Functions of Two Variables with Large Tangent Plane Sets

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Running head: C^1-FUNCTIONS AND TANGENT PLANES

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ABSTRACT

We show that there exists a C^1 function, f, of two variables and a set $E \subseteq \mathbf{R}^2$ of zero Lebesgue measure such that using the natural three-dimensional parametrization of tangent planes $z = ax+by+c$ to the surface $z = f(x,y)$ the (three-dimensional) interior of the set of parameter values, (a,b,c), of tangent planes corresponding to points (x,y) in E is non-empty. From the Morse-Sard theorem it follows that there are no such C^2 functions. We also study briefly the relationship of our example with the Denjoy-Young-Saks theorem.

INTRODUCTION

Given $f(x,y)$, a differentiable function of two variables let, $S(x_0,y_0) = (a,b,c)$ denote the parameters of the equation $z = ax+by+c$ of the tangent plane to the surface $z = f(x,y)$ at the point (x_0,y_0). We give an example of a C^1 function, $f(x,y)$, and a nowhere dense closed set $E \subseteq \mathbf{R}^2$ with zero two-dimensional measure such that the interior of $S(E) \subseteq \mathbf{R}^3$ is nonempty. This implies, for example, that $\lambda_3(S(E)) > 0$, where λ_3 is the Lebesgue measure in $\mathbf{R}^3$.

Since from $S(x_0,y_0) = (a,b,c)$ it follows that $f(x_0,y_0) - ax_0 - by_0 = c$ and the partials of $f(x,y) - ax - by$ vanish at (x_0,y_0) our result can also be interpreted in the following way. There is an open set $G \subseteq \mathbf{R}^2$ such that for any $(a,b) \in G$ the set of critical values of $f(x,y) - ax - by$ contains a nonempty interval. For C^2 functions the Morse-Sard theorem [3], [5], [7] implies that for any $(a,b) \in \mathbf{R}^2$ the set of critical values of $f(x,y) - ax - by$ is of λ_1 measure zero, and hence cannot contain an interval. Therefore Fubini's theorem implies that for C^2 functions $S(E)$ is of zero λ_3-measure for any subset E of the domain of f.

The two variable version of the gradient problem of C. E. Weil [8] is the following. Let f be a differentiable function of two variables. Is it always true that for any open set $G \subseteq \mathbf{R}^2$ the set of those (x,y)'s for which $\nabla f(x,y) = (\partial_x f(x,y), \partial_y f(x,y))$ belongs to G is either empty, or of positive λ_2-measure?

Trying to give a positive answer to the gradient problem we found the counterexample discussed in our paper. If our counterexample did not exist that would answer the gradient problem. However its existence shows that there might be a counterexample for the gradient problem as well.

The one-dimensional positive answer to the gradient problem is the, so called, Denjoy-Clarkson property of the derivative [1], [2]. It is natural to ask what is true about the question discussed in our paper when we work with functions of one variable. Thus assume that I is an open interval and $f : I \to \mathbf{R}$ is a differentiable function of one variable and at $x_0 \in \mathbf{R}$ let $y = ax + b$ be the equation of its tangent line. Denote by $S_1 : I \to \mathbf{R}^2$ the mapping which assigns to each $x_0 \in I$ the parameters (a, b). Is it possible that $S_1(I)$ has non-empty interior? Or, is it possible that $\lambda_2(S_1(I)) > 0$? Well, if f is a C^1 function then by the one-dimensional version of Sard's theorem for any $a \in \mathbf{R}$ the critical values of $f(x) - ax$ are of measure zero and hence we obtain that for any fixed a the set of b's for which $(a, b) \in S_1(I)$ is of measure zero. Therefore it would be natural to ask whether $S_1(I)$ can be "larger" if we assume only the differentiability of f. However in this case a part of the Denjoy-Young-Saks theorem [7, Ch. IX. (3.7) Theorem, p.267] answers the question. It follows from this result that if $y = a'x + b'$ is a given line in the plane, π' denotes the othogonal projection onto this line, and E' denotes the π' image of those points of the graph of f where the tangent of f is perpendicular to $y = a'x + b'$ then $\lambda_1(E') = 0$. This easily implies that $\lambda_2(S_1(I)) = 0$ for differentiable functions as well.

