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# GENERALIZED INTEGRALS AND RELATED TOPICS *

In this talk several topics, all connected to generalized integrals, are surveyed. The results mentioned here will serve as a basis for the author's "habilitation thesis" at the Eötvös University, Budapest. For more details of the basic definitions and for standard notation we refer to [33] and [30].

**Definition.** If $A \subset \mathbf{R}$ is an interval, $f : A \to \mathbf{R}$ is HK-integrable (Henstock-Kurzweil integrable) on $A$ and its integral, denoted by $(HK)\int_A f$, equals $I$ if for all $\epsilon > 0$, there exists $\delta : A \to (0, +\infty)$ such that

$$\left| \sum_{i=1}^{p} f(x_i)|A_i| - I \right| < \epsilon$$

for all $\delta$-fine partitions $P = \{(A_i, x_i)\}_{i=1}^{p}$ of $A$.

Recall that in the above definition the collection $P = \{(A_i, x_i)\}_{i=1}^{p}$ is a *partition of* $A$ if the intervals $A_i$ are non-overlapping, their union is $A$ and $x_i \in A_i$ for all $i$. If we do not assume that the union of the $A_i$'s equals $A$ we talk about *partitions in* $A$. If the points $x_i$ belong to a set $E \subset A$ we say that the partition is *anchored* in $E$. The partition $P$ is *$\delta$-fine* when $A_i \subset B(x_i, \delta(x_i))$ for all $i$.

### 1. How complicated need $\delta$ be in the $HK$-integral definition?

In [28] the answer to this question used an argument based on concepts which are related to the Denjoy-Perron integration, and work only in one-dimension. A different proof, which works also in the multidimensional case, was given in [7]. It turned out that in the definition of the HK-integral the function $\delta$ can be chosen to be *nearly upper semicontinuous.* We remind that *nearly upper semicontinuous* means that $\delta$ equals an upper semicontinuous function almost everywhere.

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C. E. Weil asked whether one can find a characterization of the "upper semicontinuously integrable" functions? By "upper semicontinuously integrable" functions we mean the class of those $HK$-integrable functions for which, given any $\epsilon > 0$, the function $\delta$ can be chosen to be upper semicontinuous. In [12] we showed that if $f \in \mathrm{HK}(A)$ then $f$ is upper semicontinuously integrable if and only if there exists a Baire 1 function $g : A \to \mathbf{R}$ such that $g > |f|$ on $A$.

**2. Ergodic theorem and generalized integrals.**

P. Major in [23] gave an example showing that with non-Lebesgue integrable functions one should be very cautious to obtain a "non-absolute ergodic theorem". He showed that *there exists a function* $f : X \to \mathbf{R}$, *and* $S_1, S_2 : X \to X$ *ergodic transformations on a probability space* $(X, \mu)$ *such that*

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} f(S_1^k x) = 0 \text{ a.e. and}$$

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} f(S_2^k x) = a \neq 0 \text{ a.e.}$$

By the ergodic theorem clearly $f \notin L^1(X, \mu)$. M. Laczkovich asked whether in the above result $S_1, S_2$ can be irrational rotations of the unit circle $\mathbf{T}$. In [17] we answer this question by showing that *there exists a Borel measurable function* $f : \mathbf{R} \to \mathbf{R}$ *periodic by* 1 *and two irrationals* $\alpha$ *and* $\beta$ *such that* $\frac{1}{n+1}\sum_{k=0}^{n} f(x + k\alpha) \to 1$ *and* $\frac{1}{n+1}\sum_{k=0}^{n} f(x + k\beta) \to 0$ *for all* $x \in \mathbf{R}$.

Related to this "rotation question" assume that $f : \mathbf{T} \to \mathbf{T}$ is measurable and set $\Gamma_f = \{\alpha : (1/(n+1))\sum_{k=0}^{n} f(x + k\alpha) \text{ converges a.e.}\}$.

In [14] we verified that $\Gamma_f$ *is of positive Lebesgue measure if and only if* $f \in L^1(\mathbf{T})$. *Furthermore, given a sequence of rationally independent irrationals* $\{\alpha_j\}_{j=1}^{\infty}$, *there exists a non-Lebesgue integrable* $f : \mathbf{T} \to \mathbf{R}$, *such that each* $\alpha_j$ *belongs to the rotation set of* $f$. This result implies that $\Gamma_f$ can be dense for non-integrable functions. In [34] R. Svetic improves this result by showing that *there exists a non-integrable* $f : \mathbf{T} \to \mathbf{R}$ *such that* $\Gamma_f$ *is c-dense in* $\mathbf{T}$. (A set $S \subset \mathbf{T}$ is *c-dense* if the cardinality of $S \cap I$ equals continuum for every open interval $I \subset \mathbf{T}$.)

