DENSITY POINTS AND BI-LIPSCHITZ FUNCTIONS IN $\mathbf{R}^m$

BY

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Abstract

If $A, B \subset \mathbf{R}^m$ and f is a bi-Lipschitz function mapping A onto B then density or dispersion points of A are mapped exactly onto density or dispersion points of B respectively.

Introduction. W.F.Pfeffer asked the author about how density points of Lebesgue measurable sets are transformed by bi-Lipschitz mappings. We call a mapping f bi-Lipschitz if f and f^{-1} are both Lipschitz. By some authors bi-Lipschitz mappings are called lipeomorphisms, we prefer the term bi-Lipschitz since in our opinion it sounds much better. In this paper we prove that density and dispersion points are preserved by bi-Lipschitz mappings. If f and f^{-1} are defined on $\mathbf{R}^m$ then this statement is an easy exercise in measure theory. Assume that the bi-Lipschitz function f is defined on the set $A \subset \mathbf{R}^m$ and maps A onto $B \subset \mathbf{R}^m$. Then it is still easy to prove that dispersion points of A are mapped into dispersion points of B. The statement about density points is more involved. Unfortunately, in general one can not find a bi-Lipschitz extension $\tilde{f} : \mathbf{R}^m \to \mathbf{R}^m$ of $f : A \to B$ and reduce the problem to the easy case. One can only obtain by Kirszbraun's theorem a Lipschitz function $\tilde{f} : \mathbf{R}^m \to \mathbf{R}^m$ such that $\tilde{f}|_A = f$ and $\tilde{f}$ is Lipschitz but not necessarily bi-Lipschitz. The most difficult part of our proof is related to the range of $\tilde{f}$ and is of topological nature. If the range of $\tilde{f}$ contains large holes then it might happen

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that a density point is transformed into a non-density point. To be more precise assume that $p \in A$ is a density point of A and denote by $B(p,r)$ the ball centered at p and of radius $r > 0$. One has to show that $f(p) = \tilde{f}(p) \in B$ is a density point of B. To prove this we have to show that the range of $\tilde{f}$ can not contain big holes near $\tilde{f}(p)$. In fact we shall prove that there exists a $c > 0$, such that c depends only on the Lipschitz properties of f, and $\tilde{f}(B(p,r))$ contains a relatively big ball, namely $B(f(p), cr) \subset \tilde{f}(B(p,r))$, whenever $r > 0$ is sufficiently small.

We also remark that the almost every version of our theorem, that is, almost every density points are mapped into density points, follows easily from the result about dispersion points and Lebesgue's density theorem. The point in our theorem is that we prove our result about all density points.

We conclude the introduction by pointing out that Theorem 1 implies that each bi-Lipschitz function maps the *essential boundary* of its domain onto the essential boundary of its image. This corollary of Theorem 1 motivated W.F.Pfeffer's original question. In conjunction with [F, Theorem 4.5.11] the above corollary implies that the bi-Lipschitz image of a *Caccioppoli set* is again a Caccioppoli set. The usual proof of this fact rests on interpreting Caccioppoli sets as *integral currents* - a technique substantially more involved than that of the present paper (cf. [F, Chapter 4]).

Preliminaries. By $\mathbf{R}^m$ we denote the m-dimensional Euclidean space. If $x \in \mathbf{R}^m$ we denote by $||x||$ the distance of $x \in \mathbf{R}^m$ from the origin that is $||x|| = \sqrt{\sum_{i=1}^m x_i^2}$. The ball in $\mathbf{R}^m$ centered at p and of radius r is denoted by $B(p,r)$, its boundary the sphere is denoted by $S(p,r)$. By $\text{int}(A)$ we denote the interior of the set $A \subset \mathbf{R}^m$. The m-dimensional Lebesgue measure of the measurable set A is denoted by $\lambda(A)$. If $A \subset \mathbf{R}^m$ is Lebesgue measurable, $p \in \mathbf{R}^m$ is called a density point of A whenever

$$\lim_{r \to 0} \frac{\lambda(B(p,r) \cap A)}{\lambda(B(p,r))} = 1.$$

Dispersion points of a measurable set A are the density points of the complement of A. Recall that if $A \subset \mathbf{R}^m$ is measurable then by Lebesgue's density theorem almost every point of $\mathbf{R}^m$ is either a density or a dispersion point of A.

