Lipeomorphisms, sets of bounded variation and integrals

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ABSTRACT

To obtain a lipeomorphic (bi-Lipschitz) change of variables theorem for generalized integrals lipeomorphic transformations of partitions are studied.

1. Introduction

For a sufficiently general setting of the Gauss-Green theorem in [P1] generalized integrals over sets of bounded variation were considered. The family of BV sets (sets of bounded variation, or sets of finite perimeter) provides the largest class of sets in $\mathbf{R}^m$ for which the surface area and exterior normal can still be defined in a way which leads to a useful integration theory. Originally in the Henstock type definition of higher dimensional integrals the integration was done on (higher dimensional) intervals and the partitions considered consisted of intervals as well. This concept was not coordinate free and one could not consider a sufficiently general class of transformations of these integrals. The current generalizations of the Henstock-Kurzweil integral are leading to interesting and difficult questions in geometric measure theory.

Instead of intervals one can consider the class of BV sets, or the class of figures (finite unions of intervals). In [BP] variations of additive functions were used to define two integrals. One is the $\mathcal{F}$ integral, based on partitions into figures. The other one is the $\mathcal{BV}$ integral, based on partitions into $\mathcal{BV}$ sets (the class $\mathcal{BV}$ consists of essentially closed BV sets, for each BV set A one can find a set $A' \in \mathcal{BV}$ such that the symmetric difference of A and A' is of measure zero). First, it is not clear how different these two concepts of integration are. In [P2] it was shown that if A is a figure then the class of $\mathcal{F}$ and $\mathcal{BV}$ integrable functions coincides. Therefore if we consider functions defined on figures we will only say that a function is integrable, since this will mean both $\mathcal{F}$ and $\mathcal{BV}$ integrability. On the other hand, a remark of [P2] shows that the $\mathcal{BV}$ integral is a proper generalization of the $\mathcal{F}$ integral. In fact, an example of this author was used to show that there is a $\mathcal{BV}$ set A, and a function $f : A \to \mathbf{R}$ which has no integrable extension to a figure containing A.

Since in most cases the integral based on a smaller family of sets turns out to be more general this relationship of the $\mathcal{BV}$ and $\mathcal{F}$ integrals is a bit unusual.

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W. F. Pfeffer suggested to consider the following class of integrable functions. Given a BV set A, consider those functions $f : A \to \mathbf{R}$ for which there is a figure K containing A such that f can be extended to be an integrable function on K. We will denote this class of functions by $\mathcal{BVF}(A)$. As the above mentioned example shows there are examples of $\mathcal{BV}$ sets when the class $\mathcal{BVF}(A)$ is a proper subset of $\mathcal{BV}(A)$. It would be interesting to see a characterization of those sets where these two concepts coincide. Clearly, figures are such sets.

One natural motivation for studying the $\mathcal{BVF}$ integral is that it is easy to see that it is additive, while for the $\mathcal{BV}$ integral an extension process, named the c-integral, is needed to obtain additivity, for details see [P1, Remark 5.2.5 and Section 9].

Working with BV sets the natural class of transformations is the class of lipeomorphisms. A transformation $\Phi : A \to \mathbf{R}^m$ is a lipeomorphism if both Φ and its inverse Φ^{-1} are Lipschitz mappings. Some authors call these mappings bi-Lipschitz mappings.

W. F. Pfeffer asked this author whether the $\mathcal{BVF}$ integral has a lipeomorphic change of variables property, that is, whether this class is invariant with respect to lipeomorphic transformations. This leads to the following, in our opinion very interesting, extension problem. Assume that $A \subseteq \mathbf{R}^m$ is a BV set and $\Phi : A \to \mathbf{R}^m$ is a lipeomorphism. Suppose furthermore, that $K \subseteq \mathbf{R}^m$ is a figure which contains $\Phi(A)$, and $f : K \to \mathbf{R}$ is an integrable function (we have this situation if we consider $\mathcal{BVF}$ integrable functions on $\Phi(A)$). Denote by $J_\Phi(x)$ the Jacobian of Φ where it exists. Recall that for lipeomorphisms J_Φ exists almost everywhere on A. Then, one can define g almost everywhere on A by setting $g(x) = J_\Phi(x) f(\Phi(x))$ at all points where the right hand side is defined. Then there is the natural question whether $g(x)$ is $\mathcal{BVF}$ integrable on A, that is, whether there is an extension of g onto a figure containing A. In the main theorem of this paper we give a positive answer to this question. This also yields the lipeomorphic change of variables formula $\mathcal{BVF} \int_{\Phi(A)} f d\lambda = \mathcal{BVF} \int_A (f \circ \Phi) J_\Phi d\lambda$ for the $\mathcal{BVF}$ integral.

This paper is organized in the following way. Studying generalized integrals, $\mathcal{BV}$-sets, and lipeomorphisms one can see several interesting and difficult problems. After some preliminaries in the "Geometric Lemmas" section we review some of these properties. This author proved in [Bu1] that if $\Phi : A \to \mathbf{R}^m$ is a lipeomorphism then Φ maps density points of A exactly onto density points of $\Phi(A)$. It is easy to see that almost every density point is mapped onto density points, the main difficulty is showing that there is no exceptional set of measure zero. The basic problem is that Φ does not always have a lipeomorphic extension from A onto $\mathbf{R}^m$. This will be the most difficult problem in this paper as well. To control the shape of the partitions only regular partitions are considered. This means that the product of the surface and the diameter of our sets cannot be too large compared to the volume of these sets. The main problem is that we will have to consider regular sets A_i such that we have no information about the regularity of $A_i \cap A$. Since Φ does not necessarily extend to A_i as a lipeomorphism we face some difficulties when we need to do estimations related to the Φ image of A_i. In the "Main lemmas about lipeomorphic transformations of partitions" section we verify two Lemmas about this lipeomorphic changes of regular partitions. In Lemma 4.1 we deal with the case when there is enough room in the complement of $\Phi(A)$ in order to "make" the images $\Phi(A_i \cap A)$ regular. In Lemma 4.2, which is the most delicate lemma of this paper, we deal with the

case when there is not enough room in the complement of $\Phi(A)$. Finally in the last section we turn to generalized integrals, state and prove the main result of the paper.

2. Preliminaries

The set of reals is denoted by $\mathbf{R}$. The Euclidean distance of $x, y \in \mathbf{R}^m$ is denoted by $\operatorname{dist}(x, y)$. The open ball centered at $x \in \mathbf{R}^m$ and of radius r is denoted by $B(x, r)$. If $A \subseteq \mathbf{R}^m$ then $B(A, r) = \{x \in \mathbf{R}^m : \operatorname{dist}(x, A) < r\}$.

An interval in $\mathbf{R}^m$ is the Cartesian product of intervals on the coordinate axes. A figure is the finite union of closed compact non-degenerate intervals. The class of figures in $\mathbf{R}^m$ is denoted by $\mathcal{F}$.

Given $x \in \mathbf{R}^m$ and $d > 0$ we denote by $Q(x, d)$ the closed cube centered at x and of side length $2d$. If a cube Q is given we denote by $\ell(Q)$ its side length, that is, $\ell(Q(x, d)) = 2d$.

Definition 2.1. The set A is η-filling in the cube Q if from $Q(x, \eta) \subseteq Q$ it follows that $Q(x, \eta) \cap A \neq 0$.

If $A \subseteq \mathbf{R}^m$ then $d(A)$, $\operatorname{cl} A$, $\operatorname{int} A$, ∂A, and $|A|$ will denote its diameter, closure, interior, boundary, and Lebesgue measure, respectively.

If $A \subseteq \mathbf{R}^m$ then the transformation $\Phi : A \to \mathbf{R}^m$ is Lipschitz when there is a constant c_Φ such that $\operatorname{dist}(\Phi(x), \Phi(y)) \leq c_\Phi \operatorname{dist}(x, y)$ holds for all $x, y \in A$. We say that Φ is a c_Φ-lipeomorphism if both Φ and its inverse Φ^{-1} are c_Φ-Lipschitz transformations, that is,

$$\operatorname{dist}(x, y)/c_\Phi \leq \operatorname{dist}(\Phi(x), \Phi(y)) \leq c_\Phi \operatorname{dist}(x, y),$$

observe that in this case $c_\Phi \geq 1$ should hold. If $\Phi : A \to \mathbf{R}^m$ is a c_Φ-Lipschitz mapping then by Kirszbraun's theorem [Fe, 2.10.43] it has a c_Φ- Lipschitz extension $\tilde{\Phi} : \mathbf{R}^m \to \mathbf{R}^m$. Unfortunately, there are no such extension theorems for c_Φ-lipeomorphisms. The best we can say, and the reader can easily check, that if $\Phi : A \to \mathbf{R}^m$ is a c_Φ- lipeomorphism then we can extend it to a c_Φ-lipeomorphism defined on $\operatorname{cl} A$.

All sets considered in this paper will be Lebesgue measurable. The sets $A, B \subseteq \mathbf{R}^m$ are non-overlapping if $|A \cap B| = 0$. The s-dimensional Hausdorff measure of the set $A \subseteq \mathbf{R}^m$ is denoted by $\mathcal{H}^s(A)$. For the special case $s = m-1$ we write simply $\mathcal{H}(A)$. A set is thin if it is of σ-finite $\mathcal{H}$-measure. Also recall that there is a constant, κ_m such that $\mathcal{H}^m(A) = \kappa_m |A|$.

Definition 2.2. Let $0 \leq s < \infty$, $A \subseteq \mathbf{R}^m$, and $a \in \mathbf{R}^m$. The upper s density of A at a is

$$\Theta^{*s}(A, a) = \limsup_{r \to 0+} \frac{\mathcal{H}^s(A \cap B(a, r))}{(2r)^s}.$$

If $A \subseteq \mathbf{R}^m$ we say that x is a dispersion point of A if $\Theta^{*m}(A, x) = 0$ and x is a density point of A if it is a dispersion point of $\mathbf{R}^m \setminus A$. The set of the density points of A is called the essential interior of A and is denoted by $\operatorname{int}^* A$. The set $\mathbf{R}^m \setminus \operatorname{int}^*(\mathbf{R}^m \setminus A)$ is called the essential closure of A, and is denoted by $\operatorname{cl}^* A$. Finally the essential boundary of A equals

$\partial^* A = \mathrm{cl}^* A \setminus \mathrm{int}^* A$. The set A is essentially closed or essentially open if $A = \mathrm{cl}^* A$, or $A = \mathrm{int}^* A$.