It is still an open question whether there exists an $f : \mathbf{T} \to \mathbf{R}$, such that $f \notin L^1(\mathbf{T})$ and the Hausdorff-dimension of $\Gamma_f$ is positive.

For further results of different nature, but related to non-Lebesgue integrable functions and the ergodic theorem, we refer to [37] and [38].

**3. Special partitions.**

The *regularity* of an $m$-dimensional interval is the ratio of its shortest and longest side. An interval is $r$-regular when its regularity is greater or equal than $r$. Given $r>0$, allowing only $r$-regular intervals in the definition of the HK-integral one can obtain a "regular" generalization for higher dimensions. However, there are examples when $A$ and $B$ are non-overlapping intervals, the generalized "regular" integrals $\int_A f$, and $\int_B f$ exist, but $\int_{A\cup B} f$ does not. W. Pfeffer in [25] suggested to solve this problem by using *special* partitions. The partition $P=\{(A_i,x_i)\}$ is special when it is a partition and $x_i$ is a *vertex* of $A_i$ for each $i$. Then the problem arises whether for any $\delta : A\to(0,\infty)$, $r>0$, there exists a $\delta$-fine $r$-regular special partition of $A$. In [6] we answered this problem for planar partitions. We showed that *if the rectangle $A\subset \mathbf{R}^2$, and $1/1000\geq r>0$ then for all $\delta : A\to(0,\infty)$ there exists a $\delta$-fine $r$-regular special partition of $A$.*

In dimensions higher than two the problem about the existence of special partitions is still unsolved.

In the later development of higher dimensional integration special partitions did not turn out to be as useful as they were expected to be. However motivated by possible applications to harmonic analysis H. Fejzić, C. Freiling and D. Rinne in [22] did work on improving the regularity constant of the previous theorem. They showed that in the above result *the constant* $1/1000$ *can be replaced by* $1/\sqrt{2}$. It would be interesting to see whether $1/\sqrt{2}$ is the best constant possible? We believe that probably not.

## 4. Application of special partitions to the (generalized) wave equation.

It is well-known that *if $f:\mathbf{R}^2\to\mathbf{R}^2$ is smooth ($C^\infty$) then $\partial^2 f/\partial x\partial y=0$ on $\mathbf{R}^2$ implies $f(x,y)=a(x)+b(y)$.*

In [1] the authors studied different generalizations of the wave equation. They used generalized concepts of differentiation and studied whether the above theorem remained valid. They also studied the connection of these results with the theory of the uniqueness for multiple trigonometric series.

Let $f$ be a function of two variables and set $\Delta f(x,y;h,k)=f(x+h,y+k)-f(x+h,y)-f(x,y+k)+f(x,y)$. Let $Df(x,y)=\lim_{h,k\to 0}\Delta f(x,y;h,k)/hk$. In [AFCR] there is a short and neat proof for the following result of K. Bogel, [2]: *if $Df(x,y)=0$ on $\mathbf{R}^2$ then $f(x,y)=a(x)+b(y)$.*

**Definition.** If there is a number $s$ such that for arbitrarily large $M$'s $\sup_{1/M\leq|h/k|\leq M}|\frac{\Delta f(x,y;h,k)}{hk}-s|\to 0$ as $\max\{|h|,|k|\}\to 0$ then $s$ is the generalized $(x,y)$-derivative of $f$ in the restricted sense, and is denoted by $D^r f(x,y)$.

In [1] it was observed that it is a corollary of the theorem about the existence of special partitions that *if $D^r f(x,y)=0$ on $\mathbf{R}^2$ then $f(x,y)=a(x)+b(y)$.*

**5. The $\mathcal{BV}$- and $\mathcal{F}$-integrals.**

A set $A \subset \mathbf{R}^m$ is a *figure*, that is, $A \in \mathcal{F}$ when it is a finite union of compact non-degenerate subintervals of $\mathbf{R}^m$.

