A question of J. Pavelich related to Diophantine approximation

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Abstract

In this paper we prove that there exist $q_1 > 0$ and a set S of uncountably many irrationals in the triadic Cantor set such that for all $x \in S$ $\left\langle 3^n/(\{3^n x\} - \frac{1}{2})\right\rangle > \frac{1}{6} + q_1$ for all but finitely many n's in $\mathbb{N}$. By $\{x\}$ and $\langle x\rangle$ we denote the fractional part of x and the distance of x from the closest integer, respectively. In fact, S is c-dense in the Cantor set. This answers a question asked by J. Pavelich. Our existence proof of irrationals uses discretization of the problem and computer programs.

1 Introduction

In this paper we will deal with a question asked by J. Pavelich [9]. This question originates from her Ph. D. thesis research at Caltech and was communicated to us by her advisor A. Kechris. The question originally looked technical, purely theoretical, and was expected to be related to number theory. Our solution has nothing to do with number theory. We needed to show that a certain exceptional set is sufficiently large. We had to find an existence proof of uncountably many irrationals with certain "bad approximation" properties. Instead of using number theory we used computer programs and discretization of the original "continuous problem" plus standard error estimation techniques to show that

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in the Cantor triadic set one can find an uncountable number of irrationals with this bad approximation properties.

Our computer programs also showed that the question is quite sharp, that is, checking the different possibilities needed several days of running time and a series of adjusted programs, plus methods of handling exceptional cases.

Here is the original question. Denote by $\{x\}$ the fractional part of x and let S^* be the set of all $x \in [0,1]$ that satisfy the following property:

For infinitely many positive integers n there is some positive integer k, such that

$$\left| \frac{\{3^n x\}}{3^n} - \frac{1}{2 \cdot 3^n} - \frac{1}{k} \right| < \frac{1}{6 \cdot k^2}. \tag{1}$$

It is not difficult to show that S^* is a dense G_δ set. Is it true that S^* is co-countable? Does it contain, for example, all the irrationals?

We give a negative answer to both of these questions. We will verify that the Cantor triadic set, C_3, contains uncountably many irrationals not in S^*.

The dynamics behind the problem is $\{3^n x\}$, that is, using base three representation of $x \in C_3$ the onesided shift on infinite sequences of zeros and twos. The main difficulty is coming from the factor 3^n in the denominator. This will make the estimation in (1) "sensitive" of at least of the first $2n+2$ digits of x.

In Section 2 of the paper we first rephrase the original question and state Theorem 1. The proof of this theorem is depending on some computer program running results. In the proof of Theorem 1 we discuss the theory behind these programs and use the results we obtained by them. In Section 3 we present some data we obtained from our computer programs and which is needed to understand the theoretical part of the paper.

After some search of the literature about earlier results related to Diophantine approximation and usage of computer programs to obtain "theoretical results" we should mention A. Deanin [4], she used computer programs to test Mahler's conjecture on best P-adic Diophantine Approximation constants and found a counterexample when $P = 83$. T. N. Langtry in [7] used simultaneous Diophantine approximation to obtain rank-1 lattice quadrature rules. On p. 1641 of [7] poorly approximated vectors were giving "good choices". We used in the purely theoretical paper [1] inhomogeneous Diophantine approximations and Kronecker's theorem to answer a long time unsolved problem of J. A. Haight and H. v. Weizsäcker, probably this motivated A. Kechris to communicate to us the approximation problem given in (1). For an example of usage of BASIC programs and discretization we refer to [8].

For classical results on badly approximable numbers we refer to Cassels [2], Davenport [3] and Schmidt [10] and [11].

Finally, a few words based on [9] on the analysis and descriptive set theory background of the problem which led to the diophantine approximation question of our paper. A natural $\prod_1^1$ rank, $|f|_{DIFF}$ on the set of differentiable functions was introduced in [6]. According to this rank $|f|_{DIFF} = 1$ if and only if $f \in C^1([0,1])$. For the concept of scales see [5]. In [9] a scale $\{\phi_{\epsilon,U} : \epsilon \in Q,\ U \in \mathcal{U}\}$ was introduced (where $Q = (0,1) \cap \mathbb{Q}$ and $\mathcal{U}$ denotes the set of $[0,1]$-relatively open intervals with rational endpoints). According to Proposition 3.4.2 of [9] if a sequence (f_n) of rank 1 functions converges in the scale $\{\phi_{\epsilon,U} : \epsilon \in Q,\ U \in \mathcal{U}\}$ and converges pointwise to a function $f \in C([0,1])$ then the limit function is differentiable and $f_n' \to f'$ uniformly. On the other hand Proposition 3.4.7 shows that there exists a sequence (f_n) of rank 2 differentiable functions which converges in the scale $\{\phi_{\epsilon,U} : \epsilon \in Q,\ U \in \mathcal{U}\}$, converges uniformly to 0, but on a comeager subset of $[0,1]$, $\lim_{n\to\infty} f_n'(x)$ does not exist. In the construction of (f_n) of this proposition approximation properties of reals became interesting and this led to the formulation of (1). To find an answer to (1) turned out to be surprisingly difficult. If we had a positive answer to (1) that would have implied that in Proposition 3.4.7 $f_n'(x)$ diverges on a cocountable set.

2 Main Result

First we rewrite (1) as

$$\frac{1}{k} - \frac{1}{6k^2} < \frac{\{3^n x\} - \frac{1}{2}}{3^n} < \frac{1}{k} + \frac{1}{6k^2}.$$

Then, by further rearrangement we obtain

$$k + \frac{k}{6k-1} > \frac{3^n}{\{3^n x\} - \frac{1}{2}} > k - \frac{k}{6k-1}. \tag{2}$$

If $x \in C_3$ then $|\{3^n x\} - \frac{1}{2}| \geq 1/6$ and hence if n and k satisfy (2) then

$$k > \frac{3^n}{12}. \tag{3}$$

Therefore, if we consider large n's for which we can find k's satisfying (1) or (2), by (3), k should also be large if $x \in C_3$.

We will denote by $\langle x \rangle$ the distance of x to the closest integer and put

$$\Phi_n(x) \stackrel{\text{def}}{=} \frac{3^n}{\{3^n x\} - \frac{1}{2}}.$$

Our aim is to show the following theorem.

Theorem 1. *There exist $q_1 > 0$ and a set S of uncountably many irrationals in C_3 such that for all $x \in S$*

$$\left\langle \frac{3^n}{\{3^n x\} - \frac{1}{2}} \right\rangle = \langle \Phi_n(x) \rangle > \frac{1}{6} + q_1 \tag{4}$$

for all but finitely many n's in $\mathbb{N}$. In fact, S is c-dense in C_3, by this we mean that any portion of C_3 contains continuum many points of S.

Since $k/(6k-1) \to 1/6$ as $k \to \infty$ from Theorem 1 it follows that $S \subset S^*$ and this gives a negative answer to J. Pavelich's question.

Proof. The small positive number q_1 can be chosen to be 3^{-100}.

We denote by $\tau_n(x)$ the n'th digit of $x \in [0,1]$ in base three, that is,

$$x = \sum_{n=1}^{\infty} \tau_n(x) 3^{-n}.$$

If $x \in C_3$ we assume that $\tau_n(x) \in \{0,2\}$. We will denote by $\mathcal{T}_\infty$ the set of infinite zero-two sequences and by $\mathcal{T}_n$ the set of n long zero-two sequences. If τ and ν are two sequences and τ is finite then the concatenation of them is simply denoted by $\tau\nu$.

Given n if $\tau = (\tau_k)_{k=1}^{\infty} \in \mathcal{T}_\infty$, or $\tau = (\tau_k)_{k=1}^{n} \in \mathcal{T}_n$ for an $m \geq n$ we denote by $\tau|n$ the first n terms of $\mathcal{T}_m$ (or of $\mathcal{T}_\infty$), while if $k < n \leq m$ is also given $\tau_{k,n}$ will denote the sequence $\tau_k \tau_{k+1} ... \tau_n$. If $\tau \in \mathcal{T}_\infty$ then $x(\tau)$ will denote the uniquely determined $x \in C_3$ with $\tau = (\tau_k(x))_{k=1}^{\infty}$. If $\tau \in \mathcal{T}_n$ then $X(\tau)$ will denote the set of those x's in C_3 for which $\tau = (\tau_k(x))_{k=1}^{n}$ and we also set

$$x(\tau) = \sum_{k=1}^{n} \tau_k 3^{-k} \text{ and } \overline{x}(\tau) = \sum_{k=1}^{n} \tau_k 3^{-k} + 0.5 \cdot 3^{-n}.$$

Observe that $X(\tau) \subset [\overline{x}(\tau) - 0.5 \cdot 3^{-n}, \overline{x}(\tau) + 0.5 \cdot 3^{-n}]$.

Based on our computer program runs we have a "forbidden ten digit sequence list", $\mathcal{F}_{10}$, which, by definition, consists of the following zero-two sequences of length ten $\{0220200002, 0220200022, 2002022200, 2002022220\}$.

For an integer $m > 10$ we will denote by $\mathcal{T}_{2m}^{\mathcal{F}}$ the set of those $2m$ long sequences in $\mathcal{T}_{2m}$ for which $\tau_{m,m+9} = \tau_m \tau_{m+1} ... \tau_{m+9} \notin \mathcal{F}_{10}$.