If $A, B \subset \mathbf{R}^m$ the function $f : A \to B$ is Lipschitz with constant L if $||f(x) - f(y)|| \leq L||x - y||$ for $x, y \in A$. The function mapping $A \subset \mathbf{R}^m$ onto $B \subset \mathbf{R}^m$ is bi-Lipschitz with constants L and L' if both f and f^{-1} are Lipschitz with constants L and L' respectively.

We recall Ascoli's theorem and its corollary see [R, Appendix A, A5, on p. 369].

Theorem A. *Suppose X is a compact space, $C(X)$ is the sup-normed Banach space of all continuous complex functions on X, and $\Phi \subset C(X)$ is pointwise bounded and equicontinuous. More explicitely,*

(a) $sup\{|f(x)| : f \in \Phi\} < \infty$ for every $x \in X$, and

(b) if $\epsilon > 0$, every $x \in X$ has a neighborhood V such that $|f(x) - f(y)| < \epsilon$ for all $y \in V$ and for all $f \in \Phi$.

Then Φ is totally bounded in $C(X)$.

Corollary of Theorem A. Since $C(X)$ is complete, the closure of Φ is compact, and every sequence in Φ contains a uniformly convergent subsequence.

We also remark that in our paper we need only the special case when X is a closed disk of $\mathbf{R}^m$ and the functions are real valued. One can obviously generalize Theorem A for mappings of $\mathbf{R}^m$ into $\mathbf{R}^m$, by applying the original version of Theorem A to the coordinate functions.

We also recall Kirszbraun's theorem [F, Theorem 2.10.43].

Theorem B. *If $F \subset \mathbf{R}^m$ and $f : F \to \mathbf{R}^m$ is a Lipschitz function with constant L then there exists a Lipschitz function $g : \mathbf{R}^m \to \mathbf{R}^m$ with constant L such that $f = g$ on F.*

For the topological part of the proof of our lemma we recall the following well-known topological fact [HW, Chapter IV, §2, B), p.40].

Theorem C. *Given $p \in \mathbf{R}^m$, and $r > 0$ there exists no continuous mapping f of $B(p, r)$ in $S(p, r)$ which keeps each point of $S(p, r)$ fixed.*

We also need Brouwer's Theorem on the Invariance of Domain [HW, Chapter VI, §6, Theorem VI 9, p.95].

Theorem D. *Let X be an arbitrary subset of $\mathbf{R}^m$ and h a homeomorphism of X on another subset $h(X)$ of $\mathbf{R}^m$. Then if x is an interior point of X, $h(x)$ is an interior point of $h(X)$.*

The main result. In this section first we state Theorem 1 and a lemma. Then we prove Theorem 1 from the lemma. Finally we shall prove the lemma.

Theorem 1. *Suppose that A and B are measurable subsets of $\mathbf{R}^m$, the function f maps A onto B, and f is $bi-Lipschitz$ with constants L and L'. Then f maps density or dispersion points of A into density or dispersion points of B respectively.*

Remark. We assumed in Theorem 1 that f is defined only on A and the statement in this case is true only about the density or dispersion points which belong to A. On the other hand if p belongs to the closure of A one can select a sequence $p_n \in A$, with $\lim_{n\to\infty} p_n = p$ and letting $f(p) = \lim_{n\to\infty} f(p_n)$ one can obtain an extension of f onto $A_p = A \cup \{p\}$. It is easy to see that this extension of f is well defined and bi-Lipschitz. Furthermore A_p and A has the same density points. A similar statement is true about B and $B_{f(p)} = B \cup \{f(p)\}$. Applying our theorem for f, A_p, and B_p we obtain that if p was a density or dispersion point of $\mathrm{cl}(A) \setminus A$ then one can find $f(p) \in \mathrm{cl}(B) \setminus B$ which is respectively a density or dispersion point of B.