A bounded measurable set is a set of bounded variation (abbreviated as BV set) if it has a finite perimeter, that is, $\mathcal{H}(\partial^* A) = ||A||$ is finite. (BV sets are usually defined otherwise by saying that A is a BV set if the distributional gradient of its characteristic function is a vector-valued measure in $\mathbf{R}^m$, whose variation, the surface measure of A, is finite. However by [Fe, 2.10.6, and 4.5.11] this definition is equivalent to the one we use.) The class of essentially closed BV sets is denoted by $\mathcal{BV}$. Observe that $\mathcal{F} \subseteq \mathcal{BV}$ and if $A \in \mathcal{F}$ then $A = \mathrm{cl}\, A = \mathrm{cl}^* A$, and $\partial A = \partial^* A$.

If $A \in BV$ then $\mathrm{cl}^* A \in \mathcal{BV}$, and the symmetric difference of A and $\mathrm{cl}^* A$ is of zero Lebesgue measure.

Given $\epsilon > 0$ we say that a BV set A is ϵ-regular if $d(A)||A|| > 0$ and $|A|/d(A)||A|| > \epsilon$. Observe that there are several slightly different concepts of regularity, but from the point of view of our integration theory these concepts are equivalent.

3. Geometric Lemmas

From [Bu2] we recall a theorem which says that if a set E is a subset of a BV set H, and H is of not too much surface then the Lipschitz image of E can also be wrapped into a figure of not too big surface.

Theorem 3.1. *Assume that $E \subseteq \mathbf{R}^m$ is compact, $\Phi : E \to \mathbf{R}^m$ is a Lipschitz c_Φ-mapping, and H is a BV set such that $E \subseteq int^* H$. Then there exists a figure F, and a constant c_L depending on c_Φ and m such that $\Phi(E) \subseteq int(F)$, $||F|| \leq c_L||H||$, and $|F| \leq c_L|H|$.*

We remark that if E is a figure then $\Phi(E)$ is not necessarily a BV set, that is $||\Phi(E)||$ can be infinite, for the details see [Bu2]. In [Bu2] we stated the above theorem with the conclusion $\Phi(E) \subseteq F$. However using a slightly bigger value of c_L we can enlarge the figure F to obtain $\Phi(E) \subseteq \mathrm{int}\,(F)$.

We will also use from [Z, Theorem 5.4.3., p.230] the

Relative isoperimetric inequality. *If $H \subseteq \mathbf{R}^m$ is a BV set then for each ball $B(x,r) \subseteq \mathbf{R}^m$ we have*

$$\min\{|B(x,r) \cap H|, |B(x,r) \setminus H|\}^{\frac{m-1}{m}} \leq c_{riso} \mathcal{H}(B(x,r) \cap \partial^* H),$$

where the constant c_{riso} depends only on the dimension.

We say that $Q \subseteq \mathbf{R}^m$ is a dyadic netcube if there exist integers $n, k_1, ..., k_m$ such that Q is the Cartesian product of intervals of the form $[(k_j - 1)/2^n, k_j/2^n]$, $j = 1, ..., m$.

The next lemma is roughly stating that for regular BV sets we have a certain uniformity of the density properties, that is, we can find sufficiently large cubes which contain a lot of the set, but not too much of its surface.

Lemma 3.2. *Assume that $\epsilon, \nu > 0$, $B \subseteq \mathbf{R}^m$ is an ϵ-regular BV set and $x \in B$ is an arbitrary point. Then there exists a constant $c_{\epsilon,\nu} < 1/4\sqrt{m}$ depending on ϵ, ν and m such that we can find a dyadic net cube $Q = Q(x', d_Q) \subseteq B(x, 4d(B)\sqrt{m})$ satisfying*

$$c_{\epsilon,\nu} d(B) < d_Q \leq 2c_{\epsilon,\nu} d(B),$$

$$\mathcal{H}(int(Q) \cap \partial^* B) \leq \nu ||Q||,$$

and

$$|Q \setminus cl^* B| < \nu |Q|.$$

Proof. Choose a cube $Q_0 \supset B$ such that it is the union of 2^m non-overlapping dyadic net cubes of side length $\ell(Q_0)/2$, $Q_0 \subseteq B(x, 4d(B)\sqrt{m})$ and $\ell_0 = \ell(Q_0) \leq 4d(B)$. Since B is ϵ-regular $|B| \geq \epsilon ||B|| d(B)$; hence

$$\ell_0^m = |Q_0| \geq |B| \geq \epsilon ||B|| \ell_0/4, \text{ that is, } 4\ell_0^{m-1}/\epsilon \geq ||B||.$$

By the (standard) isoperimetric inequality there exists a constant c_{iso} depending only on m such that $|B|^{(m-1)/m} \leq c_{iso} ||B||$. Thus for any ϵ-regular BV set B we have

$$|B| \geq \epsilon ||B|| d(B) \geq \frac{\epsilon}{c_{iso}} |B|^{\frac{m-1}{m}} d(B),$$

$$|B|^{1/m} \geq \frac{\epsilon}{c_{iso}} d(B), \tag{3.1}$$

$$|B| \geq \frac{\epsilon^m}{c_{iso}^m} d(B)^m \geq \frac{\epsilon^m}{c_{iso}^m 4^m} \ell_0^m = \beta \ell_0^m,$$

where β is a constant depending only on ϵ and m.

Choose $k \in \mathbf{N}$ and denote by $Q_1, ..., Q_{2^{km}}$ the 2^{km} non-overlapping net subcubes of Q_0 with side length $\ell_0/2^k$. We assume that k is so large that $\ell_0/2^k < 1/4\sqrt{m}$.

Denote by $[x]$ the integer part of $x \in \mathbf{R}$. Next we verify that there are at least $[\beta 2^{km-1}]$ many Q_j's satisfying

$$|Q_j \cap \mathrm{cl}^* B| \geq \beta \, |Q_j|/2 = \beta \ell_0^m / 2^{km+1}.$$

Denote by $\mathcal{J}$ the set of those j's for which the above inequality is satisfied. If $\#\mathcal{J} < \beta 2^{km-1}$ then

$$\beta \ell_0^m \leq |B| = |\mathrm{cl}^* B| = \sum_{j=1}^{2^{km}} |Q_j \cap \mathrm{cl}^* B| =$$

$$\sum_{j \notin \mathcal{J}} |Q_j \cap \mathrm{cl}^* B| + \sum_{j \in \mathcal{J}} |Q_j \cap \mathrm{cl}^* B| < 2^{km} \frac{\beta \ell_0^m}{2^{km+1}} + \beta 2^{km-1} \frac{\ell_0^m}{2^{km}} = \beta \ell_0^m$$

which is a contradiction. Thus $\#\mathcal{J} \geq [\beta 2^{km-1}]$.

Denote by $\mathcal{J}_1$ the set of those j's in $\mathcal{J}$ for which $\mathcal{H}(\text{int}\,(Q_j)\cap\partial^* B)\geq c\nu||Q_j||=\nu c'_m\ell(Q_j)^{m-1}$, where c and c'_m are constants to be determined later so that they will depend only on ν and m. Then

$$4\ell_0^{m-1}/\epsilon\geq||B||\geq\sum_{j\in\mathcal{J}_1}\mathcal{H}(\text{int}\,(Q_j)\cap\partial^* B)\geq\nu c'_m(\ell_0/2^k)^{m-1}\cdot\#\mathcal{J}_1.$$

Hence

$$\#\mathcal{J}_1\leq\frac{4\cdot 2^{k(m-1)}}{\epsilon\nu c'_m}<\#\mathcal{J}$$

if k is sufficiently large (observe that the value of k is not depending on B, it depends only on ϵ, ν and m).

Assume $Q=Q_j$ for a $j\in\mathcal{J}\setminus\mathcal{J}_1$. If $0<c<1$ then $\mathcal{H}(\text{int}\,(Q)\cap\partial^* B)<c\nu||Q||<\nu||Q||$. It is also clear that $Q\subseteq Q_0\subseteq B(x,4d(B)\sqrt{m})$. Finally, using the version of the relative isoperimetric inequality which is valid for cubes there is a constant c_{risoQ} depending only on the dimension such that

$$\min\{|Q\cap B|,|Q\setminus B|\}^{\frac{m-1}{m}}\leq c_{risoQ}\mathcal{H}(\text{int}\,(Q)\cap\partial^* B)<c_{risoQ}c\nu||Q||=$$

$$c_{risoQ}c'_m\nu\ell_0^{m-1}/2^{k(m-1)}<(\frac{\beta}{2}\ell_0^m/2^{km})^{\frac{m-1}{m}}=(\frac{\beta}{2}|Q|)^{\frac{m-1}{m}}$$

where in the last inequality we assumed that c, and hence c'_m, are both chosen sufficiently small, clearly this smallness depends on ν, m and ϵ, where the dependence on ϵ is implicit, since β depends on ϵ.

Since $j\in\mathcal{J}$ implies $|Q_j\cap B|=|Q\cap B|\geq\beta|Q|/2$ we infer

$$|Q\setminus B|<(c_{risoQ}c'_m\nu)^{\frac{m}{m-1}}|Q|\leq|Q|$$

whenever $c_{risoQ}c'_m\nu<1$, which we can assume.