Assume $n \geq 10$. We say that $\tau \in \mathcal{T}_{2(n+1)} = \mathcal{T}_{2n+2}$ is q_1-continuable if there exists $m > n+1$ and $\tau' \in \mathcal{T}_{2m}^{\mathcal{F}}$ such that $\tau'|2n+2 = \tau$ and

$$\langle \Phi_k(x) \rangle > \frac{1}{6} + q_1 \text{ for all } x \in X(\tau') \text{ and } k = n, ..., m-2. \tag{5}$$

In this case we say that τ' is a q_1-continuation of τ. Observe that in (5) we use $k = n, ..., m-2$ and not $k = 2n+2, ..., 2m$. The set of q_1-continuable $2n+2$ long sequences will be denoted by $\mathcal{C}_{2n+2}$. The set of those sequences in $\mathcal{C}_{2n+2}$ for which a fixed $m > n+1$ can be used in the definition is denoted by $\mathcal{C}_{2n+2,2m}$. The set consisting of those $\tau' \in \mathcal{T}_{2m}^{\mathcal{F}}$ which are q_1-continuations of τ will be denoted by $\mathcal{C}_{2m}(\tau)$.

We say that $\tau \in \mathcal{T}_{2n+2}$ is q_1-double-continuable if there exists $m > n+1$ and two different $\tau', \tau'' \in \mathcal{C}_{2m}(\tau)$. The set of q_1-double-continuable $2n+2$ long sequences will be denoted by $\mathcal{D}_{2n+2}$. Again, the set of those sequences in $\mathcal{D}_{2n+2}$ for which a fixed $m > n+1$ can be used in the definition is denoted by $\mathcal{D}_{2n+2,2m}$.

We will show that

$$\text{given } n \geq 100 \text{ any } \tau \in \mathcal{T}_{2n+2}^{\mathcal{F}} \text{ is } q_1\text{-double-continuable.} \tag{6}$$

It is easy to see that (6) implies Theorem 1. Indeed, starting with arbitrary $\tau \in \mathcal{T}_{2n+2}^{\mathcal{F}}$ we can choose τ^0 and τ^1 such that they are both different

q_1-continuations of τ. If γ is a zero-one sequence and τ^γ is given we can again choose $\tau^{\gamma 0}$ and $\tau^{\gamma 1}$ such that they are both different q_1-continuations of τ^γ. Repeating the above procedure for any infinite zero-one sequence $\overline{\gamma}$ we can choose an infinite zero-two sequence $\tau^{\overline{\gamma}}$ such that $\tau^{\overline{\gamma}|n+1}$ is a q_1-continuation of $\tau^{\overline{\gamma}|n}$ for all n and if $\overline{\gamma}$ and $\overline{\gamma}'$ are different then $\tau^{\overline{\gamma}}$ and $\tau^{\overline{\gamma}'}$ are also different. From the repeated application of (5) it follows that for any zero-one sequence $\overline{\gamma}$ the point $x(\tau^{\overline{\gamma}}) \in S$. By the definition of q_1-continuation $\tau^{\overline{\gamma}} \in X(\tau)$ and hence $X(\tau) \cap S$ is uncountable, which implies that S is c-dense in C_3.

To complete the proof of Theorem 1 we need to verify (6).

Assume $n \geq 100$ and $\tau = \tau_1 ... \tau_{2n+2} \in \mathcal{T}^{\mathcal{F}}_{2n+2}$ are both fixed.

We put $\alpha_k(x) = \{3^k x\} - 1/2$ for a $k \in \mathbb{N}$ and $x \in [0,1]$.

Then $\Phi_k(x) = 3^k/\alpha_k(x)$.

To understand the motivation of the above definitions and to see how our computer programs work, first we will look at what happens if we take an arbitrary $\tau' \in \mathcal{T}_{2n+4}$ such that $\tau'|2n+2 = \tau$, that is, we add two new terms (digits) τ_{2n+3} and τ_{2n+4} to the sequence τ, and "hope" that τ' is a q_1-continuation of τ.

In this case

$$\overline{x}(\tau') - \overline{x}(\tau) = \tau_{2n+3} 3^{-2n-3} + \tau_{2n+4} 3^{-2n-4} - 0.5 \cdot 3^{-2n-2} + 0.5 \cdot 3^{-2n-4}, \quad (7)$$

and

$$\begin{aligned} &\alpha_n(\overline{x}(\tau')) - \alpha_n(\overline{x}(\tau)) = \\ &\qquad \tau_{2n+3} 3^{-n-3} + \tau_{2n+4} 3^{-n-4} - 0.5 \cdot 3^{-n-2} + 0.5 \cdot 3^{-n-4}. \end{aligned} \quad (8)$$

Observe that

$$|\alpha_n(\overline{x}(\tau')) - \alpha_n(\overline{x}(\tau))| < 0.5 \cdot 3^{-n-2}. \quad (9)$$

Using a first order Taylor polynomial approximation to the function $3^n/x$ and (8) we can approximate

$$\Phi_n(\overline{x}(\tau')) - \Phi_n(\overline{x}(\tau)) \quad (10)$$

by

$$\begin{aligned} &-\frac{3^n}{\alpha_n(\overline{x}(\tau))^2} (\alpha_n(\overline{x}(\tau')) - \alpha_n(\overline{x}(\tau))) = \\ &\qquad - \frac{1}{\alpha_n(\overline{x}(\tau))^2} \cdot \frac{1}{81} \cdot (\tau_{2n+3} 3 + \tau_{2n+4} - 4.5 + 0.5) \end{aligned} \quad (11)$$

(to keep the analogy with "higher level estimates" below we do not simplify $-4.5+0.5 = -4$). By (9) the error term of the above approximation is estimated as

$$\left| \frac{3^n}{\gamma^3} (\alpha_n(\overline{x}(\tau')) - \alpha_n(\overline{x}(\tau)))^2 \right| < \frac{3^{-n-4}}{|\gamma|^3}, \quad (12)$$

where γ is in the interval determined by $\alpha_n(\overline{x}(\tau'))$ and $\alpha_n(\overline{x}(\tau))$. An easy computation shows that $\alpha_n(\overline{x}(\tau'))$ and $\alpha_n(\overline{x}(\tau))$ are of the same sign and of absolute value between 1/6 and 1/2. Hence, $1/6 \leq |\gamma| \leq 1/2$ and the error term in (12) is less than $3^{-n-4} \cdot 6^3 < 3^{-99}$ which is very small and we can use (11) instead of (10).

Unfortunately, to find $\langle \Phi_n(\overline{x}(\tau)) \rangle$ one needs all digits $\tau_{n+1}, ..., \tau_{2n+2}$ and a computation similar to the above error estimates shows that all these digits are significant, so we cannot use rough error estimates, we need all of them.

Hence the only information we have is that $\widetilde{x} = \Phi_n(\overline{x}(\tau))$ is somewhere in $\mathbb{R}$ and "hope" to be able to choose the next two digits (τ_{2n+3} and τ_{2n+4}) of τ' so that

$$\langle \Phi_n(x) \rangle > \frac{1}{6} + q_1 \text{ for any } x \in X(\tau'). \tag{13}$$

If $x \in X(\tau')$ then $\tau(x)|2n+4 = \tau'$ and estimations like (10-12) show that

$$|\Phi_n(x) - \Phi_n(\overline{x}(\tau'))| \leq \left| -\frac{1}{\alpha_n(\overline{x}(\tau'))^2} \right| \cdot \frac{1}{81} \cdot 0.5 + 3^{-99} \stackrel{\text{def}}{=} \widetilde{\epsilon}_1 + 3^{-99}. \tag{14}$$

We want to find τ_{2n+3} and τ_{2n+4} such that $\tau' \in \mathcal{T}^{\mathcal{F}}_{2n+4}$ and

$$\langle \Phi_n(\overline{x}(\tau')) \rangle > \frac{1}{6} + q_1 + \widetilde{\epsilon}_1 + 3^{-99}. \tag{15}$$

Then, by (14) we have (13).

The problem is that we do not have any information about $\widetilde{x}$. Hence we try to show that for any possible value of $\widetilde{x}$ we can find suitable τ_{2n+3} and τ_{2n+4}.