In the next few definitions we assume that $H \subset \mathbf{R}^m$ is measurable. The *essential interior* of $H$, denoted by $\mathrm{int}^* H$, consists of the Lebesgue density points of $H$. The *essential exterior* of $H$, denoted by $\mathrm{ext}^* H$, consists of the Lebesgue dispersion points of $H$. (A dispersion point is a zero density point of $H$.) The *essential closure* of $H$ is $\mathrm{cl}^* H = \mathbf{R}^m \setminus \mathrm{ext}^* H$. The *essential boundary* of $H$ is $\partial^* H = \mathbf{R}^m \setminus (\mathrm{ext}^* H \cup \mathrm{int}^* H)$. Finally $H$ is a *set of bounded variation, $H \in BV$*, when $\mathcal{H}(\partial^* H) = ||H|| < \infty$. Here $\mathcal{H}$ denotes the $(m-1)$-dimensional Hausdorff-measure, $\mathcal{H}^{m-1}$. BV sets are important in geometric measure theory, for details about their properties we refer to [21] and [39]. We denote by $\mathcal{BV}$ the set of essentially closed BV-sets, that is $H \in \mathcal{BV}$ when $H \in$BV, and $H = \mathrm{cl}^* H$. The *regularity* of an $H \in$BV is the number $r(H) = |A|/(d(A)||A||)$ when $d(A)||A|| \neq 0$, otherwise let $r(H) = 0$. We say that $H$ is $\eta$-regular when $r(H) > \eta > 0$.

Let $\mathcal{A}$ denote either $\mathcal{F}$ or $\mathcal{BV}$.

**Definition**. The function $F : \mathcal{A} \to \mathbf{R}$ is a *charge* when

i) $F$ is additive, that is, $F(A \cup B) = F(A) + F(B)$ when $A, B \in \mathcal{A}$ are non-overlapping;

ii) $F$ is continuous, that is, for every $\epsilon > 0$ there exists $\eta > 0$ such that $|F(A)| < \epsilon$ for each $A \in \mathcal{A}$ with $A \subset B(\mathbf{0}, 1/\epsilon)$, $||A|| < 1/\epsilon$, and $|A| < \eta$.

**Definition**. A set in $\mathbf{R}^m$ is *thin* when it is of $\sigma$-finite $\mathcal{H}$-measure. If $A \subset \mathbf{R}^m$ then $\delta : A \to [0, \infty)$ is a *gage function* on $A$ if its null set $\{x : \delta(x) = 0\}$ is thin.

The concepts of charges, gages, and regularity of BV sets, originating from [25] and [26], were crucial in the development of the variational integral.

**Definition**. If $A \in \mathcal{A}$, then $f : A \to \mathbf{R}$ is *$\mathcal{A}$-integrable* on $A$ if there exists a charge $F$ such that for all $\epsilon > 0$, there exists a gage $\delta$ on $A$ satisfying $\sum_{i=1}^{p} |f(x_i)|A_i| - F(A_i)| < \epsilon$ for each $\epsilon$-regular $\delta$-fine $\mathcal{A}$-partition $\{(A_i, x_i)\}_{i=1}^{p}$ in $A$.

For the above integrals a very general version of the *divergence theorem* holds. For different generalizations and proofs of the divergence theorem we refer to [25], [29], [30], and [18, Theorem 5.6].

One natural question in this field is whether the classes of $\mathcal{F}$- and $\mathcal{BV}$-integrable functions are different. W.F. Pfeffer in [31] proved that *if $K$ is a figure then $f$ is $\mathcal{F}$-integrable on $K$ if and only if it is $\mathcal{BV}$-integrable.* On the other hand, an unpublished example of this author (for details see, [29] and [31]) implies that *there exits $A \in \mathcal{BV}$ and an $f : A \to \mathbf{R}$ which is $\mathcal{BV}$-integrable on $A$ but has no $\mathcal{F}$-integrable extension onto a figure containing $A$.*

**6. Variations of additive functions.**

In his memoir [35] B. Thomson used variational measures to study certain function classes related to integration. Later this work continued in papers of B. Bongiorno, L. Di Piazza, V. Skvortsov, Pfeffer and the present author, see [4], [18], [32] and other references in this section.

