in [22, Chapter 6, Theorem 3].

Lemma 7.13. *Let $E \subset \mathbb{R}^m$ be a closed set, and let $v \in C(E; \mathbb{R}^m)$ be Lipschitz at each point of a set $B \subset E$. Then v can be extended to a $w \in C(\mathbb{R}^m; \mathbb{R}^m)$ that is still Lipschitz at each $x \in B$.*

Let $A \in \boldsymbol{\mathcal{BV}}$, and let $v \in C(\mathrm{cl}\, A; \mathbb{R}^m)$ be Lipschitz at almost all $x \in A$. In view of Lemma 7.13, v can be extended to a $w \in C(\mathbb{R}^m; \mathbb{R}^m)$ that is still Lipschitz at almost all $x \in A$. By Stepanoff's theorem, w is differentiable almost everywhere in A, and we define a function $\mathrm{div}\, v$ on A, called the *divergence* of v, by the formula

$$\mathrm{div}\, v(x) = \begin{cases} \mathrm{div}\, w(x) & \text{if } w \text{ is differentiable at } x, \\ 0 & \text{otherwise.} \end{cases}$$

If F is the flux of w, Lemmas 5.9,(2) and 7.11 imply

$$\mathrm{div}\, w(x) = F'(x) = (F \llcorner A)'(x)$$

for almost all $x \in A$. Since $F \llcorner A$ depends only on v, and not on the extension w of v, so does the equivalence class of $\mathrm{div}\, v$; the same conclusion follows also from [15, Lemma 10.5.5].

Theorem 7.14. *Let $A \in \boldsymbol{\mathcal{BV}}$ be essentially closed, let $v \in C(\mathrm{cl}_* A; \mathbb{R}^m)$, and let $S \subset \mathrm{int}_* A$ be a thin set such that v is Lipschitz at each $x \in \mathrm{int}_* A - S$. Then* $\mathrm{div}\, v$ *is R-integrable in A and*

$$\int_A \mathrm{div}\, v \, d\mathcal{L}^m = \int_{\partial_* A} v \cdot \nu_A \, d\mathcal{H}^{m-1}.$$

Proof. Using Lemma 7.13, extend v to a $w \in C(\mathbb{R}^m; \mathbb{R}^m)$ that is Lipschitz at each $x \in \mathrm{int}_* A - S$, and denote by F the flux of w. Since $S \cup \partial_* A$ is a thin set, Lemmas 5.10 and 5.9,(2) imply $(V_* F) \llcorner \mathrm{cl}_* A$ is absolutely continuous, and $F'(x) = \mathrm{div}\, w(x)$ for almost all $x \in A$. The theorem follows from Corollary 7.12 and the definition of $\mathrm{div}\, v$. $\square$

Remark 7.15. [13, Example 5.21] shows Theorem 7.14 is false without assuming the set A is essentially closed.

Remark 7.16. [15, Example 11.1.2] shows the integrability of partial derivatives of all differentiable functions is incompatible with Fubini's theorem. In particular, in view of Theorem 7.14, the Fubini theorem is false for the R-integral. This explains in part the complexity of determining the algebra $MR_{\mathrm{loc}}(\mathbb{R}^m)$. Indeed, Theorem 6.4 relies on the following transposition of integrals:

$$\int_B \left(\int_0^{g(x)} f(x)\,dt \right) dx = \int_B f(x)g(x)\,dx = \int_0^{\|g\|_\infty} \left(\int_{g(x)>t} f(x)\,dx \right) dt$$

whenever $B \in \boldsymbol{\mathcal{BV}}$, $f \in R_{\mathrm{loc}}(\mathbb{R}^m)$, and $g \in BV_{\mathrm{loc}}^\infty$ is nonnegative.

A *Lipschitz domain* is a bounded open set $\Omega \subset \mathbb{R}^m$ whose boundary $\partial\Omega$ is Lipschitz [6, Section 4.2.1]; in particular, Ω is a BV set with $\mathrm{int}_*\Omega = \Omega$, $\mathrm{cl}_*\Omega = \mathrm{cl}\,\Omega$, and $\partial_*\Omega = \partial\Omega$. If f is a BV function in a Lipschitz domain Ω, then there is a function $\mathrm{Tr}\, f \in L^1(\partial\Omega, \mathcal{H}^{m-1})$, called the *trace* of f on $\partial\Omega$, such that

$$\mathrm{Tr}\, f(x) = \lim_{r\to 0+} \frac{1}{|B[x,r]|} \int_{\Omega\cap B[x,r]} f(y)\,dy$$

for $\mathcal{H}^{m-1}$-almost all $x \in \partial\Omega$ [6, Section 5.3].