Lemma 3.3. *Assume $B\subseteq\mathbf{R}^m$ is a figure, $Q\subseteq\mathbf{R}^m$ is a cube of side length d, and $|B\cap Q|\leq|Q|/2=d^m/2$. Then there exists a constant c_F depending only on m such that*

$$\mathcal{H}(B\cap\partial Q)\leq c_F\mathcal{H}(\partial B\cap int(Q)).$$

Proof. Assume that S denotes one of the $(m-1)$-dimensional sides of Q. Clearly, it is enough to find a constant c'_F, depending only on m, such that $\mathcal{H}(B\cap S)\leq c'_F\mathcal{H}(\partial B\cap\text{int}\,(Q))$. If $x\in S$ denote by ℓ_x the line through x which is perpendicular to the hyperplane containing S. Put $S_1=\{x\in B\cap S:\ell_x\cap\partial B\cap\text{int}\,(Q)\neq\emptyset\}$, $S_2=(B\cap S)\setminus S_1$. Since projections are not increasing measure we have

$$\mathcal{H}(S_1)\leq\mathcal{H}(\partial B\cap\text{int}\,(Q)). \tag{3.2}$$

If $x \in S_2$ then $\ell_x \cap \partial B \cap \operatorname{int}(Q) = \emptyset$ and hence $|B \cap Q| \geq \mathcal{H}(S_2)d$. Using the version of the relative isoperimetric inequality which works for cubes there is a constant c depending only on m such that

$$|B \cap Q|^{\frac{m-1}{m}} \leq c\mathcal{H}(\partial B \cap \operatorname{int}(Q)),$$

where we also used the assumption $|B \cap Q| \leq d^m/2$, which, on the other hand, implies $d^m/2 \geq \mathcal{H}(S_2)d$, that is, $d \geq (2\mathcal{H}(S_2))^{1/(m-1)}$. Hence

$$c\mathcal{H}(\partial B \cap \operatorname{int}(Q)) \geq |B \cap Q|^{\frac{m-1}{m}} \geq (\mathcal{H}(S_2)d)^{\frac{m-1}{m}} \geq$$

$$(\mathcal{H}(S_2)(2\mathcal{H}(S_2))^{\frac{1}{m-1}})^{\frac{m-1}{m}} = 2^{1/m}\mathcal{H}(S_2).$$

This and (3.2) implies the required inequality.

Next we recall some results of [V]. These results are related to degree theory.

Lemma 3.4. *Assume that $\Phi : A \to \mathbf{R}^m$ is a c_Φ-lipeomorphism and $\tilde{\Phi} : \mathbf{R}^m \to \mathbf{R}^m$ denotes its extension which is obtained by Kirszbraun's theorem. Then there exist constants $\eta_\Phi > 0$, and $c_G > 0$ depending on c_Φ and m such that if Q is a cube of side length $2d$ and A is $\eta_\Phi d$-filling in Q then $\tilde{\Phi}(Q)$ contains a cube of side length $2c_G d$.*

Proof. Since A is $\eta_\Phi d$-filling in Q if $\eta_\Phi > 0$ is sufficiently small then there are points of A sufficiently close to the center of Q. Therefore, we can assume that there is an $x \in A \cap Q$ such that $B(x, d/2) \subseteq Q$. Using the argument of 3.16 of [V] for a small $\eta_\Phi > 0$ we obtain that $B(\Phi(x), d/4c_\Phi) \subseteq \tilde{\Phi}(B(x, d/2)) \subseteq \tilde{\Phi}(Q)$ and hence we can choose c_G such that $Q(\Phi(x), c_G d) \subseteq B(\Phi(x), d/4c_\Phi) \subseteq \tilde{\Phi}(Q)$.

It is not difficult to see that the following "rescaled" version of [V, Theorem 3.17] is also valid.

Theorem 3.5. *Assume $\eta, d > 0$. Suppose that $\Psi : A_0 \to \mathbf{R}^m$ is a c_Ψ-lipeomorphism, where $A_0 \subseteq B(0, \eta d)$, $0 \in A_0$, and $\Psi(0) = 0$. Suppose also that $0 < 2\sqrt{m}/\eta^2 < 1/16c_\Psi^3(c_\Psi + 1)$ and that $B(0, \eta d) \subseteq B(A_0, 2\sqrt{m}d/\eta)$. Then $B(0, \eta d/2c_\Psi) \subseteq B(\Psi(A_0), c_\Psi 2\sqrt{m}d/\eta)$.*

We use the above theorem to obtain estimates of the size of holes in lipeomorphic preimages. We state the next lemma the way it is used later.

Lemma 3.6. *Assume that $A \subseteq \mathbf{R}^m$ and $\Phi : A \to \mathbf{R}^m$ is a c_Φ-lipeomorphism. Choose $\eta > \sqrt{m}$ such that $2\sqrt{m}/\eta^2 < 1/16c_\Phi^3(c_\Phi + 1)$. Then for any $d > 0$ and $x \in A$ if $\Phi(A)$ is d/η-filling in $Q(\Phi(x), \eta d)$ then A is $c_\Phi 2\sqrt{m}d/\eta$-filling in $Q(x, \eta d/2c_\Phi\sqrt{m})$.*

Proof. Without limiting generality we can assume that $x = \Phi(x) = 0$. Denote the inverse of Φ by Ψ. Then Ψ is also a c_Φ-lipeomorphism with $\Psi(0) = 0$. Assume that $y_0 \in B(0, \eta d) \subseteq Q(0, \eta d)$. Observe that $\eta d > d > \sqrt{m}d/\eta$. It is easy to see that we can choose $y \in B(y_0, \sqrt{m}d/\eta)$ such that $B(y, \sqrt{m}d/\eta) \subseteq B(0, \eta d)$. Then $Q(y, d/\eta) \subseteq B(y, \sqrt{m}d/\eta) \subseteq$

$B(0,\eta d) \subseteq Q(0,\eta d)$. By our assumption $\emptyset \neq Q(y,d/\eta) \cap \Phi(A) \subseteq B(y_0, 2\sqrt{m}d/\eta) \cap \Phi(A)$, that is, letting $A_0 = \Phi(A) \cap B(0,\eta d)$ we have $Q(y,d/\eta) \cap A_0 \neq \emptyset$, therefore $B(0,\eta d) \subseteq B(A_0, 2\sqrt{m}d/\eta)$. Theorem 3.5 is applicable and we infer that

$$B(0,\eta d/2c_\Phi) \subseteq B(\Psi(A_0), c_\Phi 2\sqrt{m}d/\eta) \subseteq B(A, c_\Phi 2\sqrt{m}d/\eta).$$

Thus choose y' such that

$$Q(y', c_\Phi 2\sqrt{m}d/\eta) \subseteq Q(0, \eta d/2c_\Phi\sqrt{m}) \subseteq B(0,\eta d/2c_\Phi).$$

Then $B(0,\eta d/2c_\Phi) \subseteq B(A, c_\Phi 2\sqrt{m}d/\eta)$ implies that there exists $x \in A \cap B(y', c_\Phi 2\sqrt{m}d/\eta) \subseteq A \cap Q(y', c_\Phi 2\sqrt{m}d/\eta)$. This proves the Lemma.

Finally we need estimates of BV-derivatives of lipeomorphisms.

Lemma 3.7. *Assume that $K \subseteq \mathbf{R}^m$ is a figure, $A \subseteq \mathbf{R}^m$ is a BV-set, $\Phi : A \to K$ is a c_Φ-lipeomorphism. Then the following statements hold.*

For every $\epsilon_0 > 0$ if $A_0 \subseteq A$ is an ϵ_0-regular BV-set (or $\mathcal{BV}$-set) then the image $\Phi(A_0)$ is a $c_1\epsilon_0$-regular BV-set (or $\mathcal{BV}$-set), (where c_1 depends only on c_Φ and m).

There exists an exceptional set A^2 of measure zero with the property that for every $\phi : A \to [1, +\infty)$ "weight" function, and for every $\epsilon_0 > 0$ there exists a function $\eta_{\epsilon_0} > 0$ such that for all $x \in A \setminus A^2$ and for any ϵ_0-regular $\mathcal{BV}$-set $A_0 \subseteq B(x, \eta_{\epsilon_0}(x)) \cap A$ we have

$$\phi(x)\big|J_\Phi(x)|A_0| - |\Phi(A_0)|\,\big| < \epsilon_0|A_0|.$$

Proof. Since Φ is a lipeomorphism we can estimate by using [Fe, 2.10.11] the Hausdorff and Lebesgue measure of Φ images and preimages and we can choose a constant c depending only on c_Φ and m such that

$$d(A_0)/c \leq d(\Phi(A_0)) \leq cd(A_0),$$

$$|A_0|/c \leq |\Phi(A_0)| \leq c|A_0|,$$

and

$$||A_0||/c \leq ||\Phi(A_0)|| \leq c||A_0||,$$

where for the last inequality one needs the property that $\partial^*(\Phi(A_0)) = \Phi(\partial^*(A_0))$. The above three inequalities easily imply the first statement of the lemma.

From the argument of [P1, Theorem 6.7] about the BV-differentiation of lipeomorphisms it is not difficult to see that the second statement of the Lemma holds. We only remark that the set A^2 can be chosen to be the set of points in A where the BV-derivative of Φ does not exist.

4. Main Lemmas about lipeomorphic transformations of partitions

Definition 4.1. The collection $\mathcal{P} = \{(x_1, A_1), ..., (x_p, A_p)\}$ is an ϵ-regular $\mathcal{F}$, or $\mathcal{BV}$-pre-partition if the sets A_i are non-overlapping figures, or $\mathcal{BV}$ sets and the BV-sets $\{x_i\} \cup A_i$ are ϵ-regular.

If, in addition, $x_i \in A_i$ holds for all $i = 1, ..., p$ then $\mathcal{P}$ is a partition.

If $A \subseteq \mathbf{R}^m$ is a given set and $x_i \in A$ holds for $i = 1, ..., p$ then $\mathcal{P}$ is anchored in A. Finally we say that the partition $\mathcal{P}$ is in the set $B \subseteq \mathbf{R}^m$ if $\cup_{i=1}^p A_i \subseteq B$.

Definition 4.2. Assume that $c > 1$ is given. We say that $\mathcal{P} = \{(x_1, A_1), ..., (x_p, A_p)\}$ and $\mathcal{P}' = \{(x_1', A_1'), ..., (x_p', A_p')\}$ are *c-equivalent* partitions (or pre-partitions) if they are both partitions (or pre-partitions),

$$|A_i|/c \le |A_i'| \le c|A_i|,$$

$$||A_i||/c \le ||A_i'|| \le c||A_i||,$$

and

$$d(A_i \cup \{x_i\})/c \le d(A_i' \cup \{x_i'\}) \le c \cdot d(A_i \cup \{x_i\})$$

holds for $i = 1, ..., p$.

Remark. Since each partition is also a prepartition we can also say that a partition is c-equivalent to a prepartition. It is also clear that there is a constant c_1 (which depends only on c) such that if the (pre-)partition $\mathcal{P}$ in the above definition is ϵ-regular then $\mathcal{P}'$ is $c_1\epsilon$-regular.