Returning to dimension two one can consider the corresponding higher dimensional result [7, Ch. IX. (13.11) Theorem, p. 309]. "Given a set $R \subseteq \mathbf{R}^3$, let P be a subset of R at every point of which the set R has an extreme tangent plane parallel to a fixed straight line D. Then the orthogonal projection of P on the plane perpendicular to D is of plane measure zero." A variant of the above theorem would be the following. "Given a set $R \subseteq \mathbf{R}^3$, for any unit vector $\mathbf{n} \in \mathbf{R}^3$ let $P_{\mathbf{n}}$ be a subset of R at every point of which the set R has a tangent plane perpendicular to $\mathbf{n}$. Let $T_{\mathbf{n}}$ denote the orthogonal projection of $P_{\mathbf{n}}$ onto the line $\{t\mathbf{n} : t \in \mathbf{R}\}$ and T be the union of the sets $T_{\mathbf{n}}$ for all unit vectors $\mathbf{n}$. Is it true that T is always of λ_3-measure zero?" The example given in our paper provides a negative question to the above question. Let R be the graph of our function $f(x, y)$. An easy computation shows that the normal vector of the tangent plane to f at (x_0, y_0) is given by

$$\mathbf{n} = \left(\frac{-\partial_x f}{\sqrt{1+(\partial_x f)^2+(\partial_y f)^2}}, \frac{-\partial_y f}{\sqrt{1+(\partial_x f)^2+(\partial_y f)^2}}, \frac{1}{\sqrt{1+(\partial_x f)^2+(\partial_y f)^2}} \right),$$

where the derivatives should be evaluated at (x_0, y_0) and the parameter value t for which $t \cdot \mathbf{n} \in T_{\mathbf{n}}$ equals $(-x_0\partial_x f - y_0\partial_y f + f(x_0, y_0))/\sqrt{1+(\partial_x f)^2+(\partial_y f)^2}$ where, again, the derivatives should be taken at (x_0, y_0). Therefore choosing the open set $G \subseteq \mathbf{R}^2$ such that for any $(a, b) \in G$ the set of critical values of $f(x_0, y_0) - ax_0 - by_0$ contains an open interval we obtain that $T_{\mathbf{n}}$ also contains an interval for

$$\mathbf{n} = (\frac{-a}{\sqrt{1+a^2+b^2}}, \frac{-b}{\sqrt{1+a^2+b^2}}, \frac{1}{\sqrt{1+a^2+b^2}}).$$

This implies that T is of positive measure.

It is also worth mentioning some related results which show that C^1 functions in $\mathbf{R}^2$ can behave quite badly. The classical result is due to Whitney [9], showing that there are C^1 functions, f, of two variables and continuous arcs, γ, in the plane such that the gradient of f vanishes at each point of γ but f is not constant along γ. T. W. Körner [4] showed that there are functions $f(x, y)$ such that any two points (x_0, y_0) and (x_1, y_1) can be connected by an arc γ such that, apart from the endpoints, the gradient of f vanishes along γ.

NOTATION

The reals are denoted by $\mathbf{R}$. The projection of $\mathbf{R}^2$ onto the x and y-axis is denoted by π_x and π_y, respectively (that is, for example, $\pi_x(x, y) = x$). The Lebesgue measure in $\mathbf{R}^n$ is denoted by λ_n. For $A \subseteq \mathbf{R}$ we also use the alternate notation $|A| = \lambda_1(A)$.

MAIN RESULT

Theorem. *There exists a C^1-function $f : \mathbf{R}^2 \to \mathbf{R}$, a nowhere dense closed set $E \subseteq [0, 1] \times [0, 1]$ of zero two-dimensional measure and a nonempty open set $H \subseteq \mathbf{R}^3$ such that for any $(a, b, c) \in H$ we can find an $(x_0, y_0) \in E$ for which the equation of the tangent plane to the surface $z = f(x, y)$ at (x_0, y_0) is $z = ax + by + c$.*

Proof. The proof is organized into nine steps. In Step 9 the function $f(x,y)$ will be defined as the sum of two functions of one variable $g_{\ell_0,h_0}(x)$ and $g_{\ell_0',h_0'}(y)$. These functions of one variable are defined by an inductive process in Steps 1-3. The properties of these functions and of related sets are studied in Steps 4-7. In Step 8 we introduce two parallel constructions, with suitably "balanced" constants, which are used for our functions g_{ℓ_0,h_0} and $g_{\ell_0',h_0'}$.