Theorem 7.17. *Let Ω be a Lipschitz domain, $v \in C(\mathrm{cl}\,\Omega; \mathbb{R}^m)$, and let $S \subset \Omega$ be a thin set such that v is Lipschitz at each $x \in \mathrm{int}\,\Omega - S$. Then*

$$\int_\Omega g \,\mathrm{div}\, v \, d\mathcal{L}^m = \int_{\partial\Omega} (\mathrm{Tr}\, g)\, v \cdot \nu_\Omega \, d\mathcal{H}^{m-1} - \int_\Omega v \cdot Dg \tag{7.1}$$

for each $g \in BV^\infty(\Omega)$.

Proof. There is a sequence $\{v_i\}$ in $C^1(\mathbb{R}^m; \mathbb{R}^m)$ that converges to v uniformly in $\mathrm{cl}\,\Omega$. By [6, Section 5.3, Theorem 1], equality (7.1) holds for each v_i, and the dominated convergence theorem yields

$$\lim \int_{\partial\Omega} (\mathrm{Tr}\, g)\, v_i \cdot \nu_\Omega \, d\mathcal{H}^{m-1} = \int_{\partial\Omega} (\mathrm{Tr}\, g)\, v \cdot \nu_\Omega \, d\mathcal{H}^{m-1}$$

and $\lim \int_\Omega v_i \cdot Dg = \int_\Omega v \cdot Dg$. According to Theorem 7.14,

$$\begin{aligned} \left| \int_B (\mathrm{div}\, v_i - \mathrm{div}\, v) \, d\mathcal{L}^m \right| &= \left| \int_{\partial_* B} (v_i - v) \cdot \nu_B \, d\mathcal{H}^{m-1} \right| \\ &\le \|B\| \sup_{x \in \mathrm{cl}\,\Omega} \|v_i(x) - v(x)\| \end{aligned}$$

for every bounded BV set B. Thus $\lim \int_\Omega g \,\mathrm{div}\, v_i \, d\mathcal{L}^m = \int_\Omega g \,\mathrm{div}\, v \, d\mathcal{L}^m$ for each $g \in BV^\infty(\Omega)$ by Proposition 7.7. □

8 The R_*-integral

The R-integral introduced in Sections 4 and 6 is an averaging process of appreciable appeal:

- Its definition naturally extends the descriptive definition of the Lebesgue integral (Definition 5.5).
- It has a simple Riemannian description (Theorem 5.6).
- It is invariant with respect to lipeomorphic changes of variables (Theorem 7.5).
- It has an interesting algebra of multipliers (Theorem 6.4).
- Continuous linear functionals on the space of R-integrable functions have the usual integral representation (Proposition 7.6).
- It provides the divergence theorem and integration by parts formula for all differentiable vector fields (Theorems 7.14 and 7.17).

Unfortunately, the R-integral is not adequate for integration over bounded BV sets which are not essentially closed (Remark 7.15). We show, however, that a simple extension of the R-integral inherits all desirable properties and eliminates the shortcomings.

If f is a function defined on a bounded BV set A and F is a charge, denote by $\boldsymbol{\mathcal{R}}(f,F;A)$ the family of all $B \in \boldsymbol{\mathcal{BV}}(A)$ for which $f \restriction B$ belongs to $R(B)$ and $F \llcorner B$ is the indefinite R-integral of $f \restriction B$.

Definition 8.1. A function f defined on a BV set A is called *R_*-integrable* in A if there is a charge F in A such that $A \in \mathrm{cl}_{\mathcal{T}} \boldsymbol{\mathcal{R}}(f,F;A)$. The charge F, uniquely determined by f, is called the *indefinite R_*-integral* of f, and the number $F(A)$ is called the *R_*-integral* of f over A.

The linear space of all R_*-integrable functions in a bounded BV set A is denoted by $R_*(A)$. Clearly $R(A) \subset R_*(A)$, and arguments similar to those of [13, Sections 8 and 9] show the R_*-integral has the usual "integral properties," including the invariance with respect to lipeomorphic changes of coordinates (Theorem 5.11). The validity of Propositions 7.6 and 7.7 for the R_*-integral is also easy to verify. With no danger of confusion, for $f \in R_*(A)$ we denote by $\int_A f\,d\mathcal{L}^m$ the R_*-integral of f over A.

