The Non-coincidence of Ordinary and Peano Derivatives

Zoltán Buczolich*, Department of Analysis, Eötvös Loránd University, Rákóczi út 5, 1088 Budapest, Hungary

Clifford E. Weil, Mathematics Department, Michigan State University, East Lansing, MI 48824-1027, USA

Dedicated to the memory of Jan Mařík

Abstract

Let $f : H \subset \mathbb{R} \to \mathbb{R}$ be k times differentiable in both the usual (iterative) and Peano senses. We investigate when the usual derivatives and the corresponding Peano derivatives are different and the nature of the set where they are different.

1 Introduction

Professor Jan Mařík, whose death in January of 1994 ended an outstanding career, made significant contributions to several areas of mathematics including extensions of differentiable functions. (See [3].) For his enormous contributions to analysis and for our genuine affection for him, we dedicate this paper to his memory.

This paper is motivated by the following question. Assume that $H \subset \mathbb{R}$ is perfect and for a function $f : H \to \mathbb{R}$ both the kth ordinary derivative, $f^{(k)}$, and the kth Peano derivative, f_k, exist at all points of H. How large can the set E_k of those points x in H be where $f^{(k)}(x)$ and $f_k(x)$ are different?

For $k = 1$ the ordinary and Peano derivatives are the same. It follows from Theorem 2 of this paper that for a given perfect set H the set E_2 is countable. Theorem 3 implies that if, in addition, we assume that the third ordinary and Peano derivatives exist on H, then E_2 is scattered. On the other hand Example 2 shows that for $n \geq 3$, $E_n = H$ is also possible for some perfect sets H.

In our Theorems for higher values of k we must impose coincidence assumptions about lower order Peano derivatives of the ordinary derivatives of f in

*This author was supported by Grants FKFP B-07/1997 and Hungarian National Foundation for Scientific Research Grant No. T 016094

1991 Mathematics Subject Classification: Primary 26A24

Keywords: Peano derivatives, nowhere dense perfect sets, porosity.

order to obtain results. Actually the non-coincidence sets we consider are non-coincidence sets of "exact order" k, while the non-coincidence set E_k considered in the original question is of "order" less or equal k. (We give more explanation of this heuristic background in a remark following Theorem 6.)

In Section 4 the concept of γ-gap porosity is introduced. In Theorem 5 it is proved that if the set H is γ_n-gap porous at the points $x_n \in H$, then there is a $k \geq 2$ times ordinary and Peano differentiable function such that $x_n \in E_k$ for all n. Theorem 6 shows that γ-gap porosity, in a certain sense, is also a necessary condition.

2 Definitions and Other Preliminaries

Throughout this paper H will denote a perfect subset of $\mathbb{R}$, k will be a fixed element of $\mathbb{N}$, i and j will denote nonnegative integers and $f : H \to \mathbb{R}$. The usual or iterative kth derivative of f will be denoted by $f^{(k)}$. For example if $x \in H$, then $f'(x) = \lim_{\substack{y \to x \\ y \in H}} \dfrac{f(y) - f(x)}{y - x}$. Next the corresponding Peano derivative is defined.

Definition 1. Let $f : B \subset \mathbb{R} \to \mathbb{R}$ let $k \in \mathbb{N}$ and let $x \in B$. Then f is k times Peano differentiable at x means that there are numbers $f_j(x)$ for $j = 1, 2, \dots k$ and there is a function $\epsilon : B \to \mathbb{R}$ such that $\lim_{\substack{y \to x \\ y \in B}} \epsilon(y) = 0$, and for each $y \in B$

$$f(y) = f(x) + \sum_{j=1}^{k} \frac{f_j(x)}{j!}(y - x)^j + \epsilon(y)(y - x)^k.$$

If x is an isolated point of B, then the numbers $f_1(x), f_2(x), \dots, f_k(x)$ are completely arbitrary. Otherwise they are unique if they exist. Examining the above sum it is obvious that setting $f(x) = f_0(x)$ will be useful as will $f(x) = f^{(0)}(x)$. The reader unfamiliar with the notion of Peano derivatives is directed to [4]. The major conditions imposed on the sets studied in this paper are motivated by the work done in [1]. The specific theorem is as follows. (See page 395 of [1].)

Theorem 1. *Let $H \subset \mathbb{R}$ be closed, let $k \in \mathbb{N}$ and let $f : H \to \mathbb{R}$ be k times differentiable in both the usual sense and in the Peano sense on H. Suppose for each $i, j \in \mathbb{N} \cup \{0\}$ with $i + j \leq k$ we have that $f^{(i)}$ is j times Peano differentiable on H and that $\left(f^{(i)}\right)_j = f^{(i+j)}$ on H. Then there is a function $F : \mathbb{R} \to \mathbb{R}$ which is k times Peano differentiable on $\mathbb{R}$ such that $F_j = f_j$ on H for each $j = 0, 1, 2, \dots, k$.*

Simply stated, the purpose of this paper is to investigate the equality $\left(f^{(i)}\right)_j = f^{(i+j)}$ where f is defined on a nowhere dense set, H. If H is an interval, then the existence of $f^{(i)}$ implies the existence of f_i and the equality $f^{(i)} = f_i$. Consequently the equality under study holds. However for a nowhere dense set it is possible for $f_k(x)$ and $f^{(k)}(x)$ to both exist but to be different.

Since the hypotheses of Theorem 1 are used often in this paper, we introduce the following useful notation. Let

$$PD_k(H) = \{f : H \to \mathbb{R}; f \text{ is } k \text{ times differentiable in both the usual sense and in the Peano sense}\}$$

and

$$NPD_k(H) = \{f \in PD_k(H); i + j \leq k \text{ and } x \in H \text{ imply } \left(f^{(i)}\right)_j(x) \text{ exists}\}.$$

From Theorem 2 it follows that if the condition $\left(f^{(i)}\right)_j = f^{(i+j)}$ holds on H for all $i+j \leq k$, with the exception $i = 0,\ j = k$, and k is even, then the set $f_k \neq f^{(k)}$ is countable. If we have the additional information that $f \in PD_{k+1}(H)$, then in Theorem 3 we show that the previous exceptional set is scattered. However, for odd k's in Example 2 it is shown that there are non-empty perfect sets, H and functions, f which satisfy the assumptions of Theorem 2 and $f^{(k)} \neq f_k$ everywhere on H.

3 Non-coincidence Sets

We begin with a very simple but illustrative example.

Example 1. *Let $P = \{p_1, p_2, \ldots\}$ be a countable set in $\mathbb{R}$ with no isolated points, let $\{k_n\}$ be a sequence in $\mathbb{N}$ with $k_n \geq 2$ for each $n \in \mathbb{N}$ and let $\{\alpha_n\}$ be a sequence in $\mathbb{R}$. Then there is a function $f : P \to \mathbb{R}$ which is infinitely differentiable in the usual sense and in the Peano sense on P such that $f^{(k)} \equiv 0$ on P for all $k \in \mathbb{N}$ and for each $n \in \mathbb{N}$ we have $f_k(p_n) = 0$ if $k \neq k_n$ while $f_{k_n}(p_n) = \alpha_n$.*

Let $f(p_1) = 0$ and set $g_1(x) = f(p_1) + \frac{\alpha_1}{k_1!}(x - p_1)^{k_1}$. Let $a_1 = -\infty$ and $b_1 = +\infty$.