Lemma. *Suppose that $F_n \subset B(\mathbf{0}, 1)$ and*

(1)
$$\lim_{n\to\infty} \lambda(F_n) = \lambda(B(\mathbf{0}, 1)).$$

Also suppose that $f_n : B(\mathbf{0}, 1) \to \mathbf{R}^m$ such that

(2)
$$f_n(\mathbf{0}) = \mathbf{0}, \quad n = 1, 2, ...;$$

and each f_n is Lipschitz with constant L which does not depend on n. Furthermore we assume that f_n^{-1} is well defined on $f_n(F_n)$ for every $n = 1, 2, \ldots$ and is Lipschitz with constant L' where again L' does not depend on n. Then

$$\lim_{n\to\infty} \lambda(f_n(B(\mathbf{0},1)) \cap B(\mathbf{0},1/L')) = \lambda(B(\mathbf{0},1/L')). \tag{3}$$

Proof of Theorem 1. We start with the easy part of the theorem, namely by proving that dispersion points are mapped into dispersion points.

Assume that $p \in A$ is a dispersion point. Then

$$\lim_{r\to 0} \frac{\lambda(A \cap B(p,r))}{\lambda(B(p,r))} = 0. \tag{4}$$

Put $q = f(p) \in B$. We have to show that q is a dispersion point of B. Since f is bi-Lipschitz with constants L and L' we have $f^{-1}(B(q,r) \cap B) \subset B(p, L'r)$. Assume that $\epsilon > 0$ is given and $\lambda(A \cap B(p, L'r)) < \epsilon r^m$. Then cover $A \cap B(p, L'r)$ by balls $B_1', B_2', B_3', \ldots$ such that $\sum_i \lambda(B_i') < 2\epsilon r^m$ and $B_i' \cap A \cap B(p, L'r) \neq \emptyset$. Replace the balls $B_1', B_2', B_3', \ldots$ by $B_1, B_2, B_3, \ldots$ such that each B_i is centered in $A \cap B(p, L'r)$; $B_i' \subset B_i$ and $\lambda(B_i) \leq 2^m \lambda(B_i')$. Then $\cup_i f(B_i)$ covers $B(q,r) \cap B$ and $\lambda(B(q,r) \cap B) \leq \sum_i \lambda(f(B_i)) \leq L^m \sum_i \lambda(B_i) \leq L^m 2^m \sum_i \lambda(B_i') < L^m 2^m 2\epsilon r^m$. Using (4) one can let $\epsilon \to 0$ as $r \to 0$ and prove that q is a dispersion point of B.

The proof of the fact that density points are mapped into density points is more involved. Assume that $p \in A$ is a density point and $f(p) = q$. For ease of notation we can assume that $p = q = \mathbf{0}$, indeed, by replacing f by $f(x+p) - q$; A by $A + p$; and B by $B - q$ we can reduce the general case to this one. Therefore we have to show that $q = \mathbf{0}$ is a density point of B.

Since $\mathbf{0}$ is a density point of A we have

$$\lim_{r\to 0} \frac{\lambda(A \cap B(\mathbf{0},r))}{\lambda(B(\mathbf{0},r))} = 1. \tag{5}$$

We have to show that $\mathbf{0}$ is also a density point of B. For a contradiction assume that this is not the case, that is, there exist $\eta > 0$ and $r_n > 0$, $r_n \to 0$ such that

$$\lambda(B \cap B(\mathbf{0}, r_n)) < (1-\eta)\lambda(B(\mathbf{0}, r_n)) \quad n = 1, 2, \ldots . \tag{6}$$

By Theorem B there exists a Lipschitz function $\phi : \mathbf{R}^m \to \mathbf{R}^m$ with constant L such that $\phi|_A = f|_A$. Therefore one can define the function ϕ^{-1} on $B = f(A)$ by $\phi^{-1} = f^{-1}$. Obviously ϕ^{-1} is Lipschitz with constant L'.