Since each BV set in $\mathbb{R}$ is equivalent to a figure, using arguments analogous to those employed in [15, Section 11.5], it is easy to see the R-integral and R_*-integral coincide in dimension one. The next proposition shows that in any dimension, the R_*-integral and R_{**}-integral coincide — in other words, no new integral is produced by iterating Definition 8.1.

Proposition 8.2. *Let f be a function defined on a bounded BV set A, and let F be a charge. If $\boldsymbol{\mathcal{R}}_*(f,F;A)$ is the family of all $B \in \boldsymbol{\mathcal{BV}}(A)$ for which $f \restriction B$ belongs to $R_*(B)$ and $F \llcorner B$ is the indefinite* R_**-integral of $f \restriction B$, then*

$$\mathrm{cl}_{\mathcal{T}} \boldsymbol{\mathcal{R}}_*(f,F;A) = \mathrm{cl}_{\mathcal{T}} \boldsymbol{\mathcal{R}}(f,F;A)\,.$$

Proof. Since

$$\begin{aligned}\boldsymbol{\mathcal{R}}_*(f,F;A) &\subset \bigcup\{\mathrm{cl}_\tau \boldsymbol{\mathcal{R}}(f,F;B) : B \in \boldsymbol{\mathcal{R}}_*(f,F;A)\} \\ &\subset \mathrm{cl}_\tau\Big(\bigcup\{\boldsymbol{\mathcal{R}}(f,F;B) : B \in \boldsymbol{\mathcal{R}}_*(f,F;A)\}\Big) \subset \mathrm{cl}_\tau \boldsymbol{\mathcal{R}}(f,F;A),\end{aligned}$$

we have $\mathrm{cl}_\tau \boldsymbol{\mathcal{R}}_*(f,F;A) \subset \mathrm{cl}_\tau \boldsymbol{\mathcal{R}}(f,F;A)$. As $\boldsymbol{\mathcal{R}}(f,F;A) \subset \boldsymbol{\mathcal{R}}_*(f,F;A)$, the proposition follows. ☐

The reader should compare the following proposition with Corollary 7.12.

Proposition 8.3. *Let f be a function defined on* **any** *set $A \in \boldsymbol{\mathcal{BV}}$. Then $f \in R_*(A)$ whenever there is a charge F such that $(V_*F)\llcorner \mathrm{cl}_* A$ is absolutely continuous, and $F'(x) = f(x)$ for almost all $x \in A$. In this case $F\llcorner A$ is the indefinite* R_**-integral of f.*

Proof. As in dimension one the proposition coincides with Corollary 7.12, assume $m \geq 2$. According to [23], there is a sequence $\{A_i\}$ of essentially closed BV subsets of A such that $\lim \|A - A_i\| = 0$. By Corollary 7.12, the charge $F\llcorner A_i$ is the indefinite R-integral of $f \restriction A_i$. Thus each A_i belongs to $\boldsymbol{\mathcal{R}}(f, F\llcorner A; A)$, and by the isoperimetric inequality, $\{A_i\}$ τ-converges to A. ☐

Lemma 8.4. *Let $A \in \boldsymbol{\mathcal{BV}}$, and let $\boldsymbol{\mathcal{E}} \subset \boldsymbol{\mathcal{BV}}(A)$ be such that $A \in \mathrm{cl}_\tau \boldsymbol{\mathcal{E}}$. There is a sequence $\{E_i\}$ in $\boldsymbol{\mathcal{E}}$ such that $\left|A - \bigcup_{i=1}^\infty E_i\right| = 0$.*