Let $n \in \mathbb{N}$ with $n \geq 2$ and suppose for $j = 1, 2, \ldots, n-1$, $f(p_j)$ has been defined and set $g_j(x) = f(p_j) + \frac{\alpha_j}{k_j!}(x - p_j)^{k_j}$. Also suppose that for $j = 1, 2, \ldots, n-1$ numbers a_j, $b_j \notin P$ have been selected so that $p_j \in (a_j, b_j)$ and for $i = 1, 2, \ldots, (j-1)$ either $(a_j, b_j) \cap (a_i, b_i) = \emptyset$, or $(a_j, b_j) \subset (a_i, b_i)$ and in the latter case; i.e., $p_j \in (a_i, b_i)$, $|g_i(x) - g_j(x)| < \exp\left(-\frac{1}{|x - p_i|}\right)$. To define $f(p_n)$, let $j_n = \max\{j \in \{1, 2, \ldots, n-1\}; p_n \in (a_j, b_j)\}$. (Since $(a_1, b_1) = (-\infty, \infty)$, j_n is defined.) Set $f(p_n) = g_{j_n}(p_n)$ and then let $g_n(x) = f(p_n) + \frac{\alpha_n}{k_n!}(x - p_n)^{k_n}$. To define a_n and b_n first select a closed subinterval I of (a_{j_n}, b_{j_n}) with p_n

in its interior. Let $i < j_n$ with $p_n \in (a_i, b_i)$. By the induction hypotheses, $(a_{j_n}, b_{j_n}) \subset (a_i, b_i)$ and $|g_i(x) - g_{j_n}(x)| < \exp\big(-\frac{1}{|x-p_i|}\big)$ for $x \in (a_{j_n}, b_{j_n})$. It follows that there is an $\epsilon > 0$ such that for all $i < j_n$ with $p_n \in (a_i, b_i)$ and for all $x \in I$ we have $\epsilon \le \exp\big(-\frac{1}{|x-p_i|}\big) - |g_i(x) - g_{j_n}(x)|$. Because each g_j is continuous and since $g_n(p_n) - g_{j_n}(p_n) = 0$, it is not difficult to see that there are a_n, $b_n \notin P$ with $p_n \in (a_n, b_n) \subset I$ such that

$$|g_{j_n}(x) - g_n(x)| < \exp\big(-\frac{1}{|x-p_{j_n}|}\big) \text{ and } |g_n(x) - g_{j_n}(x)| < \epsilon.$$

To complete the induction step we need only consider the case $i < n$ with $p_n \in (a_i, b_i)$. By definition $i \le j_n$. If $i = j_n$, then for $x \in (a_n, b_n)$ we have $|g_{j_n}(x) - g(x)| < \exp\big(-\frac{1}{|x-p_{j_n}|}\big)$. If $i < j_n$, then

$$\begin{aligned}
|g_i(x) - g_n(x)| \le& |g_i(x) - g_{j_n}(x)| + |g_{j_n}(x) - g_n(x)| \\
<& |g_i(x) - g_{j_n}(x)| + \epsilon \\
<& |g_i(x) - g_{j_n}(x)| + \exp\big(-\frac{1}{|x-p_{j_n}|}\big) - |g_i(x) - g_{j_n}(x)|.
\end{aligned}$$

To show that the function f has the desired properties, fix $i \in \mathbb{N}$ and let $n \in \mathbb{N}$ with $n > i$ and $p_n \in (a_i, b_i)$. Then $j_n \ge i$ and by definition $f(p_n) = g_{j_n}(p_n)$. Thus

$$|f(p_n) - g_i(p_n)| = |g_{j_n}(p_n) - g_i(p_n)| < \exp\big(-\frac{1}{|x-p_{j_n}|}\big).$$

By the definition of g_i and since $k_i \ge 2$, this estimate proves that $f'(p_i) = f_1(p_i) = 0$ (and hence that $f^{(k)} \equiv 0$ on P) and that $f_k(p_i) = 0$ if $k \ne k_i$ while $f_{k_i}(p_i) = \alpha_i$. $\square$

The next example shows that for some perfect sets H for any integer larger than 2 the corresponding usual and Peano derivatives may exist and be different everywhere on H.

Example 2. *There is a perfect set H and, for each $m \in \mathbb{N}$ with $m \ge 3$, a function $f \in NPD_k(H)$ for all $k \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ with $k \ge 2$ we have $f^{(k)} = 0$ on H and $f_k = 0$ on H except for $k = m$ while $f_m = m!$ on H.*

Remark 1. If m is odd then in the above Example we have $f' = f_1 = 0$ on H which implies that $(f^{(i)})_j = f^{(i+j)}$ holds except for $i = 0$ and $j = m$.
When m is even $(f^{(i)})_j = f^{(i+j)}$ is always satisfied when $i \ge 2$, if $i = 0$, or 1 it is satisfied for all j's with the exception of $j = m - i$.

Set $\ell_0 = 1$ and for each $n \in \mathbb{N}$ let $\ell_n = 10^{-n^n}$. In fact, any sequence satisfying the following three properties can be chosen for our construction.

(i) $\ell_{n-1} - 2\ell_n > \frac{1}{2}\ell_{n-1}$ holds for $n = 2, 3, \dots$.

(ii) For m, $n_0 \in \mathbb{N}$, $m \geq 3$ letting $\sigma_{n_0} = \sum_{n=n_0+1}^{\infty} \ell_{n-1}^{m-1} \geq \sigma'_{n_0} = \sum_{n=n_0+1}^{\infty} \ell_{n-1}^{m}$ we have $\lim_{n_0\to\infty} \ell_{n_0-1}^{-n}\sigma_{n_0} = 0$ for all $n \in \mathbb{N}$.

(iii) $\dfrac{\ell_n}{\ell_{n-1}} < \ell_{n-1}^{n-1}$ for $n = 2, 3, \dots$.

Observe that the ℓ_n's we chose satisfy properties (i)–(iii).

Set $I_{0,1} = [0,1]$, $I_{1,1} = [0, \ell_1]$, and $I_{1,2} = [1-\ell_1, 1]$. Suppose $I_{n,j} = [a,b]$ has been defined for $n \in \mathbb{N}$ and for $j \in N_n = \{1, 2, \dots, 2^n\}$. Then $I_{n+1,2j-1} = [a, a+\ell_{n+1}]$ and $I_{n+1,2j} = [b - \ell_{n+1}, b]$ defines $I_{n+1,j}$ for $j \in N_{n+1}$. Let $H = \bigcap_{n\in\mathbb{N}} \bigcup_{j=1}^{2^n} I_{n,j}$. For each $x \in H$ and for each $n \in \mathbb{N}$ let $j_{n,x}$ be that integer in N_n such that $x \in I_{n,j}$. For each $n \in \mathbb{N}$ and for each $j \in N_n$ let $p_{n,j}(x) = \alpha_{n,j}(x - a_{n,j}) + \beta_{n,j}$ where $a_{n,j}$ is the left endpoint of $I_{n,j}$ and the constants $\alpha_{n,j}$ and $\beta_{n,j}$ will be defined later depending on whether m is odd or even in a way that they will satisfy

$$|\alpha_{n,j}| \leq \ell_{n-1}^{m-1} \text{ and } |\beta_{n,j}| \leq \ell_{n-1}^{m}. \tag{1}$$

For $x \in H$ set $f(x) = \sum_{n\in\mathbb{N}} p_{n,j_{n,x}}(x)$.

Note that for x, $y \in [0,1]$

$$p_{n,j_{n,y}}(y) - p_{n,j_{n,x}}(x) = p_{n,j_{n,y}}(y) - p_{n,j_{n,x}}(y) + \alpha_{n,j_{n,x}}(y - x). \tag{2}$$

For x, $y \in H$ with $x \neq y$ let $n_0(x,y) = \min\{n \in \mathbb{N}; j_{n,x} \neq j_{n,y}\}$. Consequently, for x, $y \in H$ with $x \neq y$ (denoting $n_0(x,y)$ by n_0) from (2) it follows that

$$\frac{f(y) - f(x)}{y - x} = \sum_{n<n_0} \alpha_{n,j_{n,x}} + \frac{\sum_{n\geq n_0} p_{n,j_{n,y}}(y) - p_{n,j_{n,x}}(x)}{y - x}.$$

Thus using (1) we have $|p_{n,j_{n,x}}(x)| \leq 2\ell_{n-1}^{m}$. Since $|y - x| > \ell_{n_0-1} - 2\ell_{n_0} > \ell_{n_0-1}/2$ by (i) and (ii) it follows that the second term above tends to 0 as $y \to x$ (The term $n = n_0$ must be dealt with separately, but clearly it is no more than $8\ell_{n_0-1}^{m-1}$ which tends to 0 because $m \geq 3$.) and hence $f'(x) = \sum_{n\in\mathbb{N}} \alpha_{n,j_{n,x}}$.