Next we prove that

$$\lim_{r \to 0} \frac{\lambda(\phi(B(\mathbf{0}, r) \setminus A))}{\lambda(B(\mathbf{0}, r))} = 0. \tag{7}$$

From (5) it follows that

$$\lim_{r \to 0} \frac{\lambda(B(\mathbf{0}, r) \setminus A)}{\lambda(B(\mathbf{0}, r))} = 0. \tag{8}$$

Using coverings of the set $B(\mathbf{0}, r) \setminus A$ by balls, and using $\phi(B(x, r)) \subset B(\phi(x), Lr)$ it is a simple matter to verify that $\lambda(\phi(B(\mathbf{0}, r) \setminus A)) < C \cdot \lambda(B(\mathbf{0}, r) \setminus A)$ with a constant C depending only on the dimension. This and (8) implies (7).

Put

$$\phi_n(x) = \frac{1}{L' r_n} \phi((L' r_n) \cdot x)\big|_{B(\mathbf{0},1)}$$

and $F_n = B(\mathbf{0}, 1) \cap \frac{1}{L' r_n} \cdot A$. Then (5) implies that $\lim_{n \to \infty} \lambda(F_n) = \lambda(B(\mathbf{0}, 1))$. Furthermore it is easy to see that $\phi_n(\mathbf{0}) = \mathbf{0}$ and ϕ_n is Lipschitz with constant L for $n = 1, 2, \ldots$. The definition of ϕ_n implies that $\phi_n(F_n) \subset \frac{1}{L' r_n} \cdot B$. Thus we can define ϕ_n^{-1} on $\phi_n(F_n)$ by $\phi_n^{-1} = \frac{1}{L' r_n} f^{-1}(L' r_n \cdot x)$. Obviously ϕ_n^{-1} is Lipschitz on $\phi_n(F_n)$ with constant L'. Therefore we can apply our lemma with $f_n = \phi_n$, $F_n = F_n$, $L = L$, $L' = L'$ and obtain that $\lim_{n \to \infty} \lambda(\phi_n(B(\mathbf{0}, 1)) \cap B(\mathbf{0}, 1/L')) = \lambda(B(\mathbf{0}, 1/L'))$. Thus

$$\lim_{n \to \infty} \lambda(\frac{1}{L' r_n} \phi(L' r_n \cdot B(\mathbf{0}, 1)) \cap B(\mathbf{0}, 1/L')) = \lambda(B(\mathbf{0}, 1/L')),$$

that is,

$$\lim_{n \to \infty} \frac{1}{(L' r_n)^m} \lambda(\phi(B(\mathbf{0}, L' r_n)) \cap B(\mathbf{0}, r_n)) = \lambda(B(\mathbf{0}, 1/L')).$$

Hence we obtain

$$\lim_{n \to \infty} \frac{\lambda(\phi(B(\mathbf{0}, L' r_n)) \cap B(\mathbf{0}, r_n))}{\lambda(B(\mathbf{0}, r_n))} = 1. \tag{9}$$

From (7) it follows that

$$(10) \qquad 0 = \lim_{n\to\infty} \frac{\lambda(\phi(B(\mathbf{0}, L'r_n) \setminus A))}{\lambda(B(\mathbf{0}, L'r_n))} = \frac{1}{L'^m} \lim_{n\to\infty} \frac{\lambda(\phi(B(\mathbf{0}, L'r_n) \setminus A))}{\lambda(B(\mathbf{0}, r_n))}.$$