Proof. Let $\{\boldsymbol{\mathcal{C}}_t : t \in T\}$ be the collection of all countable subfamilies of $\boldsymbol{\mathcal{E}}$, and let $\{C_i\}$ be a sequence in $\boldsymbol{\mathcal{C}} = \bigcup_{t\in T} \mathrm{cl}_\tau \boldsymbol{\mathcal{C}}_t$ which τ-converges to C. Given an integer $i \geq 1$, find a $t_i \in T$ with $C_i \in \mathrm{cl}_\tau \boldsymbol{\mathcal{C}}_{t_i}$, and observe $\bigcup_{i=1}^\infty \boldsymbol{\mathcal{C}}_{t_i} = \boldsymbol{\mathcal{C}}_s$ for an $s \in T$. Since $\bigcup_{i=1}^\infty \mathrm{cl}_\tau \boldsymbol{\mathcal{C}}_{t_i} \subset \mathrm{cl}_\tau \boldsymbol{\mathcal{C}}_s$, the set C belongs to $\mathrm{cl}_\tau \boldsymbol{\mathcal{C}}_s$, and hence to $\mathrm{cl}_\tau \boldsymbol{\mathcal{C}}$. As τ is a sequential topology (see Section 2), the family $\boldsymbol{\mathcal{C}}$ is τ-closed, and as $\boldsymbol{\mathcal{E}} \subset \boldsymbol{\mathcal{C}}$, the family $\boldsymbol{\mathcal{C}}$ contains A. Thus $A \in \mathrm{cl}_\tau \boldsymbol{\mathcal{C}}_u$ for a $u \in T$, and we can find a sequence $\{E_i\}$ with $\boldsymbol{\mathcal{C}}_u = \{E_1, E_2, \dots\}$. If $E = \bigcup \boldsymbol{\mathcal{C}}_u$, then

$$\boldsymbol{\mathcal{B}} = \big\{B \in \boldsymbol{\mathcal{BV}}(A) : |B - E|\big\}$$

contains $\boldsymbol{\mathcal{C}}_u$. Employing again τ is a sequential topology, it is easy to verify $\boldsymbol{\mathcal{B}}$ is τ-closed. We conclude $A \in \boldsymbol{\mathcal{B}}$, which proves the lemma. ☐

Lemma 8.5. *Let $A \in \boldsymbol{\mathcal{BV}}$ and $v \in C(\mathrm{cl}\, A; \mathbb{R}^m)$. Suppose there is a family $\boldsymbol{\mathcal{E}} \subset \boldsymbol{\mathcal{BV}}(A)$ such that for each $E \in \boldsymbol{\mathcal{E}}$, the restriction $v \restriction \mathrm{cl}\, E$ is Lipschitz at almost all $x \in E$. If $A \in \mathrm{cl}_\tau \boldsymbol{\mathcal{E}}$, then there is a function f, defined on A and unique up to equivalence, such that given $E \in \boldsymbol{\mathcal{E}}$,*

$$f(x) = \mathrm{div}\,(v \restriction \mathrm{cl}\, E)(x)$$

for almost all $x \in E$. In particular, $f(x) = \mathrm{div}\, v(x)$ for almost all $x \in A$ whenever $A \in \boldsymbol{\mathcal{E}}$.

Proof. Extend v to a $w \in C(\mathbb{R}^m;\mathbb{R}^m)$, and denote by F the flux of w. Given $E \in \boldsymbol{\mathcal{E}}$, use Lemma 7.13 to extend $v \restriction \mathrm{cl}\, E$ to a $w_E \in C(\mathbb{R}^m;\mathbb{R}^m)$ which is Lipschitz at almost all $x \in E$, and denote by F_E the flux of w_E. Clearly $F \llcorner E = F_E \llcorner E$, and we recall from the paragraph preceding Theorem 7.14 there is a negligible set $N_E \subset E$ such that at each $x \in E - N_E$, the charge $F \llcorner E$ is derivable and

$$\mathrm{div}\,(v \restriction \mathrm{cl}\, E)(x) = (F \llcorner E)'(x).$$

For every pair $B, C \in \boldsymbol{\mathcal{E}}$,

$$(F \llcorner B) \llcorner C = F \llcorner (B \cap C) = (F \llcorner C) \llcorner B.$$

According to Lemma 7.11, we can find a negligible set $N_{B,C} \subset B \cap C$ so that at each $x \in B \cap C - N_{B,C}$, the charge $F \llcorner (B \cap C)$ is derivable and

$$(F \llcorner B)'(x) = \big[(F \llcorner B) \llcorner C\big]'(x) = \big[(F \llcorner C) \llcorner B\big]'(x) = (F \llcorner C)'(x).$$