Similarly for x, $y \in H$ with $x \neq y$

$$\begin{aligned} f(y) - f(x) - f'(x)(y-x) - (y-x)^m &= \Big(\sum_{n\geq n_0} p_{n,j_{n,y}}(y) - p_{n,j_{n,x}}(y)\Big) - (y-x)^m \\ &= \Big(p_{n_0,j_{n_0,y}}(y) - p_{n_0,j_{n_0,x}}(y) - (y-x)^m\Big) + \\ &\quad \sum_{n>n_0} p_{n,j_{n,y}}(y) - p_{n,j_{n,x}}(y) = T(x,y) + S(x,y). \end{aligned}$$

Since $|y - x| > \ell_{n_0-1} - 2\ell_{n_0}$, and $n_0 = n_0(x,y) \to \infty$ as $y \to x$, conditions (i) and (ii) imply that $\lim_{y\to x} \frac{S(x,y)}{|y-x|^n} = 0$ for all $n > m$.

First let m be odd. Then put $\alpha_{n,j} = 0$ and $\beta_{n,j} = \begin{cases} 0 & \text{if } j \text{ is odd} \\ \ell_{n-1}^m & \text{if } j \text{ is even.} \end{cases}$ Let $x, y \in H$ with $x \neq y$. If $x < y$, then $x \in I_{n_0,2j-1}$ and $y \in I_{n_0,2j}$. Thus $T(x,y) = \ell_{n_0-1}^m - (y-x)^m$. Since $\ell_{n_0-1} - 2\ell_{n_0} < y - x < \ell_{n_0-1}$, by condition (iii)

$$\begin{aligned} 0 < \ell_{n_0-1}^m - (y-x)^m &< \ell_{n_0-1}^m - (\ell_{n_0-1} - 2\ell_{n_0})^m = \ell_{n_0-1}^m \Big[1 - \big(1 - 2\frac{\ell_{n_0}}{\ell_{n_0-1}}\big)^m\Big] \\ &< \ell_{n_0-1}^m \Big[1 - \big(1 - m \cdot 2\frac{\ell_{n_0}}{\ell_{n_0-1}}\big)\Big] < \ell_{n_0-1}^m \big(2m\ell_{n_0-1}^{n_0-1}\big) \\ &= 2m\ell_{n_0-1}^{m+n_0-1}. \end{aligned}$$

So by condition (i) for $n > m$

$$\lim_{y\to x+} \frac{|T(x,y)|}{|y-x|^n} \leq \lim_{y\to x+} 2^n \frac{2m\ell_{n_0(x,y)-1}^{m+n_0(x,y)-1}}{\ell_{n_0(x,y)-1}^n} = 0.$$

If $y < x$, then $y \in I_{n_0,2j-1}$ and $x \in I_{n_0,2j}$. Thus, since m is odd, $T(x,y) = -\ell_{n_0-1}^m + (x-y)^m$. Now as above

$$0 > (x-y)^m - \ell_{n_0-1}^m > (\ell_{n_0-1} - 2\ell_{n_0})^m - \ell_{n_0-1}^m$$

and hence $|T(x,y)| < \ell_{n_0-1}^m - (\ell_{n_0-1} - 2\ell_{n_0})^m$. So by the same argument as above, $\lim_{y\to x-} \dfrac{|T(x,y)|}{|y-x|^n} = 0$ for $n > m$.

Now, let m be even. Then put $\beta_{n,j} = 0$ and $\alpha_{n,j} = \begin{cases} 0 & \text{if } j \text{ is odd} \\ \ell_{n-1}^{m-1} & \text{if } j \text{ is even.} \end{cases}$

If $x < y$, then $x \in I_{n_0,2j-1}$ and $y \in I_{n_0,2j}$. Thus

$$|T(x,y)| = |\ell_{n_0-1}^{m-1}(y - a_{n,j_{n_0,y}}) - (y-x)^m| < |\ell_{n_0-1}^{m-1}\ell_{n_0} - (y-x)^m| < \ell_{n_0-1}^m - (y-x)^m$$

and proceeding as in the first part of the previous case yields $\lim_{y\to x+} \dfrac{|T(x,y)|}{|y-x|^n} = 0$. If $y < x$, then $y \in I_{n_0,2j-1}$ and $x \in I_{n_0,2j}$. Thus since m is even,

$$\begin{aligned} |T(x,y)| &= | - \ell_{n_0-1}^{m-1}(y - a_{n_0,j_{n_0,x}}) - (y-x)^m = |\ell_{n_0-1}^{m-1}(a_{n_0,j_{n_0,x}} - y) - (x-y)^m| \\ &< |(\ell_{n_0-1} - 2\ell_{n_0})^m - \ell_{n_0-1}^m| = \ell_{n_0-1}^m - (\ell_{n_0-1} - 2\ell_{n_0})^m \end{aligned}$$

and proceeding as in the second part of the first case yields $\lim_{y\to x-} \dfrac{|T(x,y)|}{|y-x|^n} = 0$.

Therefore in all cases if $n > m$, then

$$\lim_{y\to x} \frac{f(y) - f(x) - (y-x)f'(x) - (y-x)^m}{(y-x)^n} = 0.$$

Consequently if m is odd and $m \geq 3$, then $f'(x) = 0$ (because each $\alpha_{n,j} = 0$) for all $x \in H$ and hence $f^{(k)} = 0$ on H for all $k \geq 1$. Moreover $f_j(x) = 0$ if $j \neq m$ while $f_m(x) = m!$ for all $x \in H$.

On the other hand if $m \geq 4$ is even, then $m' = m - 1 \geq 3$ is odd and letting $\beta'_{n,j} = \alpha_{n,j}$ we have $f'(x) = \sum_{n\in\mathbb{N}} \beta'_{n,j_{n,x}}$; that is, our earlier argument for odd m's shows that $(f')'(x) = f^{(2)}(x) = 0$ for all $x \in H$ and hence $f^{(k+1)}(x) = 0$ for all $k \geq 1$. Therefore $f''(x) = 0$ on H and hence $f^{(k)} = 0$ on H for all $k \geq 2$. Moreover, as above, $f_j(x) = 0$ if $j \neq m$ while $f_m(x) = m!$ and we also have $(f')_j(x) = 0$ if $j \neq m-1$, while $(f')_{m-1}(x) = (m-1)!$ for all $x \in H$.

The preceding example shows in particular that for $n \geq 3$ the nth ordinary and the nth Peano derivatives can both exist and be different everywhere on H for some perfect sets, H. The case $n = 2$ proves to be quite different as the next theorem demonstrates. For example it shows that the second ordinary and the second Peano derivatives can differ only on a countable set.

Theorem 2. *Let $H \subset \mathbb{R}$ be perfect, let $k \in \mathbb{N}$ with $k \geq 2$ and let $f \in PD_k(H)$. Suppose $0 \leq i < k$ with $k - i$ even and put*

$$\begin{aligned} E_i = \{x \in H;\ &\textit{if } i' + j' < k \textit{ or if } i' + j' = k \textit{ and } i' > i, \textit{ then} \\ &\left(f^{(i')}\right)_{j'}(x) \textit{ exists and } = f^{(i'+j')}(x) \textit{ and} \\ &\left(f^{(i)}\right)_{k-i}(x) \neq f^{(k)}(x)\}. \end{aligned}$$

Then E_i is countable.