Step 1 d-divisions of the unit square $Q=[0,1]\times[0,1]$.

Assume that $0\le d<1/2$. A d-division of Q will consist of 64 subparallelograms of Q, denoted by Q_j, $j=0,...,63$. Put

$$T_j=\left[\frac{j}{64},\frac{j}{64}+\frac{1}{128}\right]\times\left[\frac{j}{64},\frac{j}{64}+\frac{1}{256}\right].$$

If j is even put $Q_j=T_j$. If j is odd take the above T_j and keeping the lower left corner of T_j fixed increase the slope of the horizontal side of T_j to d, that is, Q_j is bounded by the lines $x=\frac{j}{64}$, $x=\frac{j}{64}+\frac{1}{128}$, $y=\frac{j}{64}+d(x-\frac{j}{64})$, $y=\frac{j}{64}+\frac{1}{256}+d(x-\frac{j}{64})$.

Let

$$\tilde{T}_j=\left[\frac{j}{64},\frac{j}{64}+\frac{1}{128}\right]\times\left[\frac{j}{64},\frac{j}{64}+\frac{1}{128}\right].$$

From $0<d\le 1/2$ it follows that $Q_j\subseteq\tilde{T}_j$ for any $j=0,...,63$. Elementary computation yields, that if (x,y), $(x',y')\in\cup_{j=0}^{63}Q_j\subseteq\cup_{j=0}^{63}\tilde{T}_j$ and these points belong to different Q_j's (which also implies that they belong to different $\tilde{T}_j$'s) then

$$\frac{1}{3}\le\frac{y'-y}{x'-x}\le 3. \tag{1}$$

Assume that $0\le d_0<1/128$. Let $\overline{\pi}_{d_0}$ denote the affine projection onto the y-axis parallel with the line $y=d_0x$, that is, $\overline{\pi}_{d_0}(x,y)=(0,y-d_0x)$. Put $\overline{I}=\overline{\pi}_{d_0}([0,1]\times[0,1])$. Then $1\le|\overline{I}|\le 129/128$. Also put $\overline{I}_j=\overline{\pi}_{d_0}(Q_j)$. Again an easy computation shows that the $\overline{I}_j$'s are disjoint and if we take any subinterval $\overline{J}$ of $\overline{I}$ with $|\overline{J}|>6/128$ then there is an even j and an odd j' such that $\overline{I}_j\cup\overline{I}_{j'}\subseteq\overline{J}$.

Step 2 Parallelograms and the mappings ϕ_R and ψ_R.

In the sequel we need to deal with parallelograms with two sides parallel to the y-axis. We will say that the parallelogram R is of the form $(x_R,y_R),\ell,h,m$ if (x_R,y_R) is its lower left corner $|\pi_x(R)| = \ell$, two of the sides of R are parallel to the y axis and are of length h, the other two sides are parallel to the line $y = mx$, that is, R is bounded by the lines $x = x_R$, $x = x_R + \ell$, $y = y_R + m(x - x_R)$, $y = y_R + h + m(x - x_R)$. We will call m the slope of the parallelogram R.

We denote by ϕ_R the affine transformation which maps R onto Q, and for which $\phi_R(x_R, y_R) = (0, 0)$, that is,

$$\phi_R(x, y) = (x - x_R, y - y_R) \begin{pmatrix} \frac{1}{\ell} & -\frac{m}{h} \\ 0 & \frac{1}{h} \end{pmatrix}.$$

We denote by ψ_R the inverse of ϕ_R, that is,

$$\psi_R(x, y) = (x, y) \begin{pmatrix} \ell & m\ell \\ 0 & h \end{pmatrix} + (x_R, y_R).$$

It is an easy computation to verify that lines parallel to the line $y = m'x$ are mapped by ϕ_R onto lines with the slope $(m' - m)\frac{\ell}{h}$ and lines parallel to the line $y = m''x$ are mapped by ψ_R onto lines with the slope $m + \frac{hm''}{\ell}$.