The value of $\widetilde{x}$ is interesting for us only modulo one, hence we need to show that given any $\widetilde{x} \in [0,1)$ we can find τ_{2n+3} and τ_{2n+4} such that our estimated value of

$$\Phi_n(\overline{x}(\tau')) = \Phi_n(\overline{x}(\tau)) + \Big(\Phi_n(\overline{x}(\tau')) - \Phi_n(\overline{x}(\tau)) \Big) \approx$$

$$\widetilde{x} - \frac{1}{\alpha_n(\overline{x}(\tau))^2} \cdot \frac{1}{81} \cdot (-4.5 + 0.5 + \tau_{2n+3}3 + \tau_{2n+4}) \stackrel{\text{def}}{=} \widetilde{\Psi}_1(\widetilde{x}, \tau_{2n+3}, \tau_{2n+4}) \tag{16}$$

satisfies (15). We cannot use a computer program to check the existence of suitable τ_{2n+3} and τ_{2n+4} for all $\widetilde{x} \in [0,1)$. So we need to discretize our problem. We choose a large number $\widetilde{n} \geq 100$ and for any $\widetilde{x}_{\widetilde{i}} = \widetilde{i}/\widetilde{n}$, $\widetilde{i} = 0, 1, ..., (\widetilde{n}-1)$ we want to find τ_{2n+3} and τ_{2n+4} such that the following stronger version of (15) holds for our estimate from (16)

$$\langle \widetilde{\Psi}_1(\widetilde{x}_{\widetilde{i}}, \tau_{2n+3}, \tau_{2n+4}) \rangle > \frac{1}{6} + q_1 + \widetilde{\epsilon}_1 + 3^{-99} + \frac{1}{2\widetilde{n}}. \tag{17}$$

Then for an arbitrary $\widetilde{x} \in [0,1)$ we can choose $\widetilde{i}$ such that

$$\min\{|\widetilde{x} - \widetilde{x}_{\widetilde{i}}|, 1 - |\widetilde{x} - \widetilde{x}_{\widetilde{i}}|\} \leq \frac{1}{2\widetilde{n}}$$

and by (17) used with this $\widetilde{i}$ we have

$$\langle\widetilde{\Psi}_1(\widetilde{x},\tau_{2n+3},\tau_{2n+4})\rangle \geq \langle\widetilde{\Psi}_1(\widetilde{x}_{\widetilde{i}},\tau_{2n+3},\tau_{2n+4})\rangle - \frac{1}{2\widetilde{n}} > \frac{1}{6} + q_1 + \widetilde{\epsilon}_1 + 3^{-99}. \tag{18}$$

There is one more estimation we need to do.
In the definition of $\widetilde{\Psi}_1(\widetilde{x},\tau_{2n+3},\tau_{2n+4})$ and in (11) we still have a factor $-1/\alpha_n(\overline{x}(\tau))^2$ which we can only approximate in our computer programs. The exact value of $\alpha_n(\overline{x}(\tau))$ is determined by $\tau_{n+1,2n+2}$, but instead of using this exact value, for a given κ (for our computer programs κ was somewhere between 12 and 17) we use the approximate value $\tau_{n+1,n+\kappa}$, that is, we work with $\tau^\kappa \stackrel{\text{def}}{=} (\tau|n+\kappa)$, instead of τ. Then

$$|x(\tau^\kappa) - \overline{x}(\tau)| < 3^{-n-\kappa} \tag{19}$$

and

$$|\alpha_n(x(\tau^\kappa)) - \alpha_n(\overline{x}(\tau))| < 3^{-\kappa}. \tag{20}$$

Keeping in mind that $\alpha_n(x(\tau^\kappa))$ and $\alpha_n(\overline{x}(\tau))$ are of the same sign and of absolute value in $[1/6, 1/2]$ by the Mean Value Theorem we can choose a γ_0 with absolute value in $[1/6, 1/2]$ for which

$$-\frac{1}{\alpha_n(\overline{x}(\tau))^2} = -\frac{1}{\alpha_n(x(\tau^\kappa))^2} + \frac{2}{\gamma_0^3}\Big(\alpha_n(\overline{x}(\tau)) - \alpha_n(x(\tau^\kappa))\Big)$$

and hence

$$\left|\frac{-1}{\alpha_n(\overline{x}(\tau))^2} - \frac{-1}{\alpha_n(x(\tau^\kappa))^2}\right| < (2\cdot 6^3)\cdot 3^{-\kappa}. \tag{21}$$

Recalling that $\tau'|2n+2 = \tau$ one can obtain similarly

$$\left|\frac{-1}{\alpha_n(\overline{x}(\tau'))^2} - \frac{-1}{\alpha_n(x(\tau^\kappa))^2}\right| < (2\cdot 6^3)\cdot 3^{-\kappa}. \tag{22}$$

Set now

$$\begin{aligned}\Psi_1(\widetilde{x},\tau_{2n+3},\tau_{2n+4}) &= \widetilde{x} - \frac{1}{\alpha_n(x(\tau^\kappa))^2}\frac{1}{81}(-4.5+0.5+\tau_{2n+3}3+\tau_{2n+4}) = \\ &\widetilde{x} - \frac{1}{(-0.5+\frac{\tau_{n+1}}{3}+...+\frac{\tau_{n+\kappa}}{3^\kappa})^2}\cdot\frac{1}{81}(-4.5+0.5+\tau_{2n+3}3+\tau_{2n+4}).\end{aligned} \tag{23}$$

From (21) it follows that

$$|\Psi_1(\widetilde{x},\tau_{2n+3},\tau_{2n+4}) - \widetilde{\Psi}_1(\widetilde{x},\tau_{2n+3},\tau_{2n+4})| < \frac{4.5}{81}\cdot 2\cdot 6^3\cdot 3^{-\kappa} \stackrel{\text{def}}{=} \epsilon_{2,\kappa}. \tag{24}$$

In (14) $\widetilde{\epsilon}_1$ still depends on too many digits of τ', so we estimate the error we make if we use

$$\epsilon_1 \stackrel{\text{def}}{=} \left| -\frac{1}{\alpha_n(x(\tau^\kappa))^2} \right| \cdot \frac{1}{81} \cdot 0.5,$$

instead of $\widetilde{\epsilon}_1$. Then by (22)

$$\widetilde{\epsilon}_1 < \epsilon_1 + (2 \cdot 6^3) \cdot 3^{-\kappa} \frac{1}{81} 0.5 < \epsilon_1 + 3^{-\kappa+1}. \tag{25}$$

Assume that for any $\widetilde{i} \in \{0, ..., \widetilde{n} - 1\}$ we can choose τ_{2n+3} and τ_{2n+4} such that

$$\begin{aligned}\langle \Psi_1(\widetilde{x}_{\widetilde{i}}, \tau_{2n+3}, \tau_{2n+4}) \rangle > & \frac{1}{6} + q_1 + \epsilon_1 + 3^{-99} + 3^{-\kappa+1} + \frac{1}{2\widetilde{n}} + \epsilon_{2,\kappa} > \\ & \frac{1}{6} + q_1 + \widetilde{\epsilon}_1 + 3^{-99} + \frac{1}{2\widetilde{n}} + \epsilon_{2,\kappa}.\end{aligned} \tag{26}$$

Then (17) also holds and this implies that for any $\widetilde{x}$ in $[0, 1)$ we have (18). Taking the special case when $\widetilde{x} = \{\Phi_n(\widetilde{x}(\tau))\}$ we obtain (13). This, provided, the ten digit sequence $\tau_{n+2}...\tau_{n+11} \notin \mathcal{F}_{10}$, warrants that $\tau \in \mathcal{C}_{2n+2,2(n+2)}$ and $\tau' \in \mathcal{C}_{2(n+2)}(\tau)$, which shows that we have a q_1-continuation $\tau' \in \mathcal{T}^{\mathcal{F}}_{2n+4}$ of $\tau \in \mathcal{T}^{\mathcal{F}}_{2(n+1)}$.

So we have a positive answer to the q_1-continuation problem of τ if we can give a positive answer to the following discrete problem.

DP1: *We can choose a large $\widetilde{n} \geq 100$ and a (not too large) $\kappa \in \mathbb{N}$ such that for any $\widetilde{i} \in \{0, 1, ..., (\widetilde{n} - 1)\}$ we can choose $\tau_{2n+3}, \tau_{2n+4} \in \{0, 2\}$ such that* (26) *holds and $\tau_{n+2,n+11} \notin \mathcal{F}_{10}$.*

This can be easily checked by a computer program, provided our choice of $\widetilde{n}$ and κ gives a reasonable running time.

Given $k \leq n + 2$ and a sequence $\nu = \nu_1...\nu_k$ we denote by $\mathcal{T}_{2n+2}(\nu)$ the set of those $\tau \in \mathcal{T}_{2n+2}$ for which $\tau_{n+j} = \nu_j$ for $j = 1, ..., k$.

Next, observe that in the definition of Ψ_1 we use only the digits $\tau_{n+1}, ..., \tau_{n+\kappa}$ of $\tau \in \mathcal{T}^{\mathcal{F}}_{2n+2}$. Hence any positive answer by our computer program provides a positive answer to any $\widetilde{\tau} \in \mathcal{T}_{2n+2}(\tau_{n+1,...,n+\kappa})$.

Of course, we are interested in the existence of q_1-double-continuations and to obtain a positive answer to this problem we need to find for any $\widetilde{i} \in \{0, 1, ..., (\widetilde{n} - 1)\}$ two different sequences $\tau_{2n+3}\tau_{2n+4}$, and $\tau'_{2n+3}\tau'_{2n+4}$ of length two such that (26) holds with τ_{2n+3} and τ_{2n+4} and also with τ'_{2n+3} and τ'_{2n+4}. But most of the time we do not have this with the above "level one" program. This means that we need to analyze longer continuations of τ by "higher level" programs. For finding q_1-double-continuations we will need for "most parameter" values a "level three" program, and to deal with some specific exceptional values we need to run some "level four" programs as well. (The running time of these programs with required accuracy is a serious limitation and needed quite a bit of experimentation and balancing of constants plus some programming tricks to speed up these programs.)

We call a sequence $\nu \in \mathcal{T}_{10} \backslash \mathcal{F}_{10}$ universally-q_1-continuable (u-q_1-c, for short) if for any $n \geq 100$ and $\tau \in \mathcal{T}^{\mathcal{F}}_{2n+2}$ from $\tau_{n+1,n+10} = \nu$ it follows that $\tau \in \mathcal{C}_{2n+2}$.