Lemma 4.1. *Assume that $\epsilon > 0$, $A \subseteq \mathbf{R}^m$, and $\Phi : cl(A) \to \mathbf{R}^m$ is a c_Φ-lipeomorphism and $\mathcal{P}_1 = \{(x_1, A_1), ..., (x_p, A_p)\}$ is an ϵ-regular $\mathcal{F}$-partition anchored in $cl(A)$. Assume furthermore that there exists $\eta > 1$ such that for each $i \in \{1, ..., p\}$, $\Phi(A)$ is not $d(A_i)/\eta$-filling in $Q(\Phi(x_i), \eta d(A_i))$, that is, for each i we can choose y_i such that $Q(y_i, d(A_i)/\eta) \cap \Phi(A) = \emptyset$, and $Q(y_i, d(A_i)/\eta) \subseteq Q(\Phi(x_i), \eta d(A_i))$.*

Then there exists a constant $c_{\eta,\epsilon} > 1$ depending on η, ϵ, c_Φ and m such that we can choose an $\mathcal{F}$-pre-partition $\mathcal{P}_2 = \{(\Phi(x_1), B_1), ..., (\Phi(x_p), B_p)\}$ which is $c_{\eta,\epsilon}$-equivalent to $\mathcal{P}_1$ and satisfies $B_i \cap \Phi(cl^* A) = \emptyset$ for $i = 1, ..., p$.

Proof. For each i choose a dyadic net cube $B_i' = Q(y_i', d_i)$ such that $B_i' \subset \text{int}\,(Q(y_i', 2d_i)) \subset Q(y_i, d(A_i)/\eta)$ and $d(A_i)/8\eta \le d_i < d(A_i)/4\eta$.

Next we verify that there exists a constant $\kappa > 1$ depending on η, ϵ, c_Φ and m such that letting $\mathcal{J}_i = \{j : B_j' \cap B_i' \ne \emptyset\}$ we have $\#\mathcal{J}_i < \kappa$ and for $j \in \mathcal{J}_i$ we have $1/\kappa \le d(B_j')/d(B_i') \le \kappa$. We can also assume that κ is an integer. Using $B_j' \subseteq Q(y_j, d(A_j)/\eta) \subseteq Q(\Phi(x_j), \eta d(A_j))$, from $B_j' \cap B_i' \ne \emptyset$ it follows that

$$Q(\Phi(x_j), \eta d(A_j)) \cap B_i' \ne \emptyset. \tag{4.1}$$

Since $\Phi(x_j) \in \Phi(A)$ and $\Phi(A)$ is disjoint from $Q(y_i', 2d_i) \supset B_i' = Q(y_i', d_i)$, using (4.1) we infer that $\eta d(A_j) \ge d_i$. Thus $d_j \ge d(A_j)/8\eta \ge d_i/8\eta^2$; hence $d_j/d_i = d(B_j')/d(B_i') \ge 1/8\eta^2$. By a similar, symmetrical argument we also have $d_i/d_j \ge 1/8\eta^2$. Thus

$$8\eta^2 \ge d_j/d_i = d(B_j')/d(B_i') \ge 1/8\eta^2. \tag{4.2}$$

Since $d(A_i)/8\eta \le d_i < d(A_i)/4\eta$, and $d(A_j)/8\eta \le d_j < d(A_j)/4\eta$ by (4.2) we obtain

$$d(A_i)/16\eta^2 \le d(A_j) \le 16\eta^2 d(A_i).$$

Since $\{(x_1, A_1), ..., (x_p, A_p)\}$ is an ϵ-regular partition using an estimate like the one at (3.1) by the above inequality we can choose a constant $c'(\epsilon) > 0$ depending on ϵ and m such that $|A_j| \ge c'(\epsilon) d(A_j)^m$ and by the above inequalities $|A_j| \ge c(\epsilon) d(A_i)^m$ with a constant $c(\epsilon)$ depending also on ϵ and m. Since $B'_i \cap B'_j \ne \emptyset$ implies $Q(\Phi(x_i), \eta d(A_i)) \cap Q(\Phi(x_j), \eta d(A_j)) \ne \emptyset$ we have

$$\operatorname{dist}(\Phi(x_i), \Phi(x_j)) \le \eta(d(A_i) + d(A_j)) \le \eta(d(A_i) + 16\eta^2 d(A_i)) = d(A_i)\eta(1 + 16\eta^2).$$

Since Φ is a c_Φ-lipeomorphism we have

$$\operatorname{dist}(x_i, x_j) \le c_\Phi d(A_i)\eta(1 + 16\eta^2).$$

Thus

$$A_j \subseteq B(x_i, \operatorname{dist}(x_i, x_j) + d(A_j)) \subseteq B\big(x_i, c_\Phi d(A_i)\eta(1 + 16\eta^2) + 16\eta^2 d(A_i)\big). \tag{4.3}$$

Since the A_j's are non-overlapping and $|A_j| \ge c(\epsilon) d(A_i)^m$ using (4.3) we infer that $\#\mathcal{J}_i$ is bounded by a constant κ depending on η, ϵ, c_Φ and m. We can also assume that $\kappa > 8\eta^2$ and hence $\kappa \ge d(B'_j)/d(B'_i) \ge 1/\kappa$ holds for each i, j. Since B'_i and B'_j are net cubes if they overlap either $B'_i \subseteq B'_j$, or $B'_j \subseteq B'_i$. Given i choose a $j(i)$ (which might equal i) such that $B'_{j(i)} \subseteq B'_i$ and there is no proper subinterval $B'_k \subseteq B'_{j(i)}$. Since we have only finitely many B'_k's this minimal $B'_{j(i)}$ should exist. Observe that $B'_j \cap B'_i \ne \emptyset$ and hence the number of i's for which a given $j_0 = j(i)$ is bounded by κ.

It is also clear that $d(B'_{j(i)}) \ge d(B'_i)/\kappa$. Split $B'_{j(i)}$ into κ^m many equal non-overlapping subcubes and choose one of these as B_i in a way that this cube has not been used for any $i' \le i$. Using $d(A_i)/8\eta \le d_i < d(A_i)/4\eta$ and the ϵ-regularity of $\mathcal{P}_1$ it is not difficult to see that there is a constant $c_{\eta,\epsilon}$, depending on η, ϵ, c_Φ and m, such that $\{(\Phi(x_1), B_1), ..., (\Phi(x_p), B_p)\}$ is $c_{\eta,\epsilon}$- equivalent to $\mathcal{P}_1$. From $B_i \subseteq B'_i \subseteq \operatorname{int}(Q(y'_i, 2d_i)) \subseteq \operatorname{int}(Q(y_i, d(A_i)/\eta))$ it follows that $B_i \cap \Phi(\mathrm{cl}^* A) = \emptyset$ for each i.

Lemma 4.2. *Assume that $A \subseteq \mathbf{R}^m$ is a BV-set and $\Phi : cl(A) \to \mathbf{R}^m$ is a c_Φ-lipeomorphism. Given $\epsilon > 0$ there exist $\eta_0 > 2$ and a constant $c_{\eta_0} > 0$ (both depending on ϵ, c_Φ and m) which satisfy the following property. If $\mathcal{P}_1 = \{(x_1, A_1), ..., (x_p, A_p)\}$ is an ϵ-regular $\mathcal{F}$- partiton anchored in $cl(A)$ and $\Phi(A)$ is $d(A_i)/\eta_0$-filling in $Q(\Phi(x_i), \eta_0 d(A_i))$ for each i, then we can choose an $\mathcal{F}$-pre-partition $\mathcal{P}_2 = \{(\Phi(x_1), B_1), ..., (\Phi(x_p), B_p)\}$ which is c_{η_0}- equivalent to $\mathcal{P}_1$ and $\Phi(A_j \cap cl^*(A)) \cap B_i = \emptyset$ for $i \ne j$ and $i, j \in \{1, ..., p\}$.*

Proof. The A_i's are figures; hence they are closed and their essential closures, boundaries, and interiors coincide with their regular closures, boundaries, and interiors, respectively.

Since each A_i is ϵ-regular, with a constant $1/3 > \nu > 0$ use Lemma 3.2 to choose a dyadic net cube $Q_i' = Q(x_i', d_i') \subseteq B(x_i, 4d(A_i)\sqrt{m})$, satisfying

$$c_{\epsilon,\nu} d(A_i) < d_i' \leq 2c_{\epsilon,\nu} d(A_i) \leq d(A_i)/2\sqrt{m}, \tag{4.4}$$

$$\mathcal{H}(\text{int}\,(Q_i') \cap \partial A_i) \leq \nu ||Q_i'||,$$

and

$$|Q_i' \setminus A_i| < \nu |Q_i'|.$$

Assume that $c' > 0$ is a small constant (it can be chosen to be $1/30c_\Phi^2\sqrt{m}$).

Set $c_G'' = (c_G c')^m$ where c_G is the constant from Lemma 3.4.

We can assume that ν is so small that $c_L 6\nu < c_G''/2$ where c_L is the constant from Theorem 3.1. Since c_L, c_G'' depend on c_Φ and m our choice of ν will also depend only on c_Φ and m.

Assume that $\eta_0 > 2 + \sqrt{m}$ is fixed and sufficiently big, that is, it satisfies the inequalities given later in this proof. Since all these inequalities will give lower bounds for η_0 they can all simultaneously be satisfied. These bounds will make η_0 depend on ϵ, c_Φ and m.

First we assume that $2\sqrt{m}/\eta_0^2 < 1/16c_\Phi^3(c_\Phi+1)$ and observe that, by Lemma 3.6, from $\Phi(A)$ is $d(A_i)/\eta_0$-filling in $Q(\Phi(x_i), \eta_0 d(A_i))$ it follows that A is $c_\Phi 2\sqrt{m} d(A_i)/\eta_0$-filling in $Q(x_i, \eta_0 d(A_i)/2c_\Phi\sqrt{m})$.

We can assume that η_0 was chosen so large that $c_\Phi 2\sqrt{m} d(A_i)/\eta_0 < d_i'/3$ and $B(x_i, 4d(A_i)\sqrt{m}) \subseteq Q(x_i, \eta_0 d(A_i)/2c_\Phi\sqrt{m})$. Then $Q_i' \subseteq Q(x_i, \eta_0 d(A_i)/2c_\Phi\sqrt{m})$ and there exists $x_i'' \in Q(x_i', d_i'/3) \cap A$. Put $Q_i'' = Q(x_i'', c' d_i')$.