Since $\lambda(\phi(B(\mathbf{0}, L'r_n)) \cap B(\mathbf{0}, r_n)) - \lambda(\phi(B(\mathbf{0}, L'r_n) \setminus A)) \leq \lambda(\phi(B(\mathbf{0}, L'r_n) \cap B(\mathbf{0}, r_n)) \setminus \phi(B(\mathbf{0}, L'r_n) \setminus A)) \leq \lambda(\phi(A) \cap B(\mathbf{0}, r_n)) = \lambda(B \cap B(\mathbf{0}, r_n))$ we obtain from (9) and (10) that

$$1 = \lim_{n\to\infty} \frac{\lambda(\phi(B(\mathbf{0}, L'r_n)) \cap B(\mathbf{0}, r_n)) - \lambda(\phi(B(\mathbf{0}, L'r_n) \setminus A))}{\lambda(B(\mathbf{0}, r_n))} \leq$$

$$\leq \lim_{n\to\infty} \frac{\lambda(B \cap B(\mathbf{0}, r_n))}{\lambda(B(\mathbf{0}, r_n))} \leq 1,$$

where the last inequality is obvious and implies that equality holds everywhere. This contradicts (6) and hence proves our theorem.

Proof of the Lemma. For a contradiction assume that (3) is not true. then there exitsts a subsequence $n' \to \infty$ and an $\eta > 0$ such that

$$(11) \qquad \lambda(f_{n'}(B(\mathbf{0}, 1)) \cap B(\mathbf{0}, 1/L')) < (1 - \eta)\lambda(B(\mathbf{0}, 1/L'))$$

for all n'. Since the functions $f_{n'}$ are Lipschitz with constant L they are, a fortiori, equicontinuous. Furthermore by (2), $f_{n'}(\mathbf{0}) = \mathbf{0}$ and we have $f_{n'}(B(\mathbf{0}, 1)) \subset B(\mathbf{0}, L)$, that is, the sequence $f_{n'}$ is uniformly bounded. Thus by Theorem A and its Corollary there is a subsequence n'' of n' and a continuous function $f : B(\mathbf{0}, 1) \to \mathbf{R}^m$ such that $f_{n''}$ converges uniformly to f. For ease of notation in the sequel we assume that this subsequence n'' is denoted by n. Since $||f_n(x) - f_n(y)|| \leq L||x - y||$ we have $||f(x) - f(y)|| \leq \lim_{n\to\infty} ||f_n(x) - f_n(y)|| \leq L||x - y||$, that is f is Lipschitz with constant L.

Next we show that f^{-1} is well defined on $f(B(\mathbf{0}, 1))$ and is Lipschitz with constant L'. Take $u = f(x)$, $v = f(y)$, $(x, y \in B(\mathbf{0}, 1))$ and $\epsilon_1 > 0$. Choose N_1 big enough such that for $n \geq N_1$ by (1) we have $B(x, \epsilon_1) \cap F_n \neq \emptyset$, $B(y, \epsilon_1) \cap F_n \neq \emptyset$; furthermore $||f(p) - f_n(p)|| < \epsilon_1$ for $p \in B(\mathbf{0}, 1)$. Fix an $n \geq N_1$ and select $x' \in B(x, \epsilon_1) \cap F_n$ and $y' \in B(y, \epsilon_1) \cap F_n$. Put $u' = f_n(x')$, $u'' = f_n(x)$, $v' = f_n(y')$, and $v'' = f_n(y)$. Since