**Definition**. Let $E \subset \mathbf{R}^m$ and $F : \mathcal{A} \to \mathbf{R}$. Given $0 < \eta < 1/(2m)$ and a $\delta : E \to [0, \infty]$ set $V_{\eta,\delta}^{\mathcal{A}} F(E) = \sup_P \sum_{i=1}^p |F(A_i)|$ where sup is taken over all $\eta$-regular $\delta$-fine $\mathcal{A}$-partitions $P = \{(A_i, x_i)\}_{i=1}^p$ anchored in $E$.

The *$\mathcal{A}$-variation* of $F$ on $E$ is $V_*^{\mathcal{A}} F(E) = \sup_{0<\eta<1/2m} \inf_\delta V_{\eta,\delta}^{\mathcal{A}} F(E)$ where the infimum is taken over all gages.

These variations are regular Borel measures, see [18, Lemma 4.6]. One of the latest developments is the following result obtained by W. F. Pfeffer and this author in [19].

**Theorem.** *Let $F$ be a charge in a figure $A$. If $V_* F$ is $AC$, then up to a set of $\sigma$-finite measure $\mathcal{H}^{m-1}$, $A$ is the union of compact sets $K_1, K_2, \ldots$ with $V_* F(K_n) < +\infty$ for $n = 1, 2, \ldots$. In particular, $V_* F$ is $\sigma$-finite.*

(At the time of writing these notes we were also informed that independently from our work another group: B. Bongiorno, L. Di Piazza and D. Preiss also obtained similar results [3]. We do not know details of that proof, but were informed that this other proof is based on methods different from ours.)

To put the above theorem into perspective we recall from [18] that *if $F$ is a charge on $A \in \mathcal{A}$ then $DF_A$ is $\mathcal{A}$-integrable on $A$, and $F$ is its indefinite $\mathcal{A}$-integral if and only if $V_*^{\mathcal{A}} F_A$ is $\sigma$-finite and absolutely continuous.* In this result $DF_A$ is the "regular" $A$ derivative of $F$, and $V_*^{\mathcal{A}} F_A$ is the $\mathcal{A}$-variation with respect to $A$ (for the details see [18]). This, combined with the $\sigma$-finiteness result, leads to an elegant comparison of the generalized Riemann and Lebesgue integrals:

**Corollary.** *Let $F$ be a charge in a figure $A$.*

*i) If $V_* F$ is $AC$, then $F$ is the indefinite $\mathcal{F}$-integral of $DF$.*

*ii) If $V_* F$ is $AC$ and finite, then $F$ is the indefinite Lebesgue integral of $DF$.*

In dimension one the proofs are somewhat easier (see [4] and [32]).

It is worth mentioning the most important tools used in [18] and [19]. For the full descriptive definition of the $\mathcal{A}$-integral *Ward's theorem* is generalized to figures and their "regular" derivatives. The classical result (which was generalized) states that *if $F$ is an additive function of an interval and the* *upper derivative* $\overline{F}(x) < \infty$ *on a set* $E$ *then* $F$ *is derivable almost everywhere on* $E$.

To prove the $\sigma$-finiteness result in [19] we had to apply Baire's category theorem in a situation where it is not clear whether the category theorem is applicable at all. This is due to the "testy relationship" between measure and category.

We will call a set $E \subset \mathbf{R}^m$ *supporting* if $|E| > 0$ and $|E \cap U| > 0$ for each open set $U \subset \mathbf{R}^m$ with $E \cap U \neq \emptyset$.

**Theorem**[19]. *Let* $0 < s < m$, *and let* $E \subset \mathbf{R}^m$ *be a measurable set with* $|E| > 0$. *Then there exists a supporting compact set* $C \subset E$ *and a* $G_\delta$ *set* $H \subset C$ *satisfying:*

*i)* $H$ *is dense in* $C$ *and* $|H| = 0$*;*

*ii) given* $Z \subset \mathbf{R}^m$ *with* $\mathcal{H}^s(Z) = 0$, *there exists a supporting compact set* $C_Z \subset C - Z$ *so that* $H \cap C_Z$ *is dense in* $C_Z$.

In the above theorem the *same* $G_\delta$ set, $H$, works for all exceptional sets $Z$. These exceptional sets are of small measure but can be residual in $C$ and this is the measure versus category problem.

### 7. Lipschitz Transformations.

Working with sets of bounded variation the best choice for transformations is the class of *lipeomorphisms* (bi-lipschitz mappings).