Step 3 Definition of the closed sets $F(\ell_0, h_0)$ and of the functions g_{ℓ_0,h_0}.

Given ℓ_0, h_0 with $\ell_0/h_0 \leq 1/128$ we define a closed subset $F(\ell_0, h_0) \subseteq [0, \ell_0] \times [0, h_0]$ such that letting $F_x(\ell_0, h_0) = \pi_x(F(\ell_0, h_0))$ for every $x_0 \in F_x(\ell_0, h_0)$ there exists exactly one y_0 such that $(x_0, y_0) \in F(\ell_0, h_0)$. Denoting this y_0 by $g_{\ell_0,h_0}(x_0)$ we will obtain a C^1-function on $F_x(\ell_0, h_0)$ (this property is verified in Step 4).

To define $F(\ell_0, h_0)$ put $F_0 = [0, \ell_0] \times [0, h_0]$.

Assume that F_n is defined and consists of 64^n disjoint parallelograms of the form $(x, y), \ell, h, \frac{k}{2^n}$ with $\ell/h \leq 2^n/128$, and $k \in \{0, 1, ..., 2^n - 1\}$. If R is a subparallelogram of F_n of the above form then using $d = \ell/h2^{n+1} < 1/128$ take a d-division of Q. Assume that the corresponding subparallelograms are $Q_0, ..., Q_{63}$. Let $R_j = \psi_R(Q_j)$ and $F_{n+1} \cap R = \cup_{j=0}^{63} R_j$. Define this way $F_{n+1} \subseteq F_n$ for all subparallelograms of F_n. Obviously F_{n+1} will be closed and will consist of 64^{n+1} parallelograms. Observe that if R was the above subparallelogram belonging to F_n then R_j is a parallelogram of the form $(x_j, y_j), \ell_j, h_j, \frac{k_j}{2^{n+1}}$ with $x_j = x + \ell j/64$, $y_j = y + (\frac{k}{2^n} + \frac{h}{\ell})\frac{\ell j}{64}$, $\ell_j = \ell/128$, $h_j = h/256$ and $k_j = 2k$ if j is even and $k_j = 2k+1$ if j is odd. It also follows that $\ell_j/h_j = 2\ell/h \leq 2^{n+1}/128$. It is easy to verify by mathematical induction that if $n' > n$, R is a parallelogram belonging to F_n with slope

$k/2^n$ and R' is a parallelogram belonging to $F_{n'} \cap R$ and m' denotes the slope of R' then $k/2^n \le m' < (k+1)/2^n$. It is also clear that $\pi_x(F_n)$ consists of 64^n disjoint closed intervals each of length $\ell_0/128^n$ that is $|\pi_x(F_n)| = \ell_0/2^n$, and if $x_0 \in \pi_x(F_n)$ then the vertical line $x = x_0$ intersects F_n in a closed interval of length $h_0/256^n$. Set $F(\ell_0, h_0) = \cap_{n=0}^{\infty} F_n$. This clearly implies that $|F_x(\ell_0, h_0)| = |\pi_x(F(\ell_0, h_0))| = 0$.

Step 4 Verification of the fact that the function g_{ℓ_0,h_0} is C^1.

We need to verify that g_{ℓ_0,h_0} is a C^1 function on $F_x(\ell_0, h_0)$. To this end we need to investigate the process used during the definition of F_{n+1}. We use the notation introduced in Step 3. Assume that R is a given subparallelogram of F_n with slope $k/2^n$ and $(x, y), (x', y') \in \cup_{j=0}^{63} R_j$ are not belonging to the same subparallelogram R_j. Then $\phi_R(x, y), \phi_R(x', y') \in \cup_{j=0}^{63} Q_j$ are not belonging to the same Q_j and by (1) the slope m'' of the line passing through $\phi_R(x, y)$ and $\phi_R(x', y')$ satisfies $1/3 \le m'' \le 3$. Therefore (see also the end of Step 2) if $m' = \frac{y'-y}{x'-x}$, which is the slope of the line passing through $(x, y) = \psi_R(\phi_R(x, y))$ and $(x', y') = \psi_R(\phi_R(x', y'))$, then

$$\frac{h}{\ell}\frac{1}{3} + \frac{k}{2^n} \le m' \le \frac{h}{\ell}3 + \frac{k}{2^n}.$$