f_n is Lipschitz we have $||u' - u''|| \leq L||x' - x|| \leq L\epsilon_1$, and $||v' - v''|| \leq L||y' - y|| \leq L\epsilon_1$. Thus $||u' - v'|| \leq ||u'' - v''|| + 2L\epsilon_1$. It follows from $x', y' \in F_n$ that $||x' - y'|| = ||f_n^{-1}(u') - f_n^{-1}(v')|| \leq L'||u' - v'||$. Since $||x - x'|| \leq \epsilon_1$, and $||y - y'|| \leq \epsilon_1$ we have $||x - y|| \leq ||x' - y'|| + 2\epsilon_1$. Therefore $||x - y|| \leq L'||u' - v'|| + 2\epsilon_1 \leq L'(||u'' - v''|| + 2L\epsilon_1)$. Finally using that $||u'' - u|| = ||f_n(x) - f(x)|| < \epsilon_1$ and $||v'' - v|| = ||f_n(y) - f(y)|| < \epsilon_1$ we obtain that $||x - y|| \leq L'(||u'' - v''|| + 2L\epsilon_1) < L'(||u - v|| + 2\epsilon_1 + 2L\epsilon_1)$. Since $\epsilon_1 > 0$ can be arbitrarily small this implies that $||x - y|| \leq L'||u - v||$. Thus for $x, y \in B(\mathbf{0}, 1)$, $x \neq y$, $f(x) = f(y) = u = v$ one would obtain $0 < ||x - y|| \leq L'||u - v|| = 0$. This is clearly impossible and hence f^{-1} is well defined on $f(B(\mathbf{0}, 1))$. Furthermore, we showed that $||f^{-1}(u) - f^{-1}(v)|| \leq L'||u - v||$ for $u, v \in f(B(\mathbf{0}, 1))$; and hence $f|_{B(\mathbf{0},1)}$ is bi-Lipschitz with constants L and L'. This also implies that f is a homeomorphism of the closed set $B(\mathbf{0}, 1)$ onto $f(B(\mathbf{0}, 1))$. By Theorem D, applied to f and f^{-1}, f maps $S(\mathbf{0}, 1)$ onto the boundary of $f(B(\mathbf{0}, 1))$. Since f^{-1} is Lipschitz with constant L' we obtain that $dist(\mathbf{0}, f(S(\mathbf{0}, 1))) \geq 1/L'$ and hence $B(\mathbf{0}, 1/L') \subset f(B(\mathbf{0}, 1))$.

Put $\tilde{H}_n = f_n(B(\mathbf{0}, 1)) \cap B(\mathbf{0}, 1/L')$ and $H_n = B(\mathbf{0}, 1/L') \setminus \tilde{H}_n$. We remark that $B(\mathbf{0}, 1)$ is compact and its continuous image $f_n(B(\mathbf{0}, 1))$ is also compact. Thus $\tilde{H}_n$ is also compact and H_n is an open subset of $B(\mathbf{0}, 1/L')$. According to (11) $\lambda(H_n) \geq \eta\lambda(B(\mathbf{0}, 1/L'))$. Therefore one can choose $\epsilon_2 > 0$ such that

$$H_n \cap \operatorname{int}(B(\mathbf{0}, (1 - \epsilon_2)/L')) \neq \emptyset. \tag{12}$$

For a fixed $\epsilon_3 > 0$ choose a continuous function $\phi : [0, 1/L'] \to [0, 1]$ such that $\phi(t) = 0$ for $t \in [0, (1 - \epsilon_3)/L']$; $\phi(t) = 1$ for $t \in [(1 - \frac{\epsilon_3}{2})/L', 1/L')]$; and ϕ is linear on $[(1 - \epsilon_3)/L', (1 - \frac{\epsilon_3}{2})L']$. If $x \in B(\mathbf{0}, 1/L')$ put

$$\Phi_n(x) = \phi(||x||) \cdot x + (1 - \phi(||x||)) \cdot f_n \circ f^{-1}(x).$$