By Lemma 8.4, there is a sequence $\{E_i\}$ in $\boldsymbol{\mathcal{E}}$ such that $N_0 = A - \bigcup_{i=1}^{\infty} E_i$ is a negligible set. The set

$$N = N_0 \cup \left(\bigcup_{i=1}^{\infty} N_{E_i} \right) \cup \left(\bigcup_{i,j=1}^{\infty} N_{E_i,E_j} \right)$$

is negligible, and we define a function f on A by the formula

$$f(x) = \begin{cases} (F E_i)'(x) & \text{if } x \in E_i - N, \\ 0 & \text{if } x \in N. \end{cases}$$

Our choice of N shows that f is well-defined, and that given $E \in \boldsymbol{\mathcal{E}}$,

$$f(x) = \mathrm{div}\,(v \restriction \mathrm{cl}\, E)(x)$$

for each $x \in E - N \cup N_E$. Let g be a function on A such that given $E \in \boldsymbol{\mathcal{E}}$,

$$g(x) = \mathrm{div}\,(v \restriction \mathrm{cl}\, E)(x)$$

for all $x \in E - Z_E$ where Z_E is a negligible subset of E. If $Z = \bigcup_{i=1}^{\infty} Z_{E_i}$, then $g(x) = f(x)$ for each $x \in A - N \cup Z$, which establishes the equivalence class of f is unique. $\square$

Under the assumptions of Lemma 8.5, we call the function f the *divergence* of v, still denoted $\mathrm{div}\, v$. The last claim of Lemma 8.5 shows this is consistent with our previous definition of $\mathrm{div}\, v$ introduced prior to Theorem 7.14.

Theorem 8.6. *Let $A \in \boldsymbol{\mathcal{BV}}$ and $v \in C\big(\mathrm{cl}\, A; \mathbb{R}^m\big)$. Suppose there is a family $\boldsymbol{\mathcal{E}} \subset \boldsymbol{\mathcal{BV}}(A)$ with the following property: given $E \in \boldsymbol{\mathcal{E}}$, we can find a thin set $S_E \subset \mathrm{int}_* E$ so that $v \restriction \mathrm{cl}\, E$ is Lipschitz at each $x \in \mathrm{int}_* E - S_E$. If $A \in \mathrm{cl}_{\mathcal{T}} \boldsymbol{\mathcal{E}}$, then $\mathrm{div}\, v$ is R_*-integrable in A and*

$$\int_A \mathrm{div}\, v \, d\mathcal{L}^m = \int_{\partial_* A} v \cdot \nu_A \, d\mathcal{H}^{m-1}.$$

Proof. Extend v to a $w \in C(\mathbb{R}^m; \mathbb{R}^m)$, and denote by F the flux of w. Furthermore, for each $E \in \boldsymbol{\mathcal{E}}$, extend $v \restriction \operatorname{cl} E$ to a $w_E \in C(\mathbb{R}^m; \mathbb{R}^m)$ that is Lipschitz at each $x \in \operatorname{int}_* E - S_E$, and denote by F_E the flux of w_E. These extensions exist according to Lemma 7.13, and clearly $F \llcorner E = F_E \llcorner E$. The definition of $\operatorname{div} v$, together with Lemmas 5.9,(2) and 7.11, yields

$$\operatorname{div} v(x) = \operatorname{div}(v \restriction \operatorname{cl} E)(x) = (F_E \llcorner E)(x) = (F \llcorner E)'(x)$$

for almost all $x \in E$. As $S_E \cup \partial_* E$ is a thin set, $(V_* F_E) \llcorner \operatorname{cl}_* E$ is absolutely continuous by Lemmas 5.10, and Lemma 7.11 implies

$$\big[V_*(F \llcorner E)\big] \llcorner \operatorname{cl}_* E = \big[V_*(F_E \llcorner E)\big] \llcorner \operatorname{cl}_* E$$

is absolutely continuous. Now it follows from Proposition 8.3 that $(\operatorname{div} v) \restriction E$ belongs to $R_*(E)$ and $F \llcorner E$ is the indefinite R_*-integral of $(\operatorname{div} v) \restriction E$. An application of Proposition 8.2 completes the proof. □

Remark 8.7. By [2, Theorem 5.3] and [13, Proposition 7.7], each R-integrable function is *variationally integrable* according to [13, Definition 5.1]. On the other hand, Propositions 8.3 and 5.2,(2), Lemma 7.11, [2, Theorem 5.3], and [13, Proposition 7.7] imply each variationally integrable function is R_*-integrable. Thus using Proposition 8.2, we conclude the R_*-integral coincides with the *continuous integral* of [13, Definition 9.1].

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