By our definition of parallelograms belonging to F_n we have $\ell = \ell_0/128^n$, $h = h_0/256^n$ and hence we infer

$$\frac{1}{3 \cdot 2^n} + \frac{k}{2^n} \le m' \le \frac{3}{2^n} + \frac{k}{2^n}. \tag{2}$$

For each $x_0 \in F_x(\ell_0, h_0)$ there exists a unique sequence of parallelograms $R_0, R_1, ...$ such that $(x_0, g_{\ell_0,h_0}(x_0)) \in R_n$ and R_n is a subparallelogram of F_n. Then $R_0 \supset R_1 \supset ...$ and if $m_0, m_1, ...$ denotes the slopes of $R_0, R_1, ...$ respectively then for any n either $m_{n+1} = m_n$ or $m_{n+1} = m_n + \frac{1}{2^{n+1}}$. This implies that if $n' \ge n$ then

$$m_n \le m_{n'} \le m_n + \frac{1}{2^n} - \frac{1}{2^{n'}}. \tag{3}$$

Since $\pi_x(F_n)$ consists of 64^n disjoint closed intervals there is a $\delta_n > 0$ such that $(x_0 - \delta_n, x_0 + \delta_n) \cap F_x(\ell_0, h_0) \subseteq \pi_x(R_n)$. Assume that $x_1 \ne x_0$, $x_1 \in (x_0 - \delta_n, x_0 + \delta_n) \cap F_x(\ell_0, h_0)$. Then there exists $n' \ge n$ such that $(x_1, g_{\ell_0,h_0}(x_1)) \in R_{n'}$ but $(x_1, g_{\ell_0,h_0}(x_1)) \notin R_{n'+1}$. Then denoting by $R_{n',j}$ the subparallelograms of $R_{n'}$ which belong to $R_{n'} \cap F_{n'+1}$ the points

$(x_0, g_{\ell_0,h_0}(x_0))$ and $(x_1, g_{\ell_0,h_0}(x_1))$ belong to different $R_{n',j}$'s. From our construction it is clear that $m_{n'} = k_{n'}/2^{n'}$ with a suitable $0 \leq k_{n'} < 2^{n'}$ thus by (2) we have

$$\frac{1}{3 \cdot 2^{n'}} + m_{n'} \leq \frac{g_{\ell_0,h_0}(x_1) - g_{\ell_0,h_0}(x_0)}{x_1 - x_0} \leq \frac{3}{2^{n'}} + m_{n'}.$$

Using (3) we obtain

$$\frac{1}{3 \cdot 2^{n}} + m_{n} \leq \frac{g_{\ell_0,h_0}(x_1) - g_{\ell_0,h_0}(x_0)}{x_1 - x_0} \leq \frac{4}{2^{n}} + m_{n}.$$

Also from (3) it follows that m_n converges to a number, say m and we have $m_n \leq m \leq m_n + 1/2^n$. Thus

$$\frac{-2}{3 \cdot 2^{n}} + m \leq \frac{g_{\ell_0,h_0}(x_1) - g_{\ell_0,h_0}(x_0)}{x_1 - x_0} \leq \frac{4}{2^{n}} + m$$

for any $x_1 \in (x_0 - \delta_n, x_0 + \delta_n) \cap F_x(\ell_0, h_0)$, $x_1 \neq x_0$. This implies that $g'_{\ell_0,h_0}(x_0) = m$ where m is the limit of the slopes of the parallelograms R_n. An argument, similar to the above one, can imply that if $x'_0 \in (x_0 - \delta_n, x_0 + \delta_n) \cap F_x(\ell_0, h_0) \subseteq \pi_x(R_n)$ and $m' = g'_{\ell_0,h_0}(x'_0)$ then $m_n \leq m' \leq m_n + 1/2^n$ and hence $|m' - m| \leq 1/2^n$. From this it follows that g_{ℓ_0,h_0} is a C^1 function on $F_x(\ell_0, h_0)$. Using Whitney's extension theorem we can choose a C^1 extension of g_{ℓ_0,h_0} onto $\mathbf{R}$. In the sequel we will denote by g_{ℓ_0,h_0} this extended function.