Next we can also assume that η_0 is so large that using η_Φ from Lemma 3.4 we have

$$c_\Phi 2\sqrt{m} d(A_i)/\eta_0 < \eta_\Phi c' d_i'.$$

Then Lemma 3.4 implies that $\tilde{\Phi}(Q_i'')$ contains a cube of side length $2c_G c' d_i'$. Denote this cube by B_i''. Set $c_G' = (c_G c')^{m-1}$ and recall that $c_G'' = (c_G c')^m$. Then $||B_i''|| = c_G'||Q_i'|| \leq c_G'||A_i||$ and $|B_i''| = c_G''|Q_i'|$. Unfortunately, the cubes B_i'' cannot be used to define the desired pre-partition. The problem is that $Q_i'' \subseteq Q_i'$ might contain some smaller cubes of the form $Q_j'' \subseteq Q_j'$ and this implies that the sets B_i'' and B_j'' might overlap. Recall that the cubes Q_i' are dyadic netcubes and hence if Q_i' and Q_j' overlap then either $Q_i' \subseteq Q_j'$, or $Q_j' \subseteq Q_i'$. We have $|Q_i' \setminus A_i| < \nu|Q_i'| < |Q_i'|/3$, and $|Q_j' \setminus A_j| < \nu|Q_j'| < |Q_j'|/3$, thus $i \neq j$, and $Q_i' = Q_j'$ would imply that the non-overlapping figures A_i and A_j would both contain more than two-thirds of $Q_i' = Q_j'$, which is impossible; hence if Q_i' and Q_j' overlap then either $Q_i' \subset Q_j'$, or $Q_j' \subset Q_i'$. Given i denote by $\mathcal{J}_i$ the set of those indices j for which $j \neq i$ and $Q_j' \subset Q_i'$. Since we work with netcubes we can choose a suitable subset $\mathcal{J}_i^* \subseteq \mathcal{J}_i$ such that for $j, j' \in \mathcal{J}_i^*$, $j \neq j'$ the cubes Q_j' and $Q_{j'}'$ are non-overlapping and $\cup_{j\in\mathcal{J}_i^*} Q_j' = \cup_{j\in\mathcal{J}_i} Q_j'$.

Let the figure $A_i' = \text{cl}\,((Q_i' \cap A_i) \setminus \cup_{j\in\mathcal{J}_i} Q_j') = \text{cl}\,((Q_i' \cap A_i) \setminus \cup_{j\in\mathcal{J}_i^*} Q_j')$.

Since the figures A_i were non-overlapping we have $A_i' \cap \text{int}\,(A_j) = \emptyset$ for $j \in \mathcal{J}_i$. Furthermore, $|Q_j' \cap A_i| \leq |Q_j' \setminus A_j| < \nu|Q_j'| < |Q_j'|/3$ for $j \in \mathcal{J}_i$. From Lemma 3.3 we infer

that $\mathcal{H}(A_i \cap \partial Q'_j) \le c_F \mathcal{H}(\partial A_i \cap \operatorname{int}(Q'_j))$. By using (4.4) and the fact that the cubes Q'_j, $j \in \mathcal{J}_i^*$ are non-overlapping we obtain

$$||A'_i|| \le ||A_i|| + ||Q'_i|| + \sum_{j \in \mathcal{J}_i^*} \mathcal{H}(A_i \cap \partial Q'_j) \le ||A_i|| + ||Q'_i|| + c_F||A_i|| \le c'_F||A_i||$$

with a constant c'_F depending only on c_F which depends only on m.

From $|Q'_j| = |Q'_j \cap A_j| + |Q'_j \setminus A_j| < |Q'_j \cap A_j| + |Q'_j|/3$ it follows that $|Q'_j| < 3|Q'_j \cap A_j|/2$. On the other hand, the figures A_j are non-overlapping and $Q'_j \subset Q'_i$ for $j \in \mathcal{J}_i$; hence

$$\sum_{j \in \mathcal{J}_i} |Q'_j \cap A_j| = |Q'_i \cap \cup_{j \in \mathcal{J}_i}(Q'_j \cap A_j)| \le |Q'_i \setminus A_i| < \nu |Q'_i|.$$

Thus $|\cup_{j \in \mathcal{J}_i} Q'_j| \le \sum_{j \in \mathcal{J}_i} |Q'_j| < (3/2) \sum_{j \in \mathcal{J}_i} |Q'_j \cap A_j| < (3\nu/2)|Q'_i|$. Since $|Q'_i \setminus A_i| < \nu |Q'_i|$ we obtain $|\operatorname{cl}(Q'_i \setminus A'_i)| = |Q'_i \setminus A'_i| < |Q'_i \setminus A_i| + \sum_{j \in \mathcal{J}_i} |Q'_j| < 3\nu |Q'_i|$.

Let $E_i = (Q'_i \cap \cup_{j \ne i} A_j) \cup \cup_{j \in \mathcal{J}_i} Q'_j$ and $H'_i = \operatorname{cl}(Q'_i \setminus A'_i) \supset E_i$. Since Q'_i and A'_i are figures we can enlarge the figure H'_i a bit to obtain H_i such that $H'_i \subseteq \operatorname{int}(H_i)$, $||H_i|| < 2||H'_i||$ and $|H_i| < 2|H'_i|$. Applying Theorem 3.1 find a figure F_i such that $\tilde{\Phi}(E_i) \subseteq \operatorname{int}(F_i)$ and $||F_i|| \le c_L||H_i|| < 2c_L||H'_i|| \le 2c_L(||Q'_i|| + ||A'_i||) \le (2c'_F c_L + 1)||A_i||$, and $|F_i| \le c_L|H_i| \le 2c_L|H'_i| \le c_L 6\nu |Q'_i|$.

Put $B_i = \operatorname{cl}(B''_i \setminus F_i)$. Then

$$||B_i|| \le ||B''_i|| + ||F_i|| \le (c'_G + 2c'_F c_L + 1)||A_i||$$

and our assumption that $c_L 6\nu < c''_G/2$ and (4.4) imply

$$|B_i| \ge |B''_i| \setminus |F_i| \ge c''_G |Q_i| - c_L 6\nu |Q_i| \ge c''_G |Q'_i|/2 \ge \tilde{c}|A_i|,$$

where $\tilde{c}$ depends only on c_Φ and m. Recalling that η_0 depends on ϵ, c_Φ and m we can choose a constant c_D depending also on ϵ, c_Φ, and m such that

$$B_i \subseteq B''_i \subseteq \tilde{\Phi}(Q''_i) \subseteq \tilde{\Phi}(Q'_i) \subseteq \tilde{\Phi}(Q(x_i, \eta_0 d(A_i)/2c_\Phi\sqrt{m})) \subseteq B(\Phi(x_i), c_D d(A_i)).$$

The above three displayed formulae yield that there exists a constant c_{η_0} depending on ϵ, c_Φ and m such that $\mathcal{P}_1$ and $\mathcal{P}_2$ are c_{η_0}-equivalent.

Next we verify that the sets $\{B_i\}$ are non-overlapping.

Thus, proceeding towards a contradiction, assume that B_i and B_j overlap. Then the corresponding dyadic net cubes are $Q'_i = Q(x'_i, d'_i)$, and $Q'_j = Q(x'_j, d'_j)$. Without limiting generality we can assume that $d'_i \ge d'_j$.

If Q'_i and Q'_j overlap then $Q'_j \subset Q'_i$ and $j \in \mathcal{J}_i$. Then $B_j \subseteq \tilde{\Phi}(Q''_j) \subseteq \tilde{\Phi}(Q'_j) \subseteq \tilde{\Phi}(E_i) \subseteq \operatorname{int}(F_i)$ and hence $B_j \cap B_i = \emptyset$ which is impossible.

Therefore, we can assume that Q'_i and Q'_j do not overlap.

Suppose that $z \in \operatorname{cl}(A) \setminus Q'_i$. From $x''_i \in Q(x'_i, d'_i/3) \subset Q(x'_i, d'_i) = Q'_i$ it follows that $\operatorname{dist}(z, x''_i) \ge d'_i/3$. Since Φ is a c_Φ-lipeomorphism this implies that $\operatorname{dist}(\Phi(z), \Phi(x'_i)) \ge d'_i/3c_\Phi$. On the other hand $\tilde{\Phi}(Q(z, c'd'_i)) \subseteq B(\Phi(z), \sqrt{m} c_\Phi c' d'_i)$. Since c' is sufficiently

small we infer that $\tilde{\Phi}(Q(z,c'd_i')) \cap \tilde{\Phi}(Q_i'') = \tilde{\Phi}(Q(z,c'd_i')) \cap \tilde{\Phi}(Q(x_i'',c'd_i')) = \emptyset$. Since Q_i' and Q_j' do not overlap choosing $z = x_j''$ we have $x_j'' \in \mathrm{cl}\,(A) \setminus Q_i'$ and by the above argument $B_i \cap B_j \subseteq B_i'' \cap B_j'' \subseteq \tilde{\Phi}(Q_i'') \cap \tilde{\Phi}(Q_j'') \subseteq \tilde{\Phi}(Q_i'') \cap \tilde{\Phi}(Q(x_j'',c'd_j')) \subseteq \tilde{\Phi}(Q_i'') \cap \tilde{\Phi}(Q(x_j'',c'd_i')) = \emptyset$.

We verified above that if $z \in \mathrm{cl}\,(A) \setminus Q_i'$ then $\Phi(z) \notin B_i$. Therefore $\Phi((A_j \cap \mathrm{cl}^* A) \setminus Q_i') \cap B_i = \emptyset$. On the other hand, $\tilde{\Phi}(Q_i' \cap A_j) \subseteq \tilde{\Phi}(E_i) \subseteq \mathrm{int}\,(F_i)$ for $j \neq i$ and hence $B_i \cap \tilde{\Phi}(Q_i' \cap A_j) = \emptyset$. Thus for $j \neq i$

$$B_i \cap \Phi(A_j \cap \mathrm{cl}^*(A)) \subseteq B_i \cap \left(\Phi((A_j \cap \mathrm{cl}^* A) \setminus Q_i') \cup \tilde{\Phi}(Q_i' \cap A_j) \right) = \emptyset.$$

This completes the proof of the lemma.