Since $f^{-1}(x) \in B(\mathbf{0}, 1)$ the function $f_n \circ f^{-1}(x)$ is well defined. Obviously Φ_n is continuous. For a fixed $\epsilon_4 > 0$ choose an N such that

$$||f_N(x) - f(x)|| < \epsilon_4. \tag{13}$$

For any $x \in B(\mathbf{0}, 1/L')$ put $x' = f^{-1}(x)$. By using (13) we obtain $||f_N(x') - f(x')|| < \epsilon_4$, that is, $||f_N \circ f^{-1}(x) - x|| < \epsilon_4$. Therefore

$$\text{(14)} \qquad ||\Phi_N(x) - x|| = ||\phi(||x||) \cdot x + (1 - \phi(||x||)) f_N \circ f^{-1}(x) - x|| =$$

$$= (1 - \phi(||x||))||f_N \circ f^{-1}(x) - x|| < (1 - \phi(||x||))\epsilon_4 \leq \epsilon_4.$$

Choose $\epsilon_3 = \epsilon_2/2$, and $\epsilon_4 = \epsilon_3/4L'$, and an $N \in \mathbf{N}$ for ϵ_4.

It is obvious from (14) that for any $x \in B(\mathbf{0}, 1/L') \setminus B(\mathbf{0}, (1-\epsilon_3)/L')$ we have $\Phi_N(x) \in B(\mathbf{0}, 1/L') \setminus B(\mathbf{0}, (1 - \frac{5}{4}\epsilon_3)/L') \subset B(\mathbf{0}, 1/L') \setminus B(\mathbf{0}, (1 - \epsilon_2)/L')$. Thus whenever Φ_N maps an x from $B(\mathbf{0}, 1/L')$ into $B(\mathbf{0}, (1 - \epsilon_2)/L')$ then $x \in B(\mathbf{0}, (1 - \epsilon_3)/L')$ and $\Phi_N(x) = f_N \circ f^{-1}(x)$, that is,

$$\Phi_N\big(B(\mathbf{0}, 1/L')\big) \cap B(\mathbf{0}, (1 - \epsilon_2)/L') = \Phi_N\big(B(\mathbf{0}, (1 - \epsilon_3)/L')\big) \cap B(\mathbf{0}, (1 - \epsilon_2)/L') =$$

$$= f_N \circ f^{-1}(B(\mathbf{0}, (1 - \epsilon_3)/L')) \cap B(\mathbf{0}, (1 - \epsilon_2)/L') \subset f_N(B(\mathbf{0}, 1)) \cap B(\mathbf{0}, 1/L') = \tilde{H}_N.$$

By (12) we can choose $y \in H_N \cap \mathrm{int}(B(\mathbf{0}, (1-\epsilon_2)/L'))$. Recall that H_N is an open subset of $B(\mathbf{0}, 1/L')$, therefore a small neighborhood, U, of y also belongs to $H_N \cap B(\mathbf{0}, (1-\epsilon_2)/L')$. Thus $U \cap \Phi_N(B(\mathbf{0}, 1/L')) \subset U \cap \tilde{H}_N = \emptyset$, where we used that $U \subset B(\mathbf{0}, (1 - \epsilon_2)/L')$.

For any $x \in B(\mathbf{0}, 1/L')$ denote the half-line starting at y and passing through $\Phi_N(x)$ by ℓ_x. Denote by $\Phi(x)$ the intersection of ℓ_x with $S(\mathbf{0}, 1/L')$. Since $U \cap \Phi_N(B(\mathbf{0}, 1/L')) = \emptyset$ the function Φ is well defined. By the continuity of Φ_N, Φ is continuous. If $x \in S(\mathbf{0}, 1/L')$ then $x = \Phi_N(x) \in S(\mathbf{0}, 1/L')$ and hence $\Phi(x) = x$. Thus Φ is a continuous mapping of $B(\mathbf{0}, 1/L')$ onto $S(\mathbf{0}, 1/L')$ and $\Phi(x) = x$ for $x \in S(\mathbf{0}, 1/L')$. This contradicts Theorem B and concludes the proof of the lemma.

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