Step 5 Definition of the parallelograms $R^m_{i_1,\ldots,i_n}$.

In Steps 5-7 we assume that ℓ_0, h_0 are fixed. Given $m \in [0,1)$ we want to find many points in $F_x(\ell_0, h_0)$ where $g'_{\ell_0,h_0} = m$. For the above given m choose a sequence $m_0, m_1, \ldots$ such that $0 \leq m - m_n < 1/2^n$ and m_n is of the form $k/2^n$ with an integer $0 \leq k < 2^n$. It is clear that F_1 consists of 32 parallelograms of slope 0 and of 32 parallelograms of slope 1/2 and, in general, the set F_n contains 32^n parallelograms of slope $k/2^n$ for any $k \in \{0, \ldots, 2^n - 1\}$. Assume that the 32 parallelograms in F_1 with slope m_1 are denoted by $R^m_{i_1}$, $(i_1 = 0, \ldots, 31)$. If the parallelograms $R^m_{i_1,\ldots,i_n}$ are defined then each $R^m_{i_1,\ldots,i_n}$ contains 32 parallelograms belonging to F_{n+1} with slope m_{n+1}, denote these parallelograms by $R^m_{i_1,\ldots,i_n,i_{n+1}}$, $(i_{n+1} = 0, \ldots, 31)$. Thus by induction we define for each n the parallelograms $R^m_{i_1,\ldots,i_n}$ such that $i_j \in \{0, \ldots, 31\}$ for $j = 0, \ldots, n$, furthermore for any sequence $i_1, i_2, \ldots$ we have $R^m_{i_1,\ldots,i_n,i_{n+1}} \subseteq R^m_{i_1,\ldots,i_n}$, and the slope of $R^m_{i_1,\ldots,i_n}$ is m_n. It is clear from Step 4 that if $x_0 = \pi_x(\cap_{n=1}^{\infty} R^m_{i_1,\ldots,i_n})$ then $g'_{\ell_0,h_0}(x_0) = m$.

Step 6 Further study of $R^m_{i_1,...,i_n}$'s and definition of the intervals $I^m_{i_1,...,i_n}$.

Assume that $R^m_{i_1,...,i_n}$ is a parallelogram of the from $(x_0, y_0), \ell, h, m_n$ with $\ell = \ell_0/128^n$, $h = h_0/256^n$. Put $d' = 1/2^{n+1}$, $d = \ell d'/h$, $d'_0 = m - m_n$. Then $0 \le d'_0 < 1/2^n$ and we also put $d_0 = \ell d'_0/h$. For ease of notation put $\phi = \phi_{R^m_{i_1,...,i_n}}$ and $\psi = \psi_{R^m_{i_1,...,i_n}}$. Then $R^m_{i_1,...,i_n,i_{n+1}}$ is a parallelogram of the form $\psi(Q_j)$ where Q_j is a parallelogram belonging to $Q_0, ..., Q_{63}$ which is a d-division of Q (see Step 3).

Lines parallel to $y = mx$ are mapped by ϕ onto lines which are parallel to $y = d_0 x$, (see the end of Step 2). Let $\pi_m(x, y) = y - mx$. Put $I^m_{i_1,...,i_n} = \pi_m(R^m_{i_1,...,i_n})$. Recall that $\overline{\pi}_{d_0}$ was defined in Step 1 and observe that $\psi \circ \overline{\pi}_{d_0} \circ \phi(x, y) = (x_0, y - m(x - x_0))$ and hence $(\pi_y \circ \psi \circ \overline{\pi}_{d_0} \circ \phi(x, y)) - mx_0 = \pi_m(x, y)$. Since $\phi(R^m_{i_1,...,i_n}) = Q$ and by Step 1 we know that $1 \le |\overline{I}| = |\overline{\pi}_{d_0}(Q)| \le 129/128$, we infer $h \le |\psi \circ \overline{\pi}_{d_0} \circ \phi(R^m_{i_1,...,i_n})| \le 129h/128$. Using that π_y and the translation by $-mx_0$ are not changing vertical distances we obtain that

$$\frac{h_0}{256^n} \le |I^m_{i_1,...,i_n}| \le \frac{129 h_0}{128 \cdot 256^n}.$$

Step 7 The J subinterval property of the $I^m_{i_1,...,i_n}$'s.