The set of u-q_1-c sequences will be denoted by $\mathcal{UC}$. For an $m \in \{1, ..., 4\}$ we will denote by $\mathcal{UC}(m)$ the set of those $\nu \in \mathcal{UC}$ for which for any $n \geq 100$ and $\tau \in \mathcal{T}_{2n+2}^{\mathcal{F}}$ from $\tau_{n+1,n+10} = \nu$ it follows that $\tau \in \mathcal{C}_{2n+2,2(n+1+m)}$. We also put $\mathcal{UC}(\leq m) = \cup_{j=1}^{m} \mathcal{UC}(j)$. Our computer programs show that any $\nu \in \mathcal{T}_{10} \setminus \mathcal{F}_{10}$ belongs to $\mathcal{UC}(\leq 4)$, actually this is why we chose $\mathcal{F}_{10}$.

Similarly, a sequence $\nu \in \mathcal{T}_{10} \setminus \mathcal{F}_{10}$ is universally-q_1-double-continuable (u-q_1-dc, for short) if for any $n \geq 100$ and $\tau \in \mathcal{T}_{2n+2}^{\mathcal{F}}$ from $\tau_{n+1,n+10} = \nu$ it follows that $\tau \in \mathcal{D}_{2n+2}$ and the set of u-q_1-dc sequences will be denoted by $\mathcal{UD}$. The sets $\mathcal{UD}(m)$ and $\mathcal{UD}(\leq m)$ will be defined analogously to $\mathcal{UC}(m)$ and $\mathcal{UC}(\leq m)$. Our computer programs show that $\mathcal{UD}(\leq 3)$ contains at least $2^{10} - 278$ elements.

By running a computer program based on discrete problem DP1 with a $\kappa \geq 11$ one can obtain a list of sequences in $\mathcal{UC}(1)$. To do so we just need to choose a sufficiently large $\widetilde{n}$ and for a $\nu \in \mathcal{T}_{10} \setminus \mathcal{F}_{10}$ one should consider all sequences $\nu'' = \nu\nu'$, where $\nu' \in \mathcal{T}_{\kappa-10}$ and use $\tau_{n+j} = \nu''_j$ for $j = 1, ..., \kappa$ in (23), in the definition of $\Psi_1(\widetilde{x}, \tau_{2n+3}, \tau_{2n+4})$ (of course now, due to "universality", n is of no importance any more and τ_{2n+3}, τ_{2n+4} can be replaced by constants like $\widetilde{c}_1$, $\widetilde{c}_2$). If for any $\nu' \in \mathcal{T}_{\kappa-10}$ and $\widetilde{i} \in \{0, 1, ..., \widetilde{n} - 1\}$ we can find $\widetilde{c}_1$, $\widetilde{c}_2 \in \{0, 2\}$ such that

$$\langle \Psi_1(\widetilde{x}_{\widetilde{i}}, \widetilde{c}_1, \widetilde{c}_2) \rangle > \frac{1}{6} + q_1 + \epsilon_1 + 3^{-99} + 3^{-\kappa+1} + \frac{1}{2\widetilde{n}} + \epsilon_{2,\kappa} \tag{27}$$

and ν, $\nu''_{2,11} \notin \mathcal{F}_{10}$ then we know that $\nu \in \mathcal{UC}(1)$. If in the above algorithm we can always find two different $\widetilde{c}_1\widetilde{c}_2$ and $\widetilde{c}'_1\widetilde{c}'_2$ then we know that $\nu \in \mathcal{UD}(1)$.

Next we look at what should we do if for a given $l \in \{2, 3, 4\}$ we want to find sequences $\nu \in \mathcal{UC}(\leq l)$ and/or $\nu \in \mathcal{UD}(\leq l)$. We want to see how the "level l" algorithm works.

So, assume that $n \geq 100$, $\tau \in \mathcal{T}_{2n+2}^{\mathcal{F}}$ and $\tau_{n+1,n+10} = \nu$.

We want to find a suitable $\tau' \in \mathcal{T}_{2(n+1+l)}$ such that $\tau'|2n+2 = \tau$, and "hope" that $\tau' \in \mathcal{C}_{2(n+1+l)}(\tau)$.

In this case

$$\overline{x}(\tau') - \overline{x}(\tau) = -0.5 \cdot 3^{-2n-2} + 0.5 \cdot 3^{-2n-2-2l} + \sum_{j=1}^{2l} \tau_{2n+2+j} 3^{-2n-2-j}, \tag{28}$$

and for $k \in \{0, ..., l-1\}$

$$\alpha_{n+k}(\overline{x}(\tau')) - \alpha_{n+k}(\overline{x}(\tau)) = \tag{29}$$

$$-0.5 \cdot 3^{-n-2+k} + 0.5 \cdot 3^{-n-2-2l+k} + \sum_{j=1}^{2l} \tau_{2n+2+j} 3^{-n+k-2-j}.$$

Observe that

$$|\alpha_{n+k}(\overline{x}(\tau')) - \alpha_{n+k}(\overline{x}(\tau))| < 0.5 \cdot 3^{-n-2+k}. \tag{30}$$

Using a first order Taylor polynomial approximation to the function $3^{n+k}/x$ and (29) this time we approximate

$$\Phi_{n+k}(\overline{x}(\tau')) - \Phi_{n+k}(\overline{x}(\tau)) \tag{31}$$

by

$$-\frac{3^{n+k}}{\alpha_{n+k}(\overline{x}(\tau))^2}\Big(\alpha_{n+k}(\overline{x}(\tau'))-\alpha_{n+k}(\overline{x}(\tau))\Big)= \tag{32}$$
$$-\frac{1}{\alpha_{n+k}(\overline{x}(\tau))^2}\cdot\frac{1}{81}\cdot\Big(-0.5\cdot 3^{2k+2}+0.5\cdot 3^{2k+2-2l}+\sum_{j=1}^{2l}\tau_{2n+2+j}3^{2k+2-j}\Big).$$

By (30) the error term of the above approximation can be estimated as

$$\left|\frac{3^{n+k}}{\gamma_k^3}(\alpha_{n+k}(\overline{x}(\tau'))-\alpha_{n+k}(\overline{x}(\tau)))^2\right|<\frac{3^{-n-4+3k}}{|\gamma_k|^3}, \tag{33}$$

where $1/6\leq|\gamma_k|\leq 1/2$ and the error term in (33) is less than $3^{-n-4+3k}\cdot 6^3<3^{-99+3k}$ which is very small when $k\in\{1,...,4\}$ and we can use (32) instead of (31).

To find $\langle\Phi_{n+k}(\overline{x}(\tau))\rangle$ one needs $\tau_{n+1,2n+2}$.

Hence the only information we have is that $\widetilde{x}_k=\Phi_{n+k}(\overline{x}(\tau))$ is somewhere in $\mathbb{R}$ and "hope" to be able to choose the $2l$ digits, $\tau_{2n+3,2(n+1+l)}$ of τ' so that

$$\langle\Phi_{n+k}(x)\rangle>\frac{1}{6}+q_1 \text{ for any } k\in\{0,...,l-1\} \text{ and } x\in X(\tau'). \tag{34}$$

If $x\in X(\tau')$ then $\tau(x)|2(n+1+l)=\tau'$ and estimations like (31-33) show that

$$|\Phi_{n+k}(x)-\Phi_{n+k}(\overline{x}(\tau'))|\leq \tag{35}$$
$$\left|-\frac{1}{\alpha_{n+k}(\overline{x}(\tau'))^2}\right|\cdot\frac{1}{81}\cdot 0.5\cdot 3^{2k+2-2l}+3^{-99+3k}\stackrel{\text{def}}{=}\widetilde{\epsilon}_{l,k}+3^{-99+3k}.$$

We want to find $\tau_{2n+3,2(n+1+l)}\in\mathcal{T}_{2l}$ such that $\tau'\in\mathcal{T}^{\mathcal{F}}_{2(n+1+l)}$ and

$$\langle\Phi_{n+k}(\overline{x}(\tau'))\rangle>\frac{1}{6}+q_1+\widetilde{\epsilon}_{l,k}+3^{-99+3k} \text{ for } k\in\{0,...,l-1\}. \tag{36}$$

Then, by (35) we have (34).

The problem is that we do not have any information about any of the $\widetilde{x}_k$, $k\in\{0,...,l-1\}$. Hence again we try to show that for any possible value of $\widetilde{x}_k$, $k=0,...,l-1$ we can find a suitable $\tau_{2n+3,2(n+1+l)}$.