Proof. For rational numbers α and β with $\alpha > \beta$ and for $n \in \mathbb{N}$ let

$$\begin{aligned} E_{n,i}^{\alpha,\beta} = \{x \in H;\ &\left(f^{(i')}\right)_{j'}(x) = f^{(i'+j')}(x) \text{ for } i' + j' < k, \\ &\left(f^{(i')}\right)_{k-i'}(x) = f^{(k)}(x) \text{ for } i < i' < k \\ &\Big|f^{(i)}(y) - \sum_{j=0}^{k-i} \frac{\left(f^{(i)}\right)_j(x)}{j!}(y-x)^j\Big| < \frac{\alpha - \beta}{2(k-i)!}|y - x|^{k-i} \\ &\text{for } y \in H, |y - x| < \frac{1}{n}, \text{ and } f^{(k)}(x) > \alpha > \beta > \left(f^{(i)}\right)_{k-i}(x)\}. \end{aligned}$$

The theorem will be proved if it can be shown that $E_{n,i}^{\alpha,\beta}$ is an isolated set. So suppose to the contrary that x is a non-isolated point of $E_{n,i}^{\alpha,\beta}$. Let $\epsilon > 0$ and select $y \in E_{n,i}^{\alpha,\beta}$ with $|y - x| < \frac{1}{n}$ such that for $i < i' < k$

$$\left|f^{(i')}(y) - \sum_{j=0}^{k-i'} \frac{\left(f^{(i')}\right)_j(x)}{j!}(y-x)^j\right| < \epsilon|y - x|^{k-i'}. \tag{3}$$

Since both x and y belong to $E_{n,i}^{\alpha,\beta}$,

$$\left|f^{(i)}(y) - \sum_{j=0}^{k-i} \frac{\left(f^{(i)}\right)_j(x)}{j!}(y-x)^j\right| < \frac{\alpha - \beta}{2(k-i)!}|y - x|^{k-i} \tag{4}$$

and

$$\left|f^{(i)}(x)-\sum_{j=0}^{k-i}\frac{\left(f^{(i)}\right)_j(y)}{j!}(x-y)^j\right|<\frac{\alpha-\beta}{2(k-i)!}|y-x|^{k-i}. \tag{5}$$

For $u\in H$ put

$$g(u)=f(u)-\sum_{j=0}^{k}\frac{f^{(j)}(x)}{j!}(u-x)^j=f(u)-h(u)$$

where h is a polynomial of degree no more than k. Then $g^{(j)}(x)=0$ for $j=0,1,\ldots,k$ and for $0\le i'<k$

$$g^{(i')}(u)=f^{(i')}(u)-\sum_{j=0}^{k-i'}\frac{f^{(i'+j)}(x)}{j!}(u-x)^j.$$

Using the assumptions in the definition of $E_{n,i}^{\alpha,\beta}$ and (3) for $i<i'<k$ we have

$$|g^{(i')}(y)|<\epsilon|y-x|^{k-i'}. \tag{6}$$

Again using the definition of $E_{n,i}^{\alpha,\beta}$ and (4) we infer

$$\left|g^{(i)}(y)+\frac{f^{(k)}(x)-\left(f^{(i)}\right)_{k-i}(x)}{(k-i)!}(y-x)^{k-i}\right|<\frac{\alpha-\beta}{2(k-i)!}|y-x|^{k-i}. \tag{7}$$

Since h is a polynomial of degree no more than k, by (5) we have

$$\begin{aligned}\frac{\alpha-\beta}{2(k-i)!}|y-x|^{k-i}>&|g^{(i)}(x)+h^{(i)}(x)-\sum_{j=0}^{k-i}\frac{\left(g^{(i)}\right)_j(y)+\left(h^{(i)}\right)_j(y)}{j!}(x-y)^j|\\ =&\left|g^{(i)}(x)-\sum_{j=0}^{k-i}\frac{\left(g^{(i)}\right)_j(y)}{j!}(x-y)^j\right|\end{aligned}$$

(using (6) and that $\left(g^{(i)}\right)_j(y)=g^{(i+j)}(y)$ for $1<j<k-i$)

$$\begin{aligned}>&|0-g^{(i)}(y)-\frac{\left(g^{(i)}\right)_{k-i}(y)}{(k-i)!}(x-y)^{k-i}|\\ &-\epsilon\sum_{j=1}^{k-i-1}\frac{|y-x|^{k-i-j}}{j!}|x-y|^j\\ >&|-g^{(i)}(y)-\frac{\left(g^{(i)}\right)_{k-i}(y)}{(k-i)!}(x-y)^{k-i}|-\epsilon(k-i)|x-y|^{k-i}.\end{aligned}$$

Since $\left(g^{(i)}\right)_{k-i}(y) = \left(f^{(i)}\right)_{k-i}(y) - f^{(k)}(x)$, we obtain

$$\frac{\alpha-\beta}{2(k-i)!}|y-x|^{k-i} + \epsilon(k-i)|x-y|^{k-i} >$$

$$|-g^{(i)}(y) - \frac{\left(f^{(i)}\right)_{k-i}(y) - f^{(k)}(x)}{(k-i)!}(x-y)^{k-i}|.$$

Using (7) we have

$$\frac{\alpha-\beta}{2(k-i)!}|y-x|^{k-i} > \left|g^{(i)}(y) - \frac{\left(f^{(i)}\right)_{k-i}(x) - f^{(k)}(x)}{(k-i)!}(y-x)^{k-i}\right|.$$

Adding the two preceding inequalities together, canceling and keeping in mind that $k-i$ is even we obtain

$$(\alpha-\beta) + \epsilon(k-i)(k-i)! > |2f^{(k)}(x) - \left(f^{(i)}\right)_{k-i}(y) - \left(f^{(i)}\right)_{k-i}(x)| = A. \quad (8)$$

On the other hand $x, y \in E_{n,i}^{\alpha,\beta}$ implies

$$f^{(k)}(x) > \alpha > \beta > \left(f^{(i)}\right)_{k-i}(x) \text{ and } \beta > \left(f^{(i)}\right)_{k-i}(y).$$

Hence $A > 2(\alpha-\beta)$ and this contradicts (8) when ϵ is small. Thus $E_{n,i}^{\alpha,\beta}$ is countable.

In a similar fashion the set resulting from $E_{n,i}^{\alpha,\beta}$ by reversing the inequality between α and β in the definition is also clearly countable. This observation concludes the proof. □

The next theorem shows that if $i = 0$ the set defined in Theorem 2, besides being countable, is scattered when f is $k+1$ times differentiable in both senses.

Theorem 3. *Let $k \in \mathbb{N}$ be even and let $f \in PD_{k+1}(H)$. Set*

$$E = \{x \in H; \textit{if } i+j < k \textit{ or if } i+j = k \textit{ and } 0 < i, \textit{ then} \left(f^{(i)}\right)_j(x) \textit{ exists and}$$
$$= f^{(i+j)}(x) \textit{ and } f_k(x) \neq f^{(k)}(x)\}.$$

Then E is nowhere dense in each $\emptyset \neq F \subset H$ with F perfect.

Proof. By Theorem 2, E is countable. Let $\emptyset \neq F \subset H$ be perfect. Suppose there is an interval I_1 such that $F \cap I_1 \neq \emptyset$ and $F \cap E$ is dense in $I_1 \cap F$. Since f_{k+1} is a Baire one function, there is an interval $I_2 \subset I_1$ and an $M \in (0, \infty)$ such that $I_2 \cap F \neq \emptyset$ and $|f_{k+1}(x)| \leq M$ for each $x \in I_2 \cap F$. By the Baire Category Theorem there is a $\delta > 0$ such that

$$K_\delta = \{x \in I_2 \cap F; y \in H \text{ and } |x-y| < \delta \text{ implies}$$
$$|f(y) - \sum_{j=0}^{k+1} \frac{f_j(x)}{j!}(y-x)^j| < |y-x|^{k+1}\}$$

is of the second category in $I_2 \cap F$. Also there is an interval $I_3 \subset I_2$ with $I_3 \cap F \neq \emptyset$ such that K_δ is of the second category in every subportion of $I_3 \cap F$. Since $F \cap E$ is dense in $I_3 \cap F$, we may select $x \in I_3 \cap F \cap E$. Let $\epsilon > 0$. For $y \in E$, y sufficiently close to x

$$\left| f(y) - \sum_{j=0}^{k} \frac{f_j(x)}{j!}(y-x)^j \right| < \epsilon |y-x|^k \tag{9}$$

and for $0 < i < k$

$$\left| f^{(i)}(y) - \sum_{j=0}^{k-i} \frac{\left(f^{(i)}\right)_j(x)}{j!}(y-x)^j \right| < \epsilon |y-x|^{k-i}. \tag{10}$$