5. The main Theorem

To avoid repeating the same definition twice, first for figures and then for $\mathcal{BV}$-sets in the sequel $\mathcal{A}$ stands for either $\mathcal{F}$, or for $\mathcal{BV}$.

Definition 5.1. A charge is a real valued function F which satisfies the next two conditions,

i) it is *additive*, that is, $F(A \cup B) = F(A) + F(B)$ when $A, B \in \mathcal{A}$ are nonoverlapping sets.

ii) it is *continuous*, that is for any $\epsilon > 0$ there is an $\eta > 0$ such that $|F(A)| < \epsilon$ for each $A \in \mathcal{A}$ with $A \subseteq B(0, 1/\epsilon)$, $||A|| < 1/\epsilon$ and $|A| < \eta$.

Definition 5.2. A non-negative real-valued function δ defined on the set $A \subseteq \mathbf{R}^m$ is called a *gage function* on A if its null set $\{x \in A : \delta(x) = 0\}$ is thin, that is, of σ-finite $\mathcal{H}$-measure.

Definition 5.3. If $A \in \mathcal{A}$ we say that the real-valued function f defined on A is $\mathcal{A}$-*integrable* if there is a charge F such that for any $\epsilon > 0$ there is a gage δ on A satisfying

$$\sum_{i=1}^{p} \left| f(x_i)|A_i| - F(A_i) \right| < \epsilon$$

for each ϵ-regular δ-fine $\mathcal{A}$-partition $\{(x_1, A_1), ..., (x_p, A_p)\}$ in A.

Remarks. By [P2] if A is a figure then the class of $\mathcal{BV}$ and $\mathcal{F}$ integrable functions is the same. Thus, working with figures sometimes we just simply say that a function is integrable, meaning that it is both $\mathcal{F}$ and $\mathcal{BV}$ integrable.

The integrability and the value of the integral of a function is not changed if we modify its definition on a set of measure zero.

Observe that by a slight modification $\mathcal{BV}$-pre-partitions can be converted into $\mathcal{BV}$-partitions. Since the charge in the definition of the integral is continuous one can easily

see that if f is integrable on the figure K and $\epsilon > 0$ is given then we can choose a gage function δ such that

$$\sum_{i=1}^{p} |f(x_i)|A_i| - F(A_i)| < \epsilon$$

holds for any ϵ-regular δ-fine $\mathcal{BV}$-pre-partition $\{(x_1, A_1), ..., (x_p, A_p)\}$. Indeed, for $\epsilon/2 > 0$ choose a gage function δ such that

$$\sum_{i=1}^{p} |f(x_i)|A_i'| - F(A_i')| < \epsilon/2$$

holds for any $\epsilon/2$-regular δ-fine $\mathcal{BV}$-partition $\{(x_1, A_1'), ..., (x_p, A_p')\}$ anchored in K.

Using the continuity of F it is not difficult to find for an ϵ-regular δ-fine $\mathcal{BV}$-pre-partition $\{(x_1, A_1), ..., (x_p, A_p)\}$ an $\epsilon/2$-regular δ- fine $\mathcal{BV}$-partition $\{(x_1, A_1'), ..., (x_p, A_p')\}$ such that

$$\sum_{i=1}^{p} \Big| \big|f(x_i)|A_i| - F(A_i)\big| - \big|f(x_i)|A_i'| - F(A_i')\big| \Big| < \epsilon/2.$$

This implies $\sum_{i=1}^{p} |f(x_i)|A_i| - F(A_i)| < \epsilon$.

Definition 5.4. If $A \subseteq \mathbf{R}^m$ is a BV-set and $f : A \to \mathbf{R}^m$ is a given function then we say that f is $\mathcal{BVF}$ integrable on A if there exists an integrable function g on a figure A_0 such that $A \subseteq A_0$, $f = g$ almost everywhere on A and $g = 0$ outside A. Then $\mathcal{BVF} \int_A f = \mathcal{F} \int_{A_0} g$.

Theorem 5.1. *Assume that $K \subseteq \mathbf{R}^m$ is a given figure, $f : K \to \mathbf{R}$ is integrable, $A \subseteq \mathbf{R}^m$ is a BV set and $\Phi : A \to K$ is a lipeomorphism. Then $(f \circ \Phi)J_\Phi$ is $\mathcal{BVF}$ integrable on A and $\mathcal{BVF} \int_A (f \circ \Phi)J_\Phi = \mathcal{BVF} \int_{\Phi(A)} f$.*

Remark. The above theorem implies that *if A is a BV-set, $\Phi : A \to \mathbf{R}^m$ is a lipeomorphism and f is $\mathcal{BVF}$ integrable on $\Phi(A)$ then $(f \circ \Phi)J_\Phi$ is $\mathcal{BVF}$ integrable on A.*

Proof.

Step 1. Some definitions and assumptions.

Since changing our functions on sets of measure 0 does not change integrability we can assume that $A \in \mathcal{BV}$, that is, $\mathrm{cl}^*(A) = A$. If $x \in A$ and $J_\Phi(x)$ is defined put $g(x) = f(\Phi(x))J_\Phi(x)$, otherwise set $g(x) = 0$.

Assume that A_0 is a figure containing $\mathrm{cl}\,(A)$ in its interior. We need to verify that g is $\mathcal{F}$-integrable on A_0. This clearly implies that g is integrable on any subfigure of $\mathbf{R}^m$.

Assume that $0 < \epsilon < 1$ is given and fixed.

We can assume that Φ is extended onto the closure of A, that is, Φ is a $\mathrm{cl}\,(A) \to K$ lipeomorphism with constant c_Φ.

Since Φ is a lipeomorphism density points are mapped onto density points, that is, $\Phi(\mathrm{cl}^*(A)) = \mathrm{cl}^*(\Phi(A))$ and we also have $\Phi(\partial^* A) = \partial^*(\Phi(A))$.

Using that $\mathcal{F}\int_K f = \mathcal{BV}\int_K f$ exists choose an additive continuous function F satisfying the definition of the integral. By the remark after the definition of pre-partitions for $\epsilon_1 > 0$ choose a gage function $\delta \geq 0$ such that

$$\sum_{i=1}^{p} |f(x_i)|B_i| - F(B_i)| < \epsilon_1 \tag{5.1}$$

for each ϵ_1-regular δ-fine $\mathcal{BV}$-pre-partition $\{(x_1, B_1), ..., (x_p, B_p)\}$ in K. Here ϵ_1 is a small constant which will depend on ϵ, c_Φ and m.

Since $\partial K = \partial^* K$ is thin we can also assume that $\delta(x) = 0$ for $x \in \partial K$ and for $x \in \text{int}\,(K)$ we have $B(x, \delta(x)) \subseteq \text{int}\,(K)$.

Step 2. The definition of the gage function δ_1 on A_0. This function will be used to show the integrability of g.

If $x \notin \text{cl}\,(A)$ choose $\delta_1(x) > 0$ such that $B(x, \delta_1(x)) \cap \text{cl}\,(A) = \emptyset$.

It is a corollary of Theorem 6.2 of [M, p.89] that if $A' \subseteq \mathbf{R}^m$ with $\mathcal{H}(A') = \mathcal{H}^{m-1}(A') < \infty$ then the upper $(m-1)$-density $\Theta^{*m-1}(A', x) = 0$ for $\mathcal{H}$-almost every $x \notin A'$. Using this theorem with $A' = \partial^* A$ we obtain

$$\limsup_{r \to 0+} \frac{\mathcal{H}(\partial^* A \cap B(x,r))}{r^{m-1}} = 0 \text{ for } \mathcal{H}\text{-almost every } x \notin \partial^* A.$$

Denote by A^1 the set of those $x \in \text{cl}\,(A) \setminus \partial^* A$ for which the above displayed equality does not hold. Then $\mathcal{H}(A^1) = 0$, and put $\delta_1(x) = 0$ for $x \in A^1$.

For $x \in \partial^* A$ also put $\delta_1(x) = 0$, again recall that $\partial^* A$ is thin.

For $x \in \text{cl}\,(A) \setminus (\partial^* A \cup A^1)$ choose $0 < \delta_2(x)$ such that

$$\mathcal{H}(\partial^* A \cap B(x,r)) < \epsilon^{m-1} r^{m-1} \text{ holds for } 0 < r < \delta_2(x). \tag{5.2}$$

For $x \in \text{cl}\,(A) \setminus (\text{cl}^* A \cup A^1)$ choose $0 < \delta_3(x) < \delta_2(x)$ such that

$$|\text{cl}^* A \cap B(x,r)| < \frac{\epsilon_1 r^m}{|f(\Phi(x))| + 1} \text{ holds for } 0 < r < \delta_3(x). \tag{5.3}$$

For $x \in \text{int}^*(A) \setminus A^1$ choose $0 < \delta_3(x) < \delta_2(x)$ such that

$$|B(x,r) \setminus \text{cl}^*(A)| < \frac{\epsilon_1 r^m}{(|f(\Phi(x))| + 1)(J^*_\Phi(x) + 1)} \text{ holds for } 0 < r < \delta_3(x) \tag{5.4}$$

where $J^*_\Phi(x) = J_\Phi(x)$ if the latter is defined, otherwise $J^*_\Phi(x) = 0$.

Assume that $0 < c_5 < 1$, this constant will be determined later in a way that it will depend only on c_Φ and m. Using Lemma 3.7 choose a set of measure zero with the property that for $\phi(x) = |f(\Phi(x))| + 1$, $\epsilon_0 = c_5 \epsilon$, and $x \in \text{int}^*(A) \setminus A^2$ we can choose a $\delta_4(x) \leq \delta_3(x)$ such that

$$(|f(\Phi(x))|+1)\big|J_\Phi(x)|B_0| - |\Phi(B_0)|\,\big| < c_5\epsilon|B_0| \tag{5.5}$$

for any $c_5\epsilon$-regular $\mathcal{BV}$ set $B_0 \subseteq B(x,\delta_4(x)) \cap A$.

If $x \in \mathrm{cl}\,(A) \setminus (A^1 \cup \partial^* A)$ and $x \notin \mathrm{int}^*(A) \setminus A^2$ then set $\delta_4(x) = \delta_3(x)$.

Finally, for $x \in \mathrm{cl}\,(A) \setminus (A^1 \cup \partial^* A)$ choose $\delta_1(x) > 0$ such that $\delta_1(x) \leq \min\{\delta_4(x), c_2\delta(\Phi(x))\}$, where $c_2 \in (0, 1/c_\Phi)$ is a small constant, to be determined later.