The value of $\widetilde{x}_k$ is interesting for us only modulo one, hence we need to show that given any $\widetilde{x}_k\in[0,1)$, $k\in\{0,...,l-1\}$ we can find $\tau_{2n+3,2(n+1+l)}$ such that our estimated value of

$$\Phi_{n+k}(\overline{x}(\tau'))=\Phi_{n+k}(\overline{x}(\tau))+\Big(\Phi_{n+k}(\overline{x}(\tau'))-\Phi_{n+k}(\overline{x}(\tau))\Big)\approx$$
$$\widetilde{x}_k-\frac{1}{\alpha_{n+k}(\overline{x}(\tau))^2}\cdot\frac{1}{81}\cdot(-0.5\cdot 3^{2k+2}+0.5\cdot 3^{2k+2-2l}+\sum_{j=1}^{2l}\tau_{2n+2+j}3^{2k+2-j})\stackrel{\text{def}}{=} \tag{37}$$
$$\widetilde{\Psi}_{l,k}(\widetilde{x}_k,\tau_{2n+3,2(n+1+l)})$$

satisfies (36). So we need to discretize again our problem. We choose large numbers $\widetilde{n}_k \geq 100$, $(k \in \{0, ..., l-1\})$ and for any $\widetilde{x}_{k,\widetilde{i}_k} = \widetilde{i}_k/\widetilde{n}_k$, $\widetilde{i}_k = 0, 1, ..., (\widetilde{n}_k - 1)$ we want to find $\tau_{2n+3,2(n+1+l)}$ such that the following stronger version of (36) holds simultaneously for all $k \in \{0, ..., l-1\}$

$$\langle \widetilde{\Psi}_{l,k}(\widetilde{x}_{k,\widetilde{i}_k}, \tau_{2n+3,2(n+1+l)}) \rangle > \frac{1}{6} + q_1 + \widetilde{\epsilon}_{l,k} + 3^{-99+3k} + \frac{1}{2\widetilde{n}_k}. \tag{38}$$

Then for an arbitrary $\widetilde{x}_k \in [0, 1)$ we can choose $\widetilde{i}_k$ such that

$$\min\{|\widetilde{x}_k - \widetilde{x}_{k,\widetilde{i}_k}|, 1 - |\widetilde{x}_k - \widetilde{x}_{k,\widetilde{i}_k}|\} < \frac{1}{2\widetilde{n}_k}$$

and by (38) used with this $\widetilde{i}_k$ we have

$$\begin{aligned}\langle \widetilde{\Psi}_{l,k}(\widetilde{x}_k, \tau_{2n+3,2(n+1+l)}) \rangle > &\langle \widetilde{\Psi}_{l,k}(\widetilde{x}_{k,\widetilde{i}_k}, \tau_{2n+3,2(n+1+l)}) \rangle - \frac{1}{2\widetilde{n}_k} > \\ &\frac{1}{6} + q_1 + \widetilde{\epsilon}_{l,k} + 3^{-99+3k}.\end{aligned} \tag{39}$$

In the definition of $\widetilde{\Psi}_{l,k}(\widetilde{x}_k, \tau_{2n+3,2(n+1+l)})$ and in (32) we still have a factor $-1/\alpha_{n+k}(\overline{x}(\tau))^2$ which we can only approximate in our computer programs. The exact value of $\alpha_{n+k}(\overline{x}(\tau))$ is determined by $\tau_{n+1,2n+2}$, but instead of using this exact value, for a given $\kappa \in \{12, ..., 17\}$ we use again the approximate value $\tau_{n+1,n+\kappa}$, that is, we work again with $\tau^\kappa \stackrel{\text{def}}{=} (\tau | n + \kappa)$, instead of τ. Then

$$|x(\tau^\kappa) - \overline{x}(\tau)| < 3^{-n-\kappa} \tag{40}$$

and

$$|\alpha_{n+k}(x(\tau^\kappa)) - \alpha_{n+k}(\overline{x}(\tau))| < 3^{-\kappa+k}. \tag{41}$$

Keeping in mind that $\alpha_{n+k}(x(\tau^\kappa))$ and $\alpha_{n+k}(\overline{x}(\tau))$ are of the same sign and of absolute value in $[1/6, 1/2]$ by the Mean Value Theorem we can choose a $\gamma_{l,k}$ with absolute value in $[1/6, 1/2]$ for which

$$-\frac{1}{\alpha_{n+k}(\overline{x}(\tau))^2} = -\frac{1}{\alpha_{n+k}(x(\tau^\kappa))^2} + \frac{2}{\gamma_{l,k}^3}\Big(\alpha_{n+k}(\overline{x}(\tau)) - \alpha_{n+k}(x(\tau^\kappa))\Big)$$

and hence

$$\left| \frac{-1}{\alpha_{n+k}(\overline{x}(\tau))^2} - \frac{-1}{\alpha_{n+k}(x(\tau^\kappa))^2} \right| < (2 \cdot 6^3) \cdot 3^{-\kappa+k}. \tag{42}$$

Recalling that $\tau' | 2n + 2 = \tau$ one can obtain similarly

$$\left| \frac{-1}{\alpha_{n+k}(\overline{x}(\tau'))^2} - \frac{-1}{\alpha_{n+k}(x(\tau^\kappa))^2} \right| < (2 \cdot 6^3) \cdot 3^{-\kappa+k}. \tag{43}$$

Set now

$$\Psi_{l,k}(\widetilde{x}_k,\tau_{2n+3,2(n+1+l)})= \tag{44}$$
$$\widetilde{x}_k-\frac{1}{\alpha_{n+k}(x(\tau^\kappa))^2}\frac{1}{81}\Big(-0.5\cdot 3^{2k+2}+0.5\cdot 3^{2k+2-2l}+\sum_{j=1}^{2l}\tau_{2n+2+j}3^{2k+2-j}\Big)=$$
$$\widetilde{x}_k-\frac{1}{(-0.5+\frac{\tau_{n+k+1}}{3}+...+\frac{\tau_{n+\kappa}}{3^{\kappa-k}})^2}\cdot\frac{1}{81}\Big(-0.5\cdot 3^{2k+2}+0.5\cdot 3^{2k+2-2l}+$$
$$\sum_{j=1}^{2l}\tau_{2n+2+j}3^{2k+2-j}\Big).$$

From (42) and (44) it follows that

$$|\Psi_{l,k}(\widetilde{x}_k,\tau_{2n+3,2(n+1+l)})-\widetilde{\Psi}_{l,k}(\widetilde{x}_k,\tau_{2n+3,2(n+1+l)})|< \tag{45}$$
$$\frac{0.5\cdot 3^{2k+2}}{81}\cdot 2\cdot 6^3\cdot 3^{-\kappa+k}\stackrel{\text{def}}{=}\epsilon_{2,k,\kappa}.$$

In (35) $\widetilde{\epsilon}_{l,k}$ still depends on too many digits of τ', so we estimate the error we make if we use

$$\epsilon_{l,k}\stackrel{\text{def}}{=}\left|-\frac{1}{\alpha_{n+k}(x(\tau^\kappa))^2}\right|\cdot\frac{1}{81}\cdot 0.5\cdot 3^{2k+2-2l},$$

instead of $\widetilde{\epsilon}_{l,k}$. Then by (43)

$$\widetilde{\epsilon}_{l,k}<\epsilon_{l,k}+(2\cdot 6^3)\cdot 3^{-\kappa+k}\frac{1}{81}0.5\cdot 3^{2k+2-2l}<\epsilon_{l,k}+3^{-\kappa+3k+3-2l}. \tag{46}$$

Assume that for any $\widetilde{i}_k\in\{0,...,\widetilde{n}_k-1\}$ we can choose $\tau_{2n+3,2(n+1+l)}$ such that

$$\langle\Psi_{l,k}(\widetilde{x}_{k,\widetilde{i}_k},\tau_{2n+3,2(n+1+l)})\rangle>\frac{1}{6}+q_1+\epsilon_{l,k}+3^{-\kappa+3k+3-2l}+ \tag{47}$$
$$3^{-99+3k}+\frac{1}{2\widetilde{n}_k}+\epsilon_{2,k,\kappa}>$$
$$\frac{1}{6}+q_1+\widetilde{\epsilon}_{l,k}+3^{-99+3k}+\frac{1}{2\widetilde{n}_k}+\epsilon_{2,k,\kappa}.$$

Then (38) also holds and this implies that for any $\widetilde{x}_k$ in $[0,1)$ we have (39). Taking the special case when $\widetilde{x}_k=\{\Phi_{n+k}(\overline{x}(\tau))\}$ we obtain (34) for $k=0,...,l-1$. This, provided, the ten digit sequence $\tau_{n+l+1}...\tau_{n+l+10}\notin\mathcal{F}_{10}$ warrants that $\tau\in\mathcal{C}_{2n+2,2(n+1+l)}$ and $\tau'\in\mathcal{C}_{2(n+1+l)}(\tau)$, which shows that we have a q_1-continuation $\tau'\in\mathcal{T}^{\mathcal{F}}_{2(n+1+l)}$ of $\tau\in\mathcal{T}^{\mathcal{F}}_{2(n+1)}$.

So we have a positive answer to the q_1-continuation problem of τ if we can give a positive answer to the following discrete problem.

DPl: *We can choose large* $\widetilde{n}_k\geq 100$ *for* $k=0,...,l-1$ *and a (not too large)* $\kappa\in\mathbb{N}$ *such that for all* $k\in\{0,...,l-1\}$ *and for any* $\widetilde{i}_k\in\{0,1,...,(\widetilde{n}_k-1)\}$ *we* *can choose* $\tau_{2n+3,2(n+1+l)} \in \mathcal{T}_{2l}$ *such that* (47) *holds for all* $k \in \{0,...,l-1\}$ *and* $\tau_{n+l+1,n+l+10} \notin \mathcal{F}_{10}$.