Answering a question of W. F. Pfeffer in [11] we proved that lipeomorphisms map *density points onto density points*, and dispersion points onto dispersion points. It is easy to see that almost all density points are mapped onto density points, the difficulty is to show that there are no exceptional points. These questions, like the *invariance of domain theorem for lipeomorphisms*, are also related to quasiconformal/quasisymmetric maps, for the details see [36].

In Chapter 11 of [30] the *g-integral* is studied. This concept can be regarded to be an intermediate step between the standard HK-integral in $\mathbf{R}^m$ and the $\mathcal{F}$-integral discussed above. In the definition of the g-integral $\epsilon$-regular $\delta$-fine partitions consisting of intervals are used for a non-negative gage function $\delta$. Though the $g$-integral has many nice properties it has also some deficiencies. In [13] it was shown that *the g-integral is not rotation invariant.* This was among the shortcomings which motivated further generalizations. For the $\mathcal{BV}$- and $\mathcal{F}$-integrals a nice change of variables theorem holds [18, Theorem 5.7] (for other, earlier versions see [29], [30]).

**Theorem**. *Let* $A \in \mathcal{BV}$, *and let* $\Phi : A \to \mathbf{R}^m$ *be a lipeomorphism. If* $f \in \mathcal{BV}(\Phi(A))$, *then* $(f \circ \Phi)J_\Phi$ *belongs to* $\mathcal{BV}(A)$. *(Here* $J_\Phi$ *is the Jacobian of*

*$\Phi$.) If both $A$ and $\Phi(A)$ are figures and $f \in \mathcal{F}(\Phi(A))$, then $(f \circ \Phi)J_\Phi$ belongs to $\mathcal{F}(A)$. In either case,*

$$\int_{\Phi(A)} f d\lambda = \int_A (f \circ \Phi) J_\Phi d\lambda.$$

In the forthcoming paper [16] we will investigate the lipeomorphic invariance of extension properties of $\mathcal{BV}$-integrable functions.

One of the tools used in [16] is the following result from [15]: *If $E \subset \mathbf{R}^m$ compact, $\Phi : E \to \mathbf{R}^m$ is a Lipschitz-$L$ mapping, $H \in \mathcal{BV}$, and $E \subset int^*(H)$. Then there exists a figure $F$, a constant $c_{L,m}$ depending only on $L$, and on the dimension such that $\Phi(E) \subset F$, $||F|| \leq c_{L,m}||H||$, and $|F| \leq c_{L,m}|H|$.*

This theorem says that if we can put our set $E$ into a suitcase $H$, then its Lipschitz image $\Phi(E)$ can be packed into a not too much larger (both in surface and volume) suitcase $F$. The proof of this theorem is not difficult but it is not absolutely trivial. A **wrong proof** is the following: Let $\tilde{\Phi} : \mathbf{R}^m \to \mathbf{R}^m$ be a Lipschitz-$L$ extension of $\Phi$. Then "use" $||\Phi(F)|| \leq \tilde{c}_{L,m}||F||$ and pack $\Phi(E)$ into $\Phi(F)$. This does not work. In [15] we give *an example of a Lipschitz-$L$ mapping $\Phi : \mathbf{R}^2 \to \mathbf{R}^2$ and a square $Q \subset \mathbf{R}^2$ such that $dim_H(\partial^*(\Phi(Q))) > 1$; hence $||\Phi(Q)|| = \infty$.*

**8. Lebesgue integrability.**

For the usual Henstock-Kurzweil integral it is not difficult, [9], to prove that if $A \subset \mathbf{R}^m$ is an interval and $f$ is HK-integrable on $A$ then there exists a non-degenerate interval $B \subset A$ such that $f$ is Lebesgue integrable on $B$.

However this result is not necessarily true for further generalizations. In [10] it is shown that there exists a $\mathcal{BV}$-integrable function $f : [0,1]^2 \to \mathbf{R}$ such that $f$ is not Lebesgue integrable on any portion of $[0,1]^2$.

Finally, a method for constructing counterexamples was provided in [8]. The $v$-integral was essentially an intermediate name for the $\mathcal{BV}$-integral. Denoting $[0,1]^2$ by $Q$ we showed that there exists an open BV set $H \subset Q$ and a function $f : Q \to \mathbf{R}$ such that

i) $f = 0$ on $Q \setminus H$;

ii) if $A \subset H$ is an interval then $f$ is Lebesgue integrable on $A$;

iii) $f$ is $v$-integrable on $H$;

iv) $f$ is not $v$-integrable on $Q$.

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