Since $y \in E$, (10) may be rewritten as

$$\left| f^{(i)}(y) - \sum_{j=0}^{k-i} \frac{f^{(i+j)}(x)}{j!}(y-x)^j \right| < \epsilon |y-x|^{k-i}. \tag{11}$$

As in the proof of Theorem 2 for $u \in H$ let

$$g(u) = f(u) - \sum_{j=0}^{k} \frac{f^{(j)}(x)}{j!}(u-x)^j.$$

Then $g^{(j)}(x) = 0$ for $j = 0, 1, \ldots, k$. Since $x \in E$, (9) implies for $y \in E$ sufficiently close to x

$$\left| g(y) + \frac{f^{(k)}(x) - f_k(x)}{k!}(x-y)^k \right| < \epsilon |y-x|^k. \tag{12}$$

Moreover for $0 < i \leq k$

$$g^{(i)}(y) = f^{(i)}(y) - \sum_{j=0}^{k-i} \frac{f^{(i+j)}(x)}{j!}(y-x)^j.$$

So (11) and the assumption that $x \in E$ imply $|g^{(i)}(y)| < \epsilon |y-x|^{k-i}$. Also $|g^{(k)}(y)| = |f^{(k)}(y) - f^{(k)}(x)| < \epsilon$ for y sufficiently close to x since $f^{(k+1)}(x)$ exists. Because $g - f$ is a polynomial of degree at most k, for $y \in K_\delta$ sufficiently close to x we have

$$\left| g(x) - \sum_{j=0}^{k+1} \frac{g^{(j)}(y)}{j!}(x-y)^j \right| = \left| f(x) - \sum_{j=0}^{k+1} \frac{f^{(j)}(y)}{j!}(x-y)^j \right| < |x-y|^{k+1}.$$

Thus by (12)

$$\begin{aligned}
|x-y|^{k+1} >& \Big|g(x) - \sum_{j=0}^{k+1} \frac{g^{(j)}(y)}{j!}(x-y)^j\Big| \\
\geq& \Big|\frac{f^{(k)}(x) - f_k(x)}{k!}(x-y)^k\Big| - \Big|g(y) + \frac{f^{(k)}(x) - f_k(x)}{k!}(x-y)^k\Big| \\
&- \sum_{j=1}^{k} \frac{|g^{(j)}(y)|}{j!}|x-y|^j - \Big|\frac{f^{(k+1)}(y)}{(k+1)!}\Big||x-y|^{k+1} \\
\geq& \Big|\frac{f^{(k)}(x) - f_k(x)}{k!}\Big||x-y|^k - \epsilon|x-y|^k - \epsilon|x-y|^k \sum_{j=0}^{k} \frac{1}{j!} \\
&- \frac{M}{(k+1)!}|x-y|^{k+1}.
\end{aligned}$$

Dividing by $|x-y|^k$, using that y can be chosen arbitrarily close to x and that ϵ was arbitrary we obtain an inequality which contradicts $f_k(x) \neq f^{(k)}(x)$. □

4 A Porosity Condition

In this section we introduce a condition on the set H sufficient for the existence of a function in $NPD_k(H)$ for which $f^{(k)} = f_k$ fails to hold on a dense subset of H. We show that in some sense the condition is necessary.

Definition 2. Let $H \subset \mathbb{R}$, let $0 < \gamma < 1$ and let $x \in H$. Then H is γ-gap porous at x means there exist sequences $a_1 < a_2 < ... < a_\ell < ... < x < ... < b_\ell < ... < b_2 < b_1$ such that $[a_{2\ell+1}, a_{2\ell+2}] \cap H = \emptyset$, $[b_{2\ell+2}, b_{2\ell+1}] \cap H = \emptyset$, for $\ell \in \mathbb{N} \cup \{0\}$

$$\gamma|x - a_{2\ell+1}| \leq |a_{2\ell+2} - a_{2\ell+1}|, \quad \gamma|b_{2\ell+1} - x| \leq |b_{2\ell+1} - b_{2\ell+2}|$$

and

$$\lim_{\ell\to\infty} \frac{|a_{2\ell+1} - a_{2\ell}|}{|a_{2\ell+2} - a_{2\ell+1}|} = 0, \quad \lim_{\ell\to\infty} \frac{|b_{2\ell} - b_{2\ell+1}|}{|b_{2\ell+1} - b_{2\ell+2}|} = 0.$$

The first condition asserts that $[a_{2\ell+1}, a_{2\ell+2}]$ is at least a fixed portion of the interval $[a_{2\ell+1}, x]$ while the second condition can be shown to be equivalent to stating that the length of $[a_{2\ell}, a_{2\ell+1}]$ divided by the length of $[a_{2\ell}, x]$ tends to 0. Analogous statement can be made concerning the sequence $\{b_\ell\}$. These remarks are expressed in a very useful way in the following proposition.

Proposition 4. *Let $H \subset \mathbb{R}$ be perfect, let $0 < \gamma < 1$ and let $x \in H$. The set H is γ-gap porous at x if and only if for each $\epsilon > 0$ there is a $\delta > 0$ such that if $y_1, y_2 \in H$ with $|y_i - x| < \delta$ and either $y_1 < y_2 < x$ or $x < y_2 < y_1$, then we have $\frac{|y_2 - y_1|}{|x - y_1|} \geq \gamma$ or $\frac{|y_2 - y_1|}{|x - y_1|} \leq \epsilon$.*

The proof of the proposition is standard and hence is omitted.

Theorem 5. *Let H be a perfect set, let $k \in \mathbb{N}$ with $k \geq 2$ and for each $n \in \mathbb{N}$ let $x_n \in H$. Suppose for each $n \in \mathbb{N}$ there is a $\gamma_n \in (0,1)$ such that H is γ_n-gap porous at x_n. Then there is an $f \in NPD_k(H)$ such that for each $n \in \mathbb{N}$ we have $f^{(k)}(x_n) \neq f_k(x_n)$. In addition, for all n we can also insist that $i+j<k$ implies $(f^{(i)})_j(x_n) = f^{(i+j)}(x_n)$ and $0<i<k$ implies $(f^{(i)})_{k-i}(x_n) = f^{(k)}(x_n)$.*

Before proving this theorem we remark that given a $\gamma \in (0,1)$ it is easy to construct a perfect set H and a dense subset $\{x_n; n \in \mathbb{N}\}$ of H such that H is γ-porous at each x_n. Then Theorem 5 provides a function, f, which is in $NPD_k(H)$. Since $\{x_n; n \in \mathbb{N}\}$ is not scattered, by Theorem 3 f cannot belong to $PD_{k+1}(H)$.

Proof. For each $n \in \mathbb{N}$ let $a_{n,\ell} \nearrow x_n$, $b_{n,\ell} \searrow x_n$ such that for each $\ell \in \mathbb{N}$ we have

$$\gamma_n|x_n - a_{n,2\ell+1}| \leq |a_{n,2\ell+2} - a_{n,2\ell+1}|, \quad \gamma_n|b_{n,2\ell+1} - x_n| \leq |b_{n,2\ell+1} - b_{n,2\ell+2}|,$$
$$\lim_{\ell\to\infty} \frac{|a_{n,2\ell+1} - a_{n,2\ell}|}{|a_{n,2\ell+2} - a_{n,2\ell+1}|} = 0 \text{ and } \lim_{\ell\to\infty} \frac{|b_{n,2\ell} - b_{n,2\ell+1}|}{|b_{n,2\ell+1} - b_{n,2\ell+2}|} = 0.$$