We remind that δ was chosen to obtain (5.1). Since Φ is a lipeomorphism $\Phi^{-1}(\{x : \delta(x) = 0\} \cap \mathrm{cl}\,(A))$ is a thin set; hence $\{x : \delta_1(x) = 0\}$ is also thin.

Step 3. The defintion of F_0.

For $B_0 \in BV$ put $F_0(B_0) = F(\Phi(B_0 \cap \mathrm{cl}^* A))$ where we define $F(\emptyset) = 0$.

From $\mathcal{H}(\partial^* A) < \infty$ it follows that $||B_0 \cap \mathrm{cl}^* A|| \leq ||B_0|| + ||\mathrm{cl}^* A||$ and hence $B_0 \cap \mathrm{cl}^* A$ is a BV-set. By Lemma 3.7, $\Phi(B_0 \cap \mathrm{cl}^*(A))$ is also a BV-set. Since Φ is a lipeomorphism using the additivity and continuity of F it is not difficult to see that F_0 is also a charge.

Step 4. First step towards the estimation of the Riemannian sums of g, the index sets $\mathcal{I}_j$.

Recall that A_0 is a large figure such that $\mathrm{cl}\,(A) \subseteq \mathrm{int}\,(A_0)$ and we need to verify $\mathcal{F}$ integrability of g on A_0. Assume that $\mathcal{P} = \{(x_1, A_1), ..., (x_p, A_p)\}$ is an ϵ-regular δ_1-fine $\mathcal{F}$-partition in A_0.

In the next estimations we need the following index sets: $\mathcal{I}_1 = \{i : x_i \notin \mathrm{cl}\,(A)\}$, $\mathcal{I}_2 = \{i : x_i \in A^1 \cup \partial^* A\}, \mathcal{I}_3 = \{i : x_i \in \mathrm{cl}\,(A) \setminus (A^1 \cup \mathrm{cl}^* A)\}$, and $\mathcal{I}_4 = \{i : x_i \in \mathrm{int}^*(A) \setminus A^1\}$. The index sets $\mathcal{I}_j$ are disjoint and their union is $\{1, ..., p\}$. We need to estimate

$$\sum_{i=1}^{p} |g(x_i)|A_i| - F_0(A_i)|.$$

Split the above sum into four subsums $\sum_{i \in \mathcal{I}_j}$ for $j = 1, ..., 4$, (where the empty sum will equal zero by definition).

Step 5. The easy cases: $\mathcal{I}_1$, $\mathcal{I}_2$.

The sum $\sum_{i \in \mathcal{I}_1}$ equals 0 since $\mathrm{cl}\,(A) \supset \mathrm{cl}^*(A)$, and hence for $x_i \notin \mathrm{cl}\,(A)$ we have $g(x_i) = 0$, and by the choice of δ_1 we also have $A_i \cap \mathrm{cl}^*(A) = \emptyset$ and hence $F_0(A_i) = 0$ as well.

Since $\delta_1(x_i) = 0$ for $x_i \in A^1 \cup \partial^* A$ the index set $\mathcal{I}_2$ is empty and hence $\sum_{i \in \mathcal{I}_2} = 0$.

Step 6. The case of the index set $\mathcal{I}_3$.

Then $g(x_i) = 0$; hence we need to estimate

$$\sum_{i \in \mathcal{I}_3} |F_0(A_i)| = \sum_{i \in \mathcal{I}_3} |F(\Phi(A_i \cap \mathrm{cl}^* A))|.$$

Here the difficulty is that we cannot find a uniform estimate of the regularity of the sets $A_i \cap \mathrm{cl}^* A$, and of their Φ images. In the next few steps we will use Lemmas 4.1 and 4.2 to solve this problem.

Substep 6.1. The definition of partitions of type $\mathcal{P}$.

Using ϵ from this proof choose η_0 according to Lemma 4.2. Denote by $\mathcal{I}_{31}$ the set of those $i \in \mathcal{I}_3$ for which $\Phi(A)$ is $d(A_i)/\eta_0$-filling in $Q(\Phi(x_i), \eta_0 d(A_i))$ and let $\mathcal{I}_{32} = \mathcal{I}_3 \setminus \mathcal{I}_{31}$. Let $\mathcal{P}_{11} = \{(x_i, A_i)\}_{i\in\mathcal{I}_{31}}$, $\mathcal{P}_{12} = \{(x_i, A_i)\}_{i\in\mathcal{I}_{32}}$. Using Lemma 4.2 choose an $\mathcal{F}$-pre-partition $\mathcal{P}_{21} = \{(\Phi(x_i), B_i)\}_{i\in\mathcal{I}_{31}}$ which is c_{η_0}-equivalent to $\mathcal{P}_{11}$ and $\Phi(A_j \cap \mathrm{cl}^*(A)) \cap B_i = \emptyset$ for $i, j \in \mathcal{I}_{31}$, $i \neq j$. Similarly, using Lemma 4.1 with $\eta = \eta_0$ choose an $\mathcal{F}$-pre-partition $\mathcal{P}_{22} = \{(\Phi(x_i), B_i)\}_{i\in\mathcal{I}_{32}}$ which is $c_{\eta_0,\epsilon}$- equivalent to $\mathcal{P}_{12}$ and satisfies $B_i \cap \Phi(\mathrm{cl}^*(A)) = \emptyset$ for $i \in \mathcal{P}_{22}$. Observe that this implies $\Phi(A_j \cap \mathrm{cl}^*(A)) \cap B_i = \emptyset$ for $i, j \in \mathcal{I}_{32}$, $i \neq j$. In the next section we work out in detail the estimation for the partitions $\mathcal{P}_{11}$ and $\mathcal{P}_{21}$, the argument for $\mathcal{P}_{12}$ and $\mathcal{P}_{22}$ is similar (with c_{η_0} replaced by $c_{\eta_0,\epsilon}$).

Substep 6.2. The estimation for the index set $\mathcal{P}_{21}$. Regularity estimates.

Put $B_i^+ = B_i \cup \Phi(A_i \cap \mathrm{cl}^*(A))$, $B_i^- = B_i \setminus \Phi(A_i \cap \mathrm{cl}^*(A))$, $\mathcal{P}_{21}^+ = \{(\Phi(x_i), B_i^+)\}_{i\in\mathcal{I}_{31}}$, and $\mathcal{P}_{21}^- = \{(\Phi(x_i), B_i^-)\}_{i\in\mathcal{I}_{31}}$. Next we verify that these partitions are both ϵ_1- regular. Since (x_i, A_i) is ϵ-regular we have $|A_i| > \epsilon d(A_i \cup \{x_i\})||A_i||$ and by (3.1) we also have $|A_i| \geq \epsilon^m d(A_i \cup \{x_i\})^m / c_{iso}^m$. Observe that $x_i \in \mathcal{I}_3$ implies that (5.3) is applicable and hence

$$|\Phi(A_i \cap \mathrm{cl}^* A)| < c_\Phi^m |A_i \cap \mathrm{cl}^* A| < \epsilon_1 c_\Phi^m d(A_i \cup \{x_i\})^m \leq \epsilon_1 (\frac{c_\Phi c_{iso}}{\epsilon})^m |A_i|.$$

If we choose $\epsilon_1 < \epsilon^m / 2 c_{\eta_0} c_\Phi^m c_{iso}^m$ then we can continue the above estimation to obtain

$$|\Phi(A_i \cap \mathrm{cl}^* A)| < \frac{1}{2c_{\eta_0}} |A_i| < \frac{|B_i|}{2}$$

which implies

$$|B_i^+| \geq |B_i^-| \geq |B_i| - |\Phi(A_i \cap \mathrm{cl}^* A)| \geq \frac{|B_i|}{2} \geq \frac{|A_i|}{2c_{\eta_0}}.$$

We also have

$$d(B_i^- \cup \{\Phi(x_i)\}) \leq d(B_i^+ \cup \{\Phi(x_i)\}) \leq d(B_i \cup \{\Phi(x_i)\}) + d(\Phi(A_i \cap \mathrm{cl}^* A) \cup \{\Phi(x_i)\}) \leq$$

$$c_{\eta_0} d(A_i \cup \{x_i\}) + c_\Phi d(A_i \cup \{x_i\}) \leq (c_{\eta_0} + c_\Phi) d(A_i \cup \{x_i\}).$$

From (5.2) used with $r = d(A_i \cup \{x_i\}) < \delta_1(x) < \delta_2(x)$ it follows that

$$\epsilon^{m-1} d(A_i \cup \{x_i\})^{m-1} > \mathcal{H}(\partial^* A \cap B(x_i, d(A_i \cup \{x_i\}))) \geq \mathcal{H}(\partial^* A \cap A_i);$$

hence

$$||A_i \cap \mathrm{cl}^* A|| \leq ||A_i|| + \mathcal{H}(\partial^* A \cap A_i) <$$

$$||A_i|| + \epsilon^{m-1} d(A_i \cup \{x_i\})^{m-1} < \frac{|A_i|}{\epsilon d(A_i \cup \{x_i\})} + \frac{c_{iso}^m |A_i|}{\epsilon d(A_i \cup \{x_i\})},$$

where at the last estimate we used the ϵ-regularity of (x_i, A_i) and (3.1). This implies

$$||\Phi(A_i \cap \mathrm{cl}^* A)|| < c_\Phi^{m-1} \left(\frac{1 + c_{iso}^m}{\epsilon} \right) \frac{|A_i|}{d(A_i \cup \{A_i\})}.$$

Now we have

$$||B_i^\pm|| \leq ||B_i|| + ||\Phi(A_i \cap \mathrm{cl}^* A)|| \leq c_{\eta_0} ||A_i|| + ||\Phi(A_i \cap \mathrm{cl}^* A)|| \leq$$

$$\left(\frac{c_{\eta_0}}{\epsilon} + c_\Phi^{m-1} \frac{1 + c_{iso}^m}{\epsilon} \right) \frac{|A_i|}{d(A_i \cup \{x_i\})}.$$

Therefore combining the above results we have

$$|B_i^\pm| \geq \frac{|A_i|}{2c_{\eta_0}} \geq \frac{1}{2c_{\eta_0}} \left(\frac{c_{\eta_0}}{\epsilon} + c_\Phi^{m-1} \frac{1 + c_{iso}^m}{\epsilon} \right)^{-1} ||B_i^\pm|| d(A_i \cup \{x_i\}) \geq$$

$$\frac{1}{2c_{\eta_0}} (c_{\eta_0} + c_\Phi^{m-1} (1 + c_{iso})^m)^{-1} \frac{1}{c_{\eta_0} + c_\Phi} \epsilon ||B_i^\pm|| d(B_i^\pm \cup \{\Phi(x_i)\}) \geq c_2 \epsilon ||B_i^\pm|| d(B_i^\pm \cup \{\Phi(x_i)\})$$

if c_2 is a sufficiently small constant depending on ϵ, c_Φ and m. This shows that $\mathcal{P}_{21}^+$ and $\mathcal{P}_{21}^-$ are both $c_2\epsilon$-regular.