By our construction the parallelograms $R^m_{i_1,...,i_n,i_{n+1}}$ are either of the form $\psi(Q_{2i_{n+1}})$, $(i_{n+1} = 0, ..., 31)$ when $m_n \le m < m_n + \frac{1}{2^{n+1}}$, or of the form $\psi(Q_{2i_{n+1}+1})$, $(i_{n+1} = 0, ..., 31)$ when $m_n + \frac{1}{2^{n+1}} \le m < m_n + \frac{1}{2^n}$. In the first case put $\overline{J}_{i_{n+1}} = \overline{\pi}_{d_0}(Q_{2i_{n+1}})$, in the second put $\overline{J}_{i_{n+1}} = \overline{\pi}_{d_0}(Q_{2i_{n+1}+1})$. The remark at the end of Step 1 implies that if $\overline{J}$ is a subinterval of $\overline{I} = \overline{\pi}_{d_0}(Q) = \overline{\pi}_{d_0} \circ \phi(R^m_{i_1,...,i_n})$ with $|\overline{J}| > 6/128$ then there is an $i_{n+1} \in \{0, ..., 31\}$ such that $\overline{J}_{i_{n+1}} \subseteq \overline{J}$. Therefore an easy computation yields that if J is a subinterval of $I^m_{i_1,...,i_n} = (\pi_y \circ \psi \circ \overline{\pi}_{d_0} \circ \phi(R^m_{i_1,...,i_n})) - mx_0$ and $|J| > 6h/128 = 6h_0/(128 \cdot 256^n)$ then there is an $i_{n+1} \in \{0, ..., 31\}$ such that $I^m_{i_1,...,i_n,i_{n+1}} \subseteq J$. This last property will be referred to as the J subinterval property.

Step 8 Application of the J subinterval property for two parallel constructions.

Assume that $0 \le m < 1$, $0 \le m' < 1$, $\ell_0 = 1$, $h_0 = 128$, $\ell'_0 = 1$, $h'_0 = 2048 = 16h_0$. Put $R = [0, 1] \times [0, 128]$, $R' = [0, 1] \times [0, 2048]$, $I^m = \pi_m(R)$, $I^{m'} = \pi_{m'}(R')$. Then $[0, 128] \subseteq I^m \subseteq [-1, 128]$, $[0, 2048] \subseteq I^{m'} \subseteq [-1, 2048]$. If $c \in [128, 2047]$ then $J_0 = c - I^m \subseteq [0, 2048] \subseteq I^{m'}$ and $|J_0| \ge 128 > 6h'_0/128$ therefore by the J subinterval property there exists $i'_1 \in \{0, ..., 31\}$ such that $I^{m'}_{i'_1} \subseteq J_0$. Then using Step 6 we obtain

$$6 = \frac{6h_0}{128} < 8 = \frac{h'_0}{256} \leq |I^{m'}_{i'_1}| \leq \frac{129h'_0}{128 \cdot 256}.$$

Put $J'_1 = c - I^{m'}_{i'_1}$. Since $I^{m'}_{i'_1} \subseteq J_0 = c - I^m$ we have $J'_1 \subseteq I^m$. From $|J'_1| = |I^{m'}_{i'_1}| > 6h_0/128$ it follows that there exists i_1 such that $I^m_{i_1} \subseteq J'_1$. Then

$$\frac{6h'_1}{128} = \frac{6h'_0}{128 \cdot 256} = \frac{6 \cdot 16h_0}{128 \cdot 256} < \frac{h_0}{256} \leq |I^m_{i_1}| \leq \frac{129h_0}{128 \cdot 256}.$$