Further assume, as we may, that $|x_n - a_{n,1}| \leq 1$ and $|b_{n,1} - x_n| \leq 1$ for each $n \in \mathbb{N}$. For $\ell \in \mathbb{N} \cup \{0\}$ set $I_{a,n,\ell} = [a_{n,2\ell+1}, a_{n,2\ell+2}]$, $I_{b,n,\ell} = [b_{n,2\ell+2}, b_{n,2\ell+1}]$, $J_{a,n,0} = (-\infty, a_{n,1})$, $J_{b,n,0} = (b_{n,1}, \infty)$ and for $\ell \in \mathbb{N}$ set $J_{a,n,\ell} = (a_{n,2\ell}, a_{n,2\ell+1})$ and $J_{b,n,\ell} = (b_{n,2\ell+1}, b_{n,2\ell})$. Since H is γ_n-gap porous at x_n, for $\ell \in \mathbb{N} \cup \{0\}$, $I_{a,n,\ell} \cap H = \emptyset$, $I_{b,n,\ell} \cap H = \emptyset$, $\gamma_n|x_n - a_{n,2\ell+1}| \leq |I_{a,n,\ell}|$, $\gamma_n|b_{n,2\ell+1} - x_n| \leq |I_{b,n,\ell}|$, and
$$\lim_{\ell\to\infty} \frac{|J_{a,n,\ell}|}{|I_{a,n,\ell}|} + \frac{|J_{b,n,\ell}|}{|I_{b,n,\ell}|} = 0.$$

For each $n \in \mathbb{N}$ let $\alpha_n = \dfrac{\gamma_n^k}{2^n}$. Then $\sum_{n\in\mathbb{N}} \dfrac{\alpha_n}{\gamma_n^k} < \infty$. Fix $n \in \mathbb{N}$. We define $f(n) : \mathbb{R} \to \mathbb{R}$ as follows. First put $f(n)(x_n) = 0$. For $\ell \in \mathbb{N} \cup \{0\}$ the function $f(n)$ is constant on $J_{a,n,\ell}$ and on $J_{b,n,\ell}$ and is linear on $I_{a,n,\ell}$ and on $I_{b,n,\ell}$. In addition

$$f(n)(a_{n,2\ell+2}) - f(n)(a_{n,2\ell+1}) = \alpha_n\big((a_{n,2\ell+2} - x_n)^k - (a_{n,2\ell+1} - x_n)^k\big)$$

and

$$f(n)(b_{n,2\ell+1}) - f(n)(b_{n,2\ell+2}) = \alpha_n\big((b_{n,2\ell+1} - x_n)^k - (b_{n,2\ell+2} - x_n)^k\big).$$

Finally assume that $f(n)$ is continuous at x_n. Hence $f(n)$ is continuous everywhere. Since $|x_n - a_{n,2\ell+1}| \leq |I_{a,n,\ell}|/\gamma_n$ and since $|b_{n,2\ell+1} - x_n| \leq |I_{b,n,\ell}|/\gamma_n$,

$$|f(n)(a_{n,2\ell+2}) - f(n)(a_{n,2\ell+1})| \leq \alpha_n|x_n - a_{n,2\ell+1}|^k \leq \frac{\alpha_n}{\gamma_n^k}|I_{a,n,\ell}|^k$$

and

$$|f(n)(b_{n,2\ell+1}) - f(n)(b_{n,2\ell+2})| \le \alpha_n |b_{n,2\ell+1} - x_n|^k \le \frac{\alpha_n}{\gamma_n^k} |I_{b,n,\ell}|^k.$$

Let $x, y \in \cup_{\ell\in\mathbb{N}\cup\{0\}}(J_{a,n,\ell} \cup J_{b,n,\ell})$ with $x < y$. Let $h = y - x$ and let $\{L_i = [c_i, d_i]; i \in \mathcal{N}\}$ denote the set of intervals $I_{a,n,\ell}$ and $I_{b,n,\ell}$ contained in (x, y) where $\mathcal{N} \subset \mathbb{N}$. Put $h_i = |L_i|$. Then $\sum_{i\in\mathcal{N}} \dfrac{h_i}{h} \le 1$ and since $0 \le \dfrac{h_i}{h} \le 1$ implies $0 \le \dfrac{h_i^k}{h^k} \le \dfrac{h_i}{h}$, we have $\sum_{i\in\mathcal{N}} \left(\frac{h_i}{h}\right)^k \le 1$; that is, $\sum_{i\in\mathcal{N}} h_i^k \le h^k$. Because $f(n)$ is continuous at x_n and constant on each $J_{a,n,\ell}$ and on each $J_{b,n,\ell}$,

$$|f(n)(y) - f(n)(x)| \le \frac{\alpha_n}{\gamma_n^k}(y - x)^k. \tag{13}$$

Since $H \subset \cup_{\ell\in\mathbb{N}\cup\{0\}}(J_{a,n,\ell} \cup J_{b,n,\ell})$, the above inequality holds for $x, y \in H$.

For $x \in H$ put $f(x) = \sum_{n\in\mathbb{N}} f(n)(x)$. Since $f(n)$ is constant on $J_{a,n,\ell}$ and on $J_{b,n,\ell}$ and since $|x_n - a_{n,1}| \le 1$ and $|b_{n,1} - x_n| \le 1$, from (13) it follows that $|f(n)(x)| \le \dfrac{\alpha_n}{\gamma_n^k}$ for all $x \in H$. Hence the sum defining f converges for all $x \in H$.

Let $x \in H \setminus \{x_n; n \in \mathbb{N}\}$. Then for each $n \in \mathbb{N}$ there is an $\ell_n \in \mathbb{N} \cup \{0\}$ such that $x \in J_{a,n,\ell_n} \cup J_{b,n,\ell_n}$. Let $N_0 \in \mathbb{N}$. Since $f(n)$ is constant on $J_{a,n,\ell}$ and on $J_{b,n,\ell}$, there is an open interval U such that $x \in U$ and $f(n)$ is constant on U for $n \in \{1, 2, \dots, N_0 - 1\}$. Then for $y \in U$ by (13) we have

$$|f(y) - f(x)| = \Big| \sum_{n\ge N_0} \Big(f(n)(x) - f(n)(y)\Big) \Big| \le \sum_{n\ge N_0} \frac{\alpha_n}{\gamma_n^k} |y - x|^k.$$

Since $\sum_{n\in\mathbb{N}} \dfrac{\alpha_n}{\gamma_n^k} < \infty$, by definition it follows that f is k times Peano differentiable at x and that $f_j(x) = 0$ for all $1 \le j \le k$. In particular since $k \ge 2$, we have $f'(x) = f_1(x) = 0$.

To complete the proof it suffices to show that for each $n \in \mathbb{N}$, f is k times Peano differentiable at x_n with $f_j(x_n) = 0$ for $1 \le j < k$ and $f_k(x_n) = \alpha_n k!$ because then $f' = 0$ on H and hence $f^{(k)} = 0$ on H while $f_k(x_n) = \alpha_n k! \ne 0$. First we show that for each $n \in \mathbb{N}$ if $f(n)$ is k times Peano differentiable at x_n, then so is f and indeed with the same Peano derivatives. To this end fix $n_0 \in \mathbb{N}$ and choose $N_0 > n_0$. Then there is an open interval U such that $x_{n_0} \in U$ and $f(n)$ is constant on U for all $n \le N_0 - 1$ with $n \ne n_0$. Then for $y \in U$

$$f(y) - f(x_{n_0}) = f(n_0)(y) - f(n_0)(x_{n_0}) + \sum_{n\ge N_0} \Big(f(n)(y) - f(n)(x_{n_0})\Big).$$

Thus

$$|f(y) - f(x_{n_0}) - \big(f(n_0)(y) - f(n_0)(x_{n_0})\big)| \le \sum_{n\ge N_0} \frac{\alpha_n}{\gamma_n^k} |y - x_{n_0}|^k.$$

Therefore $f'(x_{n_0}) = f'(n_0)(x_{n_0})$ and for $1 \le j \le k$ we have $f_j(x_{n_0}) = f_j(n_0)(x_{n_0})$ whenever the right-hand side exists.