Substep 6.3. The estimation for the index set $\mathcal{P}_{21}$. Riemannian sum estimates.

We can assume that $1/c_2 \geq c_{\eta_0} + c_\Phi$ and hence $d(B_i^- \cup \{\Phi(x_i)\}) \leq d(B_i^+ \cup \{\Phi(x_i)\}) \leq d(A_i \cup \{x_i\})/c_2$. Recall that $A_i \subseteq B(x_i, \delta_1(x_i))$, and $\delta_1(x_i) \leq c_2 \delta(\Phi(x_i))$; hence $d(A_i \cup \{x_i\})/c_2 < \delta(\Phi(x_i))$. Therefore $B_i^- \cup \{\Phi(x_i)\} \subseteq B_i^+ \cup \{\Phi(x_i)\} \subseteq B(\Phi(x_i), \delta(\Phi(x_i))) \subseteq \mathrm{int}\,(K)$.

Thus $\mathcal{P}_{21}^+$ and $\mathcal{P}_{21}^-$ are both ϵ_1-regular δ-fine $\mathcal{BV}$-pre-partitions and hence

$$\sum_{i \in \mathcal{I}_{31}} |f(\Phi(x_i))| B_i^+| - F(B_i^+)| < \epsilon_1 \text{ and } \sum_{i \in \mathcal{I}_{31}} |f(\Phi(x_i))| B_i^-| - F(B_i^-)| < \epsilon_1,$$

therefore, using $B_i^+ \setminus B_i^- = \Phi(A_i \cap \mathrm{cl}^*(A))$ we obtain

$$\sum_{i \in \mathcal{I}_{31}} |f(\Phi(x_i))| \Phi(A_i \cap \mathrm{cl}^*(A))| - F(\Phi(A_i \cap \mathrm{cl}^*(A)))| < 2\epsilon_1.$$

Thus (5.3) used with $r = d(A_i \cup \{x_i\})$, the Lipschitz properties of Φ, the ϵ-regularity of (x_i, A_i) and (3.1) imply that there are constants c and c_3 depending on ϵ, c_Φ and m such that

$$\sum_{i \in \mathcal{I}_{31}} |F(\Phi(A_i \cap \mathrm{cl}^*(A)))| < 2\epsilon_1 + \sum_{i \in \mathcal{I}_{31}} |f(\Phi(x_i))| \cdot c \cdot \epsilon_1 \frac{d(A_i \cup \{x_i\})^m}{|f(\Phi(x_i))| + 1} \leq$$

$$2\epsilon_1 + c\epsilon_1 \sum_{i\in\mathcal{I}_{31}} d(A_i\cup\{x_i\})^m \le 2\epsilon_1 + c\epsilon_1 \sum_{i\in\mathcal{I}_{31}} \left(\frac{c_{iso}}{\epsilon}\right)^m |A_i\cup\{x_i\}| \le$$

$$2\epsilon_1 + c\epsilon_1 \left(\frac{c_{iso}}{\epsilon}\right)^m |A_0| < c_3\epsilon_1.$$

A similar estimation holds for the index set $\mathcal{I}_{32}$. Adding the estimations for $\mathcal{I}_{31}$ and $\mathcal{I}_{32}$ we can find a constant c_4 depending on ϵ, c_Φ and m such that

$$\sum_{\mathcal{I}_3} |F_0(A_i)| = \sum_{\mathcal{I}_3} |F(\Phi(A_i)\cap \mathrm{cl}^*(A))| < c_4\epsilon_1 < \epsilon,$$

where for the last estimate we can assume that ϵ_1, depending on ϵ, c_Φ and m was chosen sufficiently small.

Step 7. The case of the index set $\mathcal{I}_4$.

Put $B_i = \Phi(A_i\cap \mathrm{cl}^* A)$. By the choice of δ_1 we have $B_i \subseteq B(\Phi(x_i), c_\Phi\delta_1(x_i)) \subseteq B(\Phi(x_i), \delta(\Phi(x_i))) \subseteq \mathrm{int}\,(K)$.

Inequalities (5.4) and $\delta_3 > \delta_1 > r = d(A_i\cup\{x_i\})$ imply $|B(x_i, d(A_i\cup\{x_i\}))\setminus \mathrm{cl}^*(A)| < \epsilon_1 d(A_i\cup\{x_i\})^m < \epsilon d(A_i\cup\{x_i\})^m$, (5.2) and $\delta_2 > \delta_1$ imply $\mathcal{H}(\partial^* A\cap B(x_i, d(A_i))) < \epsilon^{m-1} d(A_i\cup\{x_i\})^{m-1}$. Recall that (x_i, A_i) is ϵ-regular. Clearly, there exists a constant $\tilde{c} > 0$ depending only on m such that if $\epsilon < \tilde{c}$ then the previous two estimations imply that $|A_i\cap \mathrm{cl}^* A| > |A_i|/2$ and $||A_i\cap \mathrm{cl}^* A|| < 2||A_i||$. Without limiting generality we can assume that $\epsilon < \tilde{c}$. Since $\{(x_i, A_i)\}_{i\in\mathcal{I}_4}$ is ϵ-regular and Φ is a lipeomorphism there is a constant $c_5 > 0$ depending on c_Φ and m such that $\{(\Phi(x_i), B_i)\}_{i\in\mathcal{I}_4}$ is a $c_5\epsilon$-regular partition. We can also assume that ϵ_1 is so small that $c_5\epsilon > \epsilon_1$.

For $x_i \in \mathrm{int}^*(A)\setminus A^2$ by the choice of $\delta_4(x_i) > \delta_1(x_i)$ we can use (5.5) with $B_0 = A_i$. Since A^2 is of measure zero and one cannot change integrability of a function by changing it on a set of measure zero we can assume that $g(x) = f(\Phi(x)) = 0$ for $x \in A^2$.

Denote by $\mathcal{I}_{41}$ the set of those i's in $\mathcal{I}_4$ for which $x_i \notin A^2$. We also put $\mathcal{I}_{42} = \mathcal{I}_4\setminus\mathcal{I}_{41}$.

Then

$$\sum_{\mathcal{I}_4} |g(x_i)|A_i| - F_0(A_i)| = \sum_{\mathcal{I}_{41}} |g(x_i)|A_i| - F_0(A_i)| + \sum_{\mathcal{I}_{42}} |g(x_i)|A_i| - F_0(A_i)|.$$

Using (5.4) with $r = d(A_i\cup\{x_i\})$, $x = x_i$, (5.5) with $B_{0,i} = A_i\cap \mathrm{cl}^* A$, $\Phi(B_{0,i}) = \Phi(A_i\cap \mathrm{cl}^* A) = B_i$, $x = x_i$ for $i \in \mathcal{I}_{41}$ and (5.1) we obtain that there exists a constant c_6 depending on c_Φ and m such that

$$\sum_{\mathcal{I}_{41}} |g(x_i)|A_i| - F_0(A_i)| = \sum_{\mathcal{I}_{41}} |f(\Phi(x_i))J_\Phi(x_i)|A_i| - F(\Phi(A_i\cap \mathrm{cl}^* A))| \le$$

$$\sum_{\mathcal{I}_{41}} |f(\Phi(x_i))| \cdot \Big(\big|J_\Phi(x_i)|A_i| - J_\Phi(x_i)|A_i\cap \mathrm{cl}^* A|\big| + \big|J_\Phi(x_i)|A_i\cap \mathrm{cl}^* A| - |B_i|\big|\Big) +$$

$$\sum_{\mathcal{I}_{41}} |f(\Phi(x_i))|B_i| - F(B_i)| \le \left(\sum_{\mathcal{I}_{41}} \epsilon_1 d(A_i\cup\{x_i\})^m + c_5\epsilon|A_i\cap \mathrm{cl}^* A|\right) + \epsilon_1 \le$$

$$\left(\sum_{\mathcal{I}_{41}} \epsilon_1 \left(\frac{c_{iso}}{\epsilon}\right)^m |A_i \cup \{x_i\}| + c_5\epsilon|A_i|\right) + \epsilon_1 \le c_6\epsilon \max\{1, |A_0|\},$$

where at the last estimate we also assumed that ϵ_1 was chosen so small that $\epsilon_1(c_{iso}/\epsilon)^m < \epsilon$. On the other hand,

$$\sum_{\mathcal{I}_{42}} |g(x_i)|A_i| - F_0(A_i)| = \sum_{\mathcal{I}_{42}} |F(\Phi(A_i \cap \mathrm{cl}^* A))| =$$

$$\sum_{\mathcal{I}_{42}} |F(B_i)| = \sum_{\mathcal{I}_{42}} |f(\Phi(x_i))|B_i| - F(B_i)| < \epsilon_1 < \epsilon.$$

Therefore there exists a constant c_7 depending on c_Φ and m such that

$$\sum_{j=1}^{4} \sum_{i \in \mathcal{I}_j} |g(x_i)|A_i| - F_0(A_i)| = \sum_{i=1}^{p} |g(x_i)|A_i| - F_0(A_i)| < c_7\epsilon.$$

This completes the proof of the $\mathcal{F}$ integrability of $(f \circ \Phi)J_\Phi$ on A_0. From what we verified it follows that $\mathcal{BVF} \int_A (f \circ \Phi)J_\Phi = F_0(A_0) = F(\Phi(A_0 \cap \mathrm{cl}^* A)) = F(\Phi(A)) = \mathcal{BVF} \int_{\Phi(A)} f$, this is the change of variables formula for the $\mathcal{BVF}$ integral.

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