By running a computer program based on discrete problem DPl with a $\kappa \geq 11$ one can obtain a list of sequences in $\mathcal{UC}(l)$. To do so we just need to choose sufficiently large $\widetilde{n}_k$'s ($k = 0,...,l-1$) and for a $\nu \in \mathcal{T}_{10} \setminus \mathcal{F}_{10}$ one should consider all sequences $\nu'' = \nu\nu'$, where $\nu' \in \mathcal{T}_{\kappa-10}$ and use $\tau_{n+j} = \nu''_j$ for $j = 1,...,\kappa$ in (44) in the definition of $\Psi_{l,k}(\widetilde{x}_k, \tau_{2n+3,2(n+1+l)})$ (of course, now, due to "universality" n is of no importance anymore and $\tau_{2n+3,2(n+1+l)} \in \mathcal{T}_{2l}$ can be replaced by digits like $\widetilde{c}_1...\widetilde{c}_{2l}$). If for any $\nu' \in \mathcal{T}_{\kappa-10}$, $k \in \{0,...,l-1\}$, and $\widetilde{i}_k \in \{0,1,...,\widetilde{n}_k - 1\}$ we can find $\widetilde{c}_1...\widetilde{c}_{2l} \in \mathcal{T}_{2l}$ such that

$$\langle \Psi_{l,k}(\widetilde{x}_{k,\widetilde{i}_k}, \widetilde{c}_1...\widetilde{c}_{2l})\rangle > \frac{1}{6} + q_1 + \epsilon_{l,k} + 3^{-\kappa+3k+3-2l} + 3^{-99+3k} + \frac{1}{2\widetilde{n}_k} + \epsilon_{2,k,\kappa} \tag{48}$$

$$\text{for } k = 0,...,l-1$$

and $\nu, \nu''_{l+1,l+10} \notin \mathcal{F}_{10}$ then we know that $\nu \in \mathcal{UC}(l)$. If in the above algorithm we can always find two different $\widetilde{c}_1...\widetilde{c}_{2l}$ and $\widetilde{c}'_1...\widetilde{c}'_{2l}$ then we know that $\nu \in \mathcal{UD}(l)$.

However, instead of using this computer program which might run too long for $l = 3$ or 4, we use a slightly modified version which can give us a list of sequences in $\mathcal{UC}(\leq l)$ or $\mathcal{UD}(\leq l)$. We call them nested programs of level l, (abbreviated as NPl for finding sequences in $\mathcal{UC}(\leq l)$, and as NPDl for finding sequences in $\mathcal{UD}(\leq l)$).

So we assume $\kappa \geq 10 + l$ is fixed and choose sufficiently large $\widetilde{n}_k$'s for $k = 0,...,l-1$ and for a $\nu \in \mathcal{T}_{10} \setminus \mathcal{F}_{10}$ we consider all sequences $\nu'' = \nu\nu'$, where $\nu' \in \mathcal{T}_{\kappa-10}$ and use $\tau_{n+j} = \nu''_j$ for $j = 1,...,\kappa$ in (44) in the definition of $\Psi_{l,k}$.

For each $\nu' \in \mathcal{T}_{\kappa-10}$ first we choose $k = 0$ and try to find for each $\widetilde{i}_0 \in \{0,...,\widetilde{n}_0 - 1\}$ a suitable $\widetilde{c}_1\widetilde{c}_2 \in \mathcal{T}_2$ such that (48) holds with $l = 1$, $k = 0$ and $\nu''_{2,11} \notin \mathcal{F}_{10}$. If we can always find suitable $\widetilde{c}_1\widetilde{c}_2$ then we are done with this value of ν', and if we are done with all $\nu' \in \mathcal{T}_{\kappa-10}$ then we showed that $\nu \in \mathcal{UC}(\leq l)$ (in fact, $\nu \in \mathcal{UC}(1)$). (For $\nu \in \mathcal{UD}(\leq l)$ in NPDl, we always need two different $\widetilde{c}_1\widetilde{c}_2$.)

Of course, there might be values of $\widetilde{i}_0$ for which we cannot find (two for NPDl) suitable $\widetilde{c}_1\widetilde{c}_2 \in \mathcal{T}_2$ as above. Denote the set of such $\widetilde{i}_0$'s by $\mathcal{I}_{1,0}$.

Now we can launch a "level 2" algoritm with only these exceptional values used. So we try to find for each $\widetilde{i}_0 \in \mathcal{I}_{1,0}$ and for each $\widetilde{i}_1 \in \{0,...,\widetilde{n}_1 - 1\}$ a suitable $\widetilde{c}_1...\widetilde{c}_4 \in \mathcal{T}_4$ such that (48) holds with $l = 2$, $k = 0,1$ and $\nu''_{3,12} \notin \mathcal{F}_{10}$. (For NPD$l$ we always need two different $\widetilde{c}_1...\widetilde{c}_4 \in \mathcal{T}_4$.) Of course, there might be values $\widetilde{i}_1 \in \{0,...,\widetilde{n}_1 - 1\}$ (denote these values by $\mathcal{I}_{2,1}(\widetilde{i}_0)$) such that there is no (two for NPDl) suitable $\widetilde{c}_1...\widetilde{c}_4 \in \mathcal{T}_4$. If $\mathcal{I}_{2,1}(\widetilde{i}_0) = \emptyset$ then we are done checking this parameter value $\widetilde{i}_0$ of level 1. Denote by $\mathcal{I}_{2,0}$ the set of those $\widetilde{i}_0$ for which $\mathcal{I}_{2,1}(\widetilde{i}_0) \neq \emptyset$.

If $\mathcal{I}_{2,0} \neq \emptyset$ then we can launch a "level 3" algorithm with only these exceptional values. (This turns out to be a big advantage in the running time of our programs, since "most" values can be treated at levels 1 or 2 and we need to use the slower running level 3 or 4 programs only for the "few" exceptional values.)

So we have to find for each $\widetilde{i}_0 \in \mathcal{I}_{2,0}$, $\widetilde{i}_1 \in \mathcal{I}_{2,1}(\widetilde{i}_0)$ and $\widetilde{i}_2 \in \{0, ..., \widetilde{n}_2 - 1\}$ a suitable $\widetilde{c}_1 ... \widetilde{c}_6 \in \mathcal{T}_6$ such that (48) holds with $l = 3$, $k = 0, 1, 2$ and $\nu''_{4,13} \notin \mathcal{F}_{10}$ (again for NPDl we need two different sequences $\widetilde{c}_1 ... \widetilde{c}_6 \in \mathcal{T}_6$). There might be values $\widetilde{i}_0$ (denote them by $\mathcal{I}_{3,0}$) for which there are some $\widetilde{i}_1$ (denote them by $\mathcal{I}_{3,1}(\widetilde{i}_0)$) for which there are some $\widetilde{i}_2$ (denote them by $\mathcal{I}_{3,2}(\widetilde{i}_0, \widetilde{i}_1)$) for which we cannot find a suitable $\widetilde{c}_1 ... \widetilde{c}_6 \in \mathcal{T}_6$ (or two suitable $\widetilde{c}_1 ... \widetilde{c}_6$'s for NPD$l$).

For NP3 (or NPD3) we stop at this level and can obtain a list of ν's for which $\mathcal{I}_{3,0} = \emptyset$ for all $\nu' \in \mathcal{T}_{\kappa-10}$. These ν's are in $\mathcal{UC}(\leq 3)$ or in $\mathcal{UD}(\leq 3)$. In fact, after the actual computer runs it turned out that we obtained a list

$$\begin{aligned}\mathcal{F}^* = \{&0200002200, 0200220200, 0220020222, 2002202000,\\ &2022002022, 2022220022\}\end{aligned}$$

such that any $\nu \in \mathcal{T}_{10} \setminus (\mathcal{F}^* \cup \mathcal{F}_{10})$ belongs to $\mathcal{UC}(\leq 3)$. As we mentioned earlier using NPD3 we obtained a list $\mathcal{L}_3 \subset \mathcal{T}_{10} \setminus \mathcal{F}_{10}$ of 274 sequences (this list is given in Section 3.3) such that if $\nu \in \mathcal{T}_{10} \setminus (\mathcal{L}_3 \cup \mathcal{F}_{10})$ then $\nu \in \mathcal{UD}(\leq 3)$.

In NP4 we try to find for each $\widetilde{i}_0 \in \mathcal{I}_{3,0}$, $\widetilde{i}_1 \in \mathcal{I}_{3,1}(\widetilde{i}_0)$, $\widetilde{i}_2 \in \mathcal{I}_{3,2}(\widetilde{i}_0, \widetilde{i}_1)$ and $\widetilde{i}_3 \in \{0, ..., \widetilde{n}_3 - 1\}$ a suitable $\widetilde{c}_1 ... \widetilde{c}_8 \in \mathcal{T}_8$ such that (48) holds with $l = 4$ for $k = 0, ..., 3$ and $\nu''_{5,14} \notin \mathcal{F}_{10}$. Like at NP3 (or at NPD3) we can determine whether a sequence $\nu \in \mathcal{T}_{10}$ belongs to $\mathcal{UC}(\leq 4)$ or $\mathcal{UD}(\leq 4)$. The easiest thing would be to run NPD4 on each item from $\mathcal{L}_3$ but unfortunately the running time required for this higher precision program does not make it possible. For the six sequences in $\mathcal{F}^*$ it is not that difficult to run NP4 which shows that all of them belong to $\mathcal{UC}(\leq 4)$. Hence with the exception of $\mathcal{F}_{10}$ all sequences in $\mathcal{T}_{10}$ belong to $\mathcal{UC}(\leq 4)$.