Finally it will be shown that for each $n \in \mathbb{N}$, $f(n)$ is k times Peano differentiable at x_n with $f(n)_j(x_n) = 0$ for $1 \le j < k$ and $f(n)_k(x_n) = \alpha_n k!$. Now fix $n \in \mathbb{N}$ and let $\epsilon > 0$. Because $0 < \gamma_n < 1$, there is $K \in \mathbb{N}$ such that $(1-\gamma_n)^K < \epsilon$ and there exists $\ell_0 \in \mathbb{N}$ such that $\ell \ge \ell_0$ implies

$$|J_{a,n,\ell}| < |I_{a,n,\ell}|\frac{\epsilon}{K} \text{ and } |J_{b,n,\ell}| < |I_{b,n,\ell}|\frac{\epsilon}{K}$$

Let $x < x_n$ with $x \in H$ so close to x_n that if $x \in J_{a,n,\ell'}$, then $\ell' \ge \ell_0$. Set $t = |x_n - a_{n,2\ell'+1}|$; the distance between x_n and the right endpoint of $J_{a,n,\ell'}$. Since $|J_{a,n,\ell'}| < |I_{a,n,\ell'}|\frac{\epsilon}{K} < t\frac{\epsilon}{K}$, we conclude that $t < |x_n - x| \le |J_{a,n,\ell'}| + t < |I_{a,n,\ell'}|\frac{\epsilon}{K} + t < t\big(1 + \frac{\epsilon}{K}\big)$. Since $|x_n - a_{n,2\ell'+1}| \le |I_{a,n,\ell'}|/\gamma_n$, we have

$$|x_n - a_{n,2\ell'+2}| = |x_n - a_{n,2\ell'+1} - (a_{n,2\ell'+2} - a_{n,2\ell'+1})| < t - \gamma_n t = (1-\gamma_n)t.$$

Moreover

$$|x_n - a_{n,2\ell'+4}| \le (1-\gamma_n)|x_n - a_{n,2\ell'+3}| < (1-\gamma_n)|x_n - a_{n,2\ell'+2}| < (1-\gamma_n)^2 t$$

and in general

$$|x_n - a_{n,2(\ell'+K)}| \le (1-\gamma_n)^K t < \epsilon t.$$

In the interval $(a_{n,2\ell'+1}, a_{n,2(\ell'+K)})$ there are K intervals $I_{a,n,\ell}$; namely $I_{a,n,\ell'}$, $I_{a,n,\ell'+1}, \ldots I_{a,n,\ell'+K-1}$. Since $|J_{a,n,\ell}| < |I_{a,n,\ell}|\frac{\epsilon}{K}$ for all $\ell = \ell'+1, \ell'+2, \ldots, \ell'+K-1$, and since for these same values of ℓ we have $|I_{a,n,\ell}| < t$, it follows that $|J_{a,n,\ell}| < \frac{\epsilon}{K}t$ and consequently

$$\sum_{\ell=\ell'+1}^{\ell'+K-1} |J_{a,n,\ell}| < (K-1)\frac{\epsilon}{K}t < \epsilon t.$$

The function $f(n)$ changes by $\alpha_n\big((a_{n,2\ell+2} - x_n)^k - (a_{n,2\ell+1} - x_n)^k\big)$ on $I_{a,n,\ell} = [a_{n,2\ell+1}, a_{n,2\ell+2}]$, later it will also be useful to keep in mind that the sign of this change equals that of $(-1)^{k+1}$. From $|x_n - a_{n,2(\ell'+K)}| < \epsilon t$ and $\displaystyle\sum_{\ell=\ell'+1}^{\ell'+K-1} |J_{a,n,\ell}| < \epsilon t$ it is easy to see that (Recall that $t = |x_n - a_{n,2\ell'+1}|$.)

$$\begin{aligned}
\alpha_n t^k &> (-1)^{k+1}(f(n)(a_{n,2(\ell'+K)}) - f(n)(a_{n,2\ell'+1})) \\
&= (-1)^{k+1} \sum_{\ell=\ell'}^{\ell'+K-1} f(n)(a_{n,2\ell+2}) - f(n)(a_{n,2\ell+1}) \\
&= (-1)^k \sum_{\ell=\ell'}^{\ell'+K-1} \alpha_n((a_{n,2\ell+1} - x_n)^k - (a_{n,2\ell+2} - x_n)^k) > \alpha_n((1-2\epsilon)t)^k.
\end{aligned}$$

From (13) and from the continuity of $f(n)$ it follows that

$$|f(n)(a_{n,2(\ell'+K)}) - f(n)(x_n)| \le \frac{\alpha_n}{\gamma_n^k}|a_{n,2(\ell'+K)} - x_n|^k \le \frac{\alpha_n}{\gamma_n^k}\epsilon^k t^k.$$

Since $f(n)$ is constant on $J_{a,n,\ell'}$, we have $f(n)(x) = f(n)(a_{n,2\ell'+1})$; hence

$$\alpha_n t^k + \frac{\alpha_n}{\gamma_n^k}\epsilon^k t^k > (-1)^k\big(f(n)(x) - f(n)(x_n)\big) > \alpha_n(1-2\epsilon)^k t^k - \frac{\alpha_n}{\gamma_n^k}\epsilon^k t^k.$$

Using the above inequality, $t < |x_n - x| < (1+\frac{\epsilon}{K})t$, and the fact that the above argument is valid for any $\epsilon > 0$ one can easily verify that

$$\frac{f(n)(x) - f(n)(x_n)}{(x-x_n)^k} \to \alpha_n \text{ as } x \to x_n,\ x < x_n, \text{ and } x \in H.$$

A similar argument is valid when $x > x_n$. This implies $f_j(n)(x_n) = 0$ for $0 \le j < k$ and $f_k(n)(x_n) = \alpha_n k! \neq 0$. Therefore for each $n_0 \in \mathbb{N}$ we have $f'(x_{n_0}) = f'(n_0)(x_{n_0}) = 0$, $f_j(x_{n_0}) = f_j(n_0)(x_{n_0}) = 0$ for $1 \le j < k$, and $f_k(x_{n_0}) = f_k(n_0)(x_{n_0}) = \alpha_{n_0} k! \neq 0 = f^{(k)}(x_{n_0})$. □

Theorem 6. *Let H be a non empty perfect set and let $f \in NPD_k(H)$. Assume that $f_j = f^{(j)}$ on H for all $2 \le j \le k-1$ and let*

$$E = \{x \in H;\ \text{if } i+j < k,\ \text{or if } i+j = k \text{ and } i > 0,\ \text{then}$$
$$\big(f^{(i)}\big)_j(x) = f^{(i+j)}(x) \text{ and } f_k(x) \neq f^{(k)}(x)\}.$$

Suppose E is dense in H. Then there is a (non empty) portion $I \cap H$ of H such that for each $x \in I \cap E$ there is a $\gamma \in (0,1)$ such that H is γ-gap porous at x.

The above theorem seems to be too restrictive, but if E is not dense in H, then we can still obtain some information about its size. It is clear that the union of dense in itself sets is dense in itself. Let E_0 denote a maximal dense in itself subset of E and H_0 be the closure of E_0. Then $E \setminus H_0$ is a scattered set, and our theorem is applicable to H_0. Here we also point out that the assumptions in the definition of the set E are not unnatural either. If $f_k(x) \neq f^{(k)}(x)$ and $x \notin E$, then there is a $k' < k$ and $0 < i < k$ such that for $g = f^{(i)}$ the point x belongs to a set of non-coincidence, E', which is defined analogously to E by using g and k'. This means that one should think of E as an "exact" non-coincidence set of order k.

Theorems 5 and 6 imply that E_2 can be dense in H if and only if the set of gap porosity points is dense in H. Furthermore the set $\{f^{(k)} \neq f_k\}$ can be dense in H for some $f \in NPD_k(H)$ if and only if the set of gap porosity points of H is dense in H.