Put $J_1 = c - I^m_{i_1}$. Since $I^m_{i_1} \subseteq J'_1 = c - I^{m'}_{i'_1}$ we have $J_1 \subseteq I^{m'}_{i'_1}$. From $|J_1| = |I^m_{i_1}| > 6h'_1/128$ it follows that there exists i'_2 such that $I^{m'}_{i'_1,i'_2} \subseteq J_1$. Then

$$\frac{6h_1}{128} = \frac{6h_0}{128 \cdot 256} < \frac{h'_0}{256 \cdot 256} = \frac{h'_1}{256} \leq |I^{m'}_{i'_1,i'_2}| \leq \frac{129h'_1}{128 \cdot 256}.$$

Put $J'_2 = c - I^{m'}_{i'_1,i'_2}$...

From the above steps it should be clear that by mathematical induction we can choose sequences $i_1, i_2, ...,$ and $i'_1, i'_2, ...$ such that $I^m_{i_1,...,i_n} \subseteq c - I^{m'}_{i'_1,...,i'_n}$, and $I^{m'}_{i'_1,...,i'_n,i'_{n+1}} \subseteq c - I^m_{i_1,...,i_n}$ for any n. Since $|I^m_{i_1,...,i_n}| \to 0$, and $|I^{m'}_{i'_1,...,i'_n}| \to 0$ as $n \to \infty$ there are unique c_0 and c'_0 such that $c_0 = \cap_{n=1}^{\infty} I^m_{i_1,...,i_n}$ and $c'_0 = \cap_{n=1}^{\infty} I^{m'}_{i'_1,...,i'_n}$. It is also clear that $c_0 + c'_0 = c$. Let $(x_0, y_0) = \cap_{n=1}^{\infty} R^m_{i_1,...,i_n}$ and $(x'_0, y'_0) = \cap_{n=1}^{\infty} R^{m'}_{i'_1,...,i'_n}$. Then $y_0 - mx_0 = \pi_m(x_0, y_0) = c_0$ and $y'_0 - m'x'_0 = \pi_{m'}(x'_0, y'_0) = c'_0$. Using the properties of the functions g_{ℓ_0,h_0} and $g_{\ell'_0,h'_0}$ we have $y_0 = g_{\ell_0,h_0}(x_0)$, $y'_0 = g_{\ell'_0,h'_0}(x'_0)$, $g'_{\ell_0,h_0}(x_0) = m$, and $g'_{\ell'_0,h'_0}(x'_0) = m'$.

Step 9. Definition of $f(x, y)$.

Put $f(x, y) = g_{\ell_0,h_0}(x) + g_{\ell'_0,h'_0}(y)$. Then by the argument of Step 8 for any $m, m' \in [0, 1)$ and $c \in [128, 2047]$ there exist $x_0 \in F_x(\ell_0, h_0)$ and $x'_0 \in F_x(\ell'_0, h'_0)$ such that letting $a = m = g'_{\ell_0,h_0}(x_0) = \partial_x f(x_0, x'_0)$, and $b = m' = g'_{\ell'_0,h'_0}(x'_0) = \partial_y f(x_0, x'_0)$ we have $c = c_0 + c'_0 = g_{\ell_0,h_0}(x_0) - ax_0 + g_{\ell'_0,h'_0}(x'_0) - bx'_0 = f(x_0, x'_0) - ax_0 - bx'_0$ for any $(a, b, c) \in [0, 1) \times [0, 1) \times [128, 2047]$. Since it is easy to see that the set $E = F_x(\ell_0, h_0) \times F_x(\ell'_0, h'_0)$ is a nowhere dense closed set and $\lambda_2(E) = 0$ our proof is complete.

Questions and remarks. It is not difficult to verify that the set E in our Theorem is of Hausdorff dimension less than two. Since f is a C^1 function and for a fixed $(a, b, c) \in H$

the set $\{f(x,y) - ax - by : (x,y) \in E\}$ contains an interval it is not difficult to see that the set E should be at least of Hausdorff dimension 1. By changing some constants in our construction for any $\epsilon > 0$ one can obtain an example when the Hausdorff dimension of E is less than $1 + \epsilon$. However it is not clear for us whether there exists an example when E is of Hausdorff dimension 1.

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