Next we need to show how to deal with $\mathcal{L}_3$. First observe that

$$\mathcal{L}_3 \text{ does not contain sequences starting with 002 or 220.} \tag{49}$$

Based on running a computer program based on NP1 we obtained $\mathcal{L}_1$ (see Section 3.1) which is a list of sequences such that $\mathcal{T}_{10} \setminus \mathcal{L}_1 \subset \mathcal{UC}(1)$, hence

$$\text{all sequences starting with 000 or 222 belong to } \mathcal{UC}(1). \tag{50}$$

Using (49) and (50) one can deal with any $\nu \in \mathcal{L}_3$ starting with 000 or 222 provided $\nu \neq 0000000000$ or 2222222222. Indeed, assume $n \geq 100$, $\tau \in \mathcal{T}^{\mathcal{F}}_{2n+2}$ and $\tau_{n+1,n+10} = \nu$. Then $\nu \in \mathcal{UC}(1)$ and hence we can choose $\tau_{2n+3}\tau_{2n+4}$ such that $\tau\tau_{2n+3}\tau_{2n+4} \in \mathcal{T}^{\mathcal{F}}_{2n+4}$ and $\tau_{n+2,n+11} = \nu_{2,10}\tau_{n+11}$. If $\nu_{2,10}\tau_{n+11}$ starts with 002, or with 220 then we know $\nu_{2,10}\tau_{n+11} \in \mathcal{UD}(\leq 3)$ and hence τ is q_1-double-continuable. If $\nu_{2,10}\tau_{n+11}$ starts with 000 or with 222 then we can repeat the above procedure. Since $\nu \neq 0000000000$ or 2222222222 in less than eight steps we can see that τ is q_1-double-continuable. Our computer programs (which are modified versions of NP4) show that if $\nu = 0000000000$, or 2222222222, $n \geq 100$ and $\tau \in \mathcal{T}^{\mathcal{F}}_{2n+2}$ and $\tau_{n+1,n+10} = \nu$ then we can find τ', a q_1-continuation of τ

which is of length $2m$, $(m \leq n+5)$ such that $\tau'_{2n+3,2m}$ contains a non 0, or non 2 digit, respectively. This means that using (50) repeatedly we can find a q_1-continuation τ'' of τ such that $\tau'' \in \mathcal{T}^{\mathcal{F}}_{2m+2}$ for an $n+2 \leq m \leq 2(n+5)$ and $\nu'' = \tau''_{m+1,m+10}$ starts with 002 (or with 220) but then we know that τ'' is q_1-double-continuable and hence τ is also q_1-double-continuable.

So, denote by $\mathcal{L}^0_3$ the set of those sequences in $\mathcal{L}_3$ which start with 02 or 20. We know that if $\nu \in \mathcal{T}_{10} \setminus (\mathcal{F}_{10} \cup \mathcal{L}^0_3)$ then ν is q_1-double-continuable.

For a $\nu \in \mathcal{T}_{10}$ denote by $\Sigma_j(\nu)$ the set of those $\nu' \in \mathcal{T}_{10}$ for which $\nu'_k = \nu_{k+j}$ for $k = 1, ..., 10-j$.

If $\nu \in \mathcal{UC}(\leq l) \cap \mathcal{L}^0_3$ for an $l \leq 4$ and $\cup_{j=1}^{l} \Sigma_j(\nu) \cap \mathcal{L}^0_3 = \emptyset$ then from $n \geq 100$, $\tau \in \mathcal{T}^{\mathcal{F}}_{2n+2}$ and $\tau_{n+1,n+10} = \nu$ it follows that for a $j \in \{1, ..., l\}$ we can find a $\tau' \in \mathcal{T}^{\mathcal{F}}_{2(n+1+j)}$, a q_1-continuation of τ such that $\tau'_{n+1+j,n+10+j} \notin \mathcal{L}^0_3$ and hence τ' and τ are q_1-double-continuable.

Denote by $\overline{\mathcal{L}}^0_3$ a suitably chosen subset of those $\nu \in \mathcal{L}^0_3$ for which $\nu \in \mathcal{UC}(\leq l)$ for an $l \leq 4$ and $\cup_{j=1}^{l} \Sigma_j(\nu) \cap \mathcal{L}^0_3 = \emptyset$. (We use these "suitably" chosen subsets instead of all sequences with the given properties since it is faster to work with an easy to find subset instead of locating all possible sequences.) We have seen above that $\overline{\mathcal{L}}^0_3 \subset \mathcal{UD}$.

Set $\mathcal{L}^1_3 = \mathcal{L}^0_3 \setminus \overline{\mathcal{L}}^0_3$.

Now we can repeat the above procedure. Given $\mathcal{L}^k_3$ for a $k \geq 1$ we can denote by $\overline{\mathcal{L}}^k_3$ a suitably chosen subset of those $\nu \in \mathcal{L}^k_3$ for which $\nu \in \mathcal{UC}(\leq l)$ for an $l \leq 4$ and $\cup_{j=1}^{l} \Sigma_j(\nu) \cap \mathcal{L}^k_3 = \emptyset$. We can see again that $\overline{\mathcal{L}}^k_3 \subset \mathcal{UD}$. Set $\mathcal{L}^{k+1}_3 = \mathcal{L}^k_3 \setminus \overline{\mathcal{L}}^k_3$.

It is not difficult to write a simple computer program which can give the information whether $\Sigma_j(\nu) \cap \mathcal{L}^k_3 = \emptyset$ for a sequence ν in $\mathcal{L}^k_3$. Checking this information "by hand" against our lists of $\mathcal{UC}(\leq l)$, $l = 2, 3, 4$ we can find sequences in a suitable subset $\overline{\mathcal{L}}^k_3$. The lists $\mathcal{UC}(\leq l)$ are the following: in Section 3.1 there is $\mathcal{L}_1$ which we have already mentioned at (50). In Section 3.2 there is also $\mathcal{L}_2$ generated by a program based on NP2 and $\mathcal{T}_{10} \setminus \mathcal{L}_2 \subset \mathcal{UC}(\leq 2)$. We have used $\mathcal{F}^*$, for which $\mathcal{T}_{10} \setminus (\mathcal{F}^* \cup \mathcal{F}_{10}) \subset \mathcal{UC}(\leq 3)$. Finally, $\mathcal{T}_{10} \setminus \mathcal{F}_{10} \subset \mathcal{UC}(\leq 4)$. As an illustration to the process we give an example. The sequence $\nu = 0202022002$ appears on $\mathcal{L}^2_3$ but not on $\mathcal{L}^3_3$. This is justified by the property that there are no sequences in $\mathcal{L}^2_3$ starting with 202022002 or with 02022002 which shows that $\cup_{j=1}^{2} \Sigma_j(\nu) \cap \mathcal{L}^2_3 = \emptyset$ and ν is not listed in $\mathcal{L}_2$, which implies $\nu \in \mathcal{UC}(\leq 2)$.

After eight steps we found a set $\mathcal{L}^8_3 = \{\nu^t : t = 1, ..., 10\}$ where

$$
\begin{aligned}
\nu^1 &= \quad 0200220200\\
\nu^2 &= \quad 0220020220\\
\nu^3 &= \quad 0220022002\\
\nu^4 &= \quad 0220022020\\
\nu^5 &= \quad 0220200220\\
\nu^6 &= \quad 2002022002\\
\nu^7 &= \quad 2002200202\\
\nu^8 &= \quad 2002200220\\
\nu^9 &= \quad 2002202002\\
\nu^{10} &= \quad 2022002022
\end{aligned}
$$

For this set $\overline{\mathcal{L}}_3^8 = \emptyset$, so we cannot decrease $\mathcal{L}_3^8$ by the methods used above. But this set is small enough to be tested by NPD4 and it turned out that ν^2, ν^3, ν^8, ν^9 belong to $\mathcal{UD}(\leq 4)$.

So now we are down to a list $\mathcal{L}_9 = \{\nu^1, \nu^4, \nu^5, \nu^6, \nu^7, \nu^{10}\}$. Then for $\nu \in \{\nu^4, \nu^7\}$ we have $\cup_{j=1}^3 \Sigma_j(\nu) \cap \mathcal{L}_9 = \emptyset$ and $\nu \in \mathcal{UC}(\leq 3)$ and hence ν^4, $\nu^7 \in \mathcal{UD}$. Set $\mathcal{L}_{10} = \{\nu^1, \nu^5, \nu^6, \nu^{10}\}$. Now, $\nu^1 \in \Sigma_3(\nu^5)$ and $\nu^{10} \in \Sigma_3(\nu^6)$. Using a modified version of NP4, which is not stopping at level 3 we can see that $\nu^5 \in \mathcal{UC}(1) \cup \mathcal{UC}(2) \cup \mathcal{UC}(4)$ and $\Sigma_1(\nu^5) \cup \Sigma_2(\nu^5) \cup \Sigma_4(\nu^5) \cap \mathcal{L}_{10} = \emptyset$. This shows that ν^5 is q_1-double-continuable. A similar argument works for ν^{10}. Finally we have a list $\mathcal{L}_{11} = \{\nu^1, \nu^{10}\}$. Then for any $\nu \in \mathcal{L}_{11}$ we have $\cup_{j=1}^4 \Sigma_j(\nu) \cap \mathcal{L}_{11} = \emptyset$ and $\nu \in \mathcal{UC}(\leq 4)$ showing that $\mathcal{L}_{11} \subset \mathcal{UD}$ as well.

Hence if we take (6) then $\nu = \tau_{n+1,n+10} \in \mathcal{T}_{10} \setminus \mathcal{F}_{10}$ and by what we have shown this implies that τ is q_1-double-continuable.