Proof. By Baire Category Theorem there is a portion $I \cap H$, a $\delta_0 > 0$, an $M' \in (0,\infty)$ and a set $F \subset I \cap H$ dense in $I \cap H$ such that for each $x \in I \cap H$ we have $|f_k(x)| \le M'$ and for each $x \in F$ and each $y \in H$ with $|y - x| < \delta_0$

$$\Big| f(y) - \sum_{j=0}^{k} \frac{f_j(x)}{j!}(y-x)^j \Big| \le |y-x|^k. \tag{14}$$

Let $p \in I \cap E$. As in the proof of Theorem 2 for $x \in H$ we let

$$g(x) = f(x) - \sum_{j=0}^{k} \frac{f^{(j)}(p)}{j!}(x-p)^j.$$

Then $g^{(j)}(p) = g_j(p) = 0$ for all $j = 0, 1, \ldots, k-1$, $g^{(k)}(p) = 0$ and $g_k(p) = f_k(p) - f^{(k)}(p) \neq 0$. In general for each $y \in H$ we have $g_k(y) = f_k(y) - f^{(k)}(p)$ and hence $|g_k(y)| \leq M' + |f^{(k)}(p)| = M$ for all $y \in I \cap H$. Moreover since $g - f$ is a polynomial of degree no more than k, for $x, y \in H$

$$g(y) - \sum_{j=0}^{k} \frac{g_j(x)}{j!}(y-x)^j = f(y) - \sum_{j=0}^{k} \frac{f_j(x)}{j!}(y-x)^j.$$

Thus by (14) for $x \in F$ and $y \in H$ with $|y - x| < \delta_0$

$$\left| g(y) - \sum_{j=0}^{k} \frac{g_j(x)}{j!}(y-x)^j \right| < |y-x|^k. \tag{15}$$

Let $A = \left(\frac{|g_k(p)|}{2(M+k!)} \right)^{\frac{1}{k-1}}$ and set $\gamma = \frac{A}{1+A}$. We will now show that H is γ-gap porous at p using Proposition 4.

Let $\epsilon > 0$. By hypothesis, $g_j(x) = g^{(j)}(x)$ for each $x \in H$ and $1 \leq j < k$ and since $p \in E$, if $i + j < k$ or if $i + j = k$ and $i > 0$, then $\left(g^{(i)}\right)_j(p) = g^{(i+j)}(p)$. Thus there is a $0 < \delta_1 < \delta_0$ such that $y \in H$, $|y - p| < \delta_1$ and $1 \leq i < k$ imply

$$\left| g_i(y) - \sum_{j=0}^{k-i} \frac{g^{(i+j)}(p)}{j!}(y-p)^j \right| < \epsilon_i |y-p|^{k-i}$$

where $\epsilon_i = \frac{i!|g_k(p)|}{k!} \left(\frac{1-\gamma}{\gamma} \right)^{i-1}$. Since $g^{(j)}(p) = 0$, for $j = 0, 1, \ldots, k$ and $|y - p| < \delta_1$ we obtain

$$|g_i(y)| < \epsilon_i |y-p|^{k-i}. \tag{16}$$

Since $g_j(p) = 0$ for $j = 0, 1, \ldots, k-1$, there is a $0 < \delta_2 < \delta_1$ such that $y \in H$ and $|y - p| < \delta_2$ implies

$$|g(y) - \frac{g_k(p)}{k!}(y-p)^k| < \epsilon_k |y-p|^k \tag{17}$$

where $\epsilon_k = \frac{\epsilon(1-\gamma)^{k-1}|g_k(p)|}{\left(1 + (1-\gamma)^{k-1}\right)2k!}$. Let $x, y \in H$ with $|p - y| < \delta_2$. Further suppose $y < x < p$. (The argument for $p < x < y$ is similar.) In addition assume that $x \in F$. By (15), (16) and (17)

$$\left| \frac{g_k(p)}{k!} \right| |(y-p)^k - (x-p)^k| \leq \left| \frac{g_k(p)}{k!}(y-p)^k - g(y) \right| + |g(y) - g(x)|$$

$$
\begin{aligned}
&+\left|g(x)-\frac{g_k(p)}{k!}(x-p)^k\right| \\
<&\epsilon_k|y-p|^k+\left|g(y)-\sum_{j=0}^{k}\frac{g_j(x)}{j!}(y-x)^j\right| \\
&+\sum_{j=1}^{k}\frac{|g_j(x)|}{j!}|y-x|^j+\epsilon_k|x-p|^k \\
<&\epsilon_k(p-y)^k+|y-x|^k+\sum_{j=1}^{k-1}\frac{\epsilon_j|x-p|^{k-j}|y-x|^j}{j!} \\
&+\frac{M}{k!}|y-x|^k+\epsilon_k(p-x)^k.
\end{aligned}
$$

Since

$$
\begin{aligned}
|(y-p)^k-(x-p)^k| &= |y-x|\left|\sum_{j=1}^{k}(y-p)^{k-j}(x-p)^{j-1}\right| \\
&= (x-y)\sum_{j=1}^{k}(p-y)^{k-j}(p-x)^{j-1},
\end{aligned}
$$

$$
\begin{aligned}
&\left|\frac{g_k(p)}{k!}\right|(x-y)\sum_{j=1}^{k}(p-y)^{k-j}(p-x)^{j-1} \qquad (18)\\
<&\epsilon_k\big((p-y)^k+(p-x)^k\big)+\sum_{j=1}^{k-1}\frac{\epsilon_j(p-x)^{k-j}(x-y)^j}{j!}+\Big(\frac{M+k!}{k!}\Big)(x-y)^k.
\end{aligned}
$$

Suppose $\epsilon < \frac{x-y}{p-y} < \gamma$. Since $p-y = p-x+x-y < p-x+\gamma(p-y)$, $(1-\gamma)(p-y) < (p-x)$. So for $j = 1, 2, \ldots, k-1$

$$
\begin{aligned}
\epsilon_j(p-x)^{k-j}(x-y)^j &< \epsilon_j(x-y)(p-y)^{k-j}(p-y)^{j-1}\gamma^{j-1} \\
&< \epsilon_j(x-y)(p-y)^{k-j}(p-x)^{j-1}\Big(\frac{\gamma}{1-\gamma}\Big)^{j-1} \qquad (19)\\
&= \frac{|g_k(p)|}{k!}j!(x-y)(p-y)^{k-j}(p-x)^{j-1}.
\end{aligned}
$$

Also

$$\begin{aligned}
&\epsilon_k\big((p-y)^k+(p-x)^k\big)+\Big(\frac{M+k!}{k!}\Big)(x-y)^k\\
<&\epsilon_k(x-y)\Big(\frac{(p-x)^{k-1}}{\epsilon(1-\gamma)^{k-1}}+\frac{(p-x)^{k-1}}{\epsilon}\Big)\\
&+(x-y)\Big(\frac{M+k!}{k!}\Big)\Big(\frac{\gamma}{1-\gamma}\Big)^{k-1}(p-x)^{k-1}\\
=&\frac{|g_k(p)|}{2k!}(x-y)(p-x)^{k-1}+\frac{|g_k(p)|}{2k!}(x-y)(p-x)^{k-1}\\
=&\frac{|g_k(p)|}{k!}(x-y)(p-x)^{k-1}.
\end{aligned}$$

Summing inequality (19) multiplied by $1/j!$ for $j=1,2,\ldots,k-1$, and adding to it the above estimate contradicts inequality (18). Thus $\frac{x-y}{p-y}\le\epsilon$ or $\frac{x-y}{p-y}\ge\gamma$. Since F is dense in $I\cap H$, we may assume that $x\in I\cap H$. So by Proposition 4 the set H is γ-gap porous at p. □

References

[1] H. Fejzić, J. Mařík, and C. E. Weil, *Extending Peano derivatives*, Math. Bohemica, **119** (1994), 387–406.

[2] V. Jarník, *Sur l'extension du domaine de definition des fonctions d'une variable, qui laisse intacte la derivabilité de la fonction*, Bull international de l'Acad Sci de Boheme, (1923).

[3] J. Mařík, *Derivatives and closed sets*, Acta. Math. Acad. Sci. Hungar., **43** (1998), 25–29.

[4] Clifford E. Weil, *The Peano notion of higher order differentiation*, Math. Japonica, **42** (1995), 587–600.