Finally a few words about the parameters we used in our computer programs. The program of type NPD3 which produced $\mathcal{L}_3$ was using the parameters $\kappa = 14$, $\widetilde{n}_0 = 750$, $\widetilde{n}_1 = 150$, $\widetilde{n}_2 = 300$.

The "high precision-low running time" four level program of type NPD4 which was used for individual checking of the sequences ν^2, ν^3, ν^8, ν^9 used $\kappa = 17$, $\widetilde{n}_0 = 2000$, $\widetilde{n}_1 = 400$, $\widetilde{n}_2 = 200$, $\widetilde{n}_3 = 300$.

$\square$

3 Data obtained from our computer programs:

3.1 $\mathcal{L}_1$

The set of 388 sequences belonging to $\mathcal{L}_1$.

0022020000	0022020002	0200000000	0200000002	0200000020	0200000022	0200000200
0200000202	0200000220	0200000222	0200002000	0200002002	0200002020	0200002022
0200002200	0200002202	0200002220	0200002222	0200020000	0200020002	0200020020
0200020022	0200020200	0200020202	0200020220	0200020222	0200022000	0200022002
0200022020	0200022022	0200022200	0200022202	0200022220	0200022222	0200200000
0200200002	0200200020	0200200022	0200200200	0200200202	0200200220	0200200222
0200202000	0200202002	0200202020	0200202022	0200202200	0200202202	0200202220
0200202222	0200220000	0200220002	0200220020	0200220022	0200220200	0200220202
0200220220	0200220222	0200222000	0200222002	0200222020	0200222022	0200222200
0200222202	0200222220	0200222222	0220000000	0220000002	0220000020	0220000022
0220000200	0220000202	0220000220	0220000222	0220002000	0220002002	0220002020
0220002022	0220002200	0220002202	0220002220	0220002222	0220020000	0220020002
0220020020	0220020022	0220020200	0220020202	0220020220	0220020222	0220022000
0220022002	0220022020	0220022022	0220022200	0220022202	0220022220	0220022222
0220200000	0220200002	0220200020	0220200022	0220200200	0220200202	0220200220
0220200222	0220202000	0220202002	0220202020	0220202022	0220202200	0220202202
0220202220	0220202222	0220220000	0220220002	0220220020	0220220022	0220220200
0220220202	0220220220	0220220222	0220222000	0220222002	0220222020	0220222022
0220222200	0220222202	0220222220	0220222222	0222000000	0222000002	0222000020
0222000022	0222000200	0222000202	0222000220	0222000222	0222002000	0222002002
0222002020	0222002022	0222002200	0222002202	0222002220	0222002222	0222020000
0222020002	0222020020	0222020022	0222020200	0222020202	0222020220	0222020222
0222022000	0222022002	0222022020	0222022022	0222022200	0222022202	0222022220
0222022222	0222200000	0222200002	0222200020	0222200022	0222200200	0222200202
0222200220	0222200222	0222202000	0222202002	0222202020	0222202022	0222202200
0222202202	0222202220	0222202222	0222220000	0222220002	0222220020	0222220022
0222220200	0222220202	0222220220	0222220222	0222222000	0222222002	0222222020
0222222022	0222222200	0222222202	0222222220	0222222222	2000000000	2000000002
2000000020	2000000022	2000000200	2000000202	2000000220	2000000222	2000002000
2000002002	2000002020	2000002022	2000002200	2000002202	2000002220	2000002222
2000020000	2000020002	2000020020	2000020022	2000020200	2000020202	2000020220
2000020222	2000022000	2000022002	2000022020	2000022022	2000022200	2000022202
2000022220	2000022222	2000200000	2000200002	2000200020	2000200022	2000200200
2000200202	2000200220	2000200222	2000202000	2000202002	2000202020	2000202022
2000202200	2000202202	2000202220	2000202222	2000220000	2000220002	2000220020
2000220022	2000220200	2000220202	2000220220	2000220222	2000222000	2000222002
2000222020	2000222022	2000222200	2000222202	2000222220	2000222222	2002000000
2002000002	2002000020	2002000022	2002000200	2002000202	2002000220	2002000222
2002002000	2002002002	2002002020	2002002022	2002002200	2002002202	2002002220
2002002222	2002020000	2002020002	2002020020	2002020022	2002020200	2002020202
2002020220	2002020222	2002022000	2002022002	2002022020	2002022022	2002022200
2002022202	2002022220	2002022222	2002200000	2002200002	2002200020	2002200022
2002200200	2002200202	2002200220	2002200222	2002202000	2002202002	2002202020
2002202022	2002202200	2002202202	2002202220	2002202222	2002220000	2002220002
2002220020	2002220022	2002220200	2002220202	2002220220	2002220222	2002222000
2002222002	2002222020	2002222022	2002222200	2002222202	2002222220	2002222222
2022000000	2022000002	2022000020	2022000022	2022000200	2022000202	2022000220
2022000222	2022002000	2022002002	2022002020	2022002022	2022002200	2022002202
2022002220	2022002222	2022020000	2022020002	2022020020	2022020022	2022020200
2022020202	2022020220	2022020222	2022022000	2022022002	2022022020	2022022022
2022022200	2022022202	2022022220	2022022222	2022200000	2022200002	2022200020
2022200022	2022200200	2022200202	2022200220	2022200222	2022202000	2022202002
2022202020	2022202022	2022202200	2022202202	2022202220	2022202222	2022220000
2022220002	2022220020	2022220022	2022220200	2022220202	2022220220	2022220222
2022222000	2022222002	2022222020	2022222022	2022222200	2022222202	2022222220
2022222222	2200202220	2200202222				

3.2 $\mathcal{L}_2$

The set of 162 sequences belonging to $\mathcal{L}_2$.

0022020000	0022020002	0200000000	0200000002	0200000020	0200000022
0200000200	0200000202	0200000220	0200000222	0200002000	0200002002
0200002020	0200002022	0200002200	0200002202	0200002220	0200002222
0200020000	0200020002	0200020020	0200020022	0200020200	0200020202
0200020220	0200020222	0200022000	0200022002	0200022020	0200022022
0200022200	0200022202	0200022220	0200022222	0200200000	0200200002
0200200020	0200200022	0200200200	0200200202	0200200220	0200200222
0200202002	0200202020	0200202022	0200202200	0200202202	0200202220
0200202222	0200220000	0200220200	0200220202	0200222220	0200222222
0220020222	0220200000	0220200002	0220200020	0220200022	0220200200
0220200202	0220200220	0220200222	0222200000	0222200002	0222200020
0222200022	0222200200	0222200202	0222200220	0222200222	0222202000
0222202002	0222202020	0222202022	0222202200	0222202202	0222202220
0222202222	0222220000	0222220002	2000002220	2000002222	2000020000
2000020002	2000020020	2000020022	2000020200	2000020202	2000020220
2000020222	2000022000	2000022002	2000022020	2000022022	2000022200
2000022202	2000022220	2000022222	2002022000	2002022002	2002022020
2002022022	2002022200	2002022202	2002022220	2002022222	2002202000
2022000000	2022000002	2022002020	2022002022	2022002222	2022020000
2022020002	2022020020	2022020022	2022020200	2022020202	2022020220
2022022000	2022022002	2022022020	2022022022	2022022200	2022022202
2022022220	2022022222	2022200000	2022200002	2022200020	2022200022
2022200200	2022200202	2022200220	2022200222	2022202000	2022202002
2022202020	2022202022	2022202200	2022202202	2022202220	2022202222
2022220000	2022220002	2022220020	2022220022	2022220200	2022220202
2022220220	2022220222	2022222000	2022222002	2022222020	2022222022
2022222200	2022222202	2022222220	2022222222	2200202220	2200202222

3.3 $\mathcal{L}_3$

The list of the 274 sequences belonging to $\mathcal{L}_3$:

0000000000	0000000002	0000000020	0000000022	0000000200	0000000202
0000000220	0000000222	0000002000	0000002002	0000002020	0000002022
0000002200	0000002202	0000002220	0000002222	0000020000	0000020002
0000020020	0000020022	0000020200	0000020202	0000020220	0000020222
0000022000	0000022002	0000022020	0000022022	0000022200	0000022202
0000022220	0000022222	0000200000	0000200002	0000200020	0000200022
0000200200	0000200202	0000200220	0000200222	0000202000	0000202002
0000202020	0000202022	0000202200	0000202202	0000202220	0000202222
0000220200	0002000000	0002000002	0002000020	0002000022	0002000200
0002000202	0002000220	0002000222	0002002022	0002002222	0002202000
0200000000	0200000002	0200000020	0200000022	0200000220	0200000222
0200002020	0200002022	0200002200	0200002202	0200020202	0200020220
0200022200	0200022202	0200022220	0200200222	0200220200	0202002022
0202022002	0202202000	0202202220	0202202222	0202220000	0202220002
0202220020	0202220022	0202220200	0202220202	0202220220	0202220222
0220020200	0220020202	0220020220	0220020222	0220022000	0220022002
0220022020	0220022022	0220022200	0220022202	0220022220	0220022222
0220200000	0220200020	0220200200	0220200202	0220200220	0220200222
0220202000	0220202002	0220202020	0220202022	0220202200	0220202202
0220202220	0220202222	0220220000	0220220002	0220220022	0220220200
0222002022	0222200000	0222200002	0222200020	0222200022	0222200200
0222200202	0222200220	0222200222	0222202000	0222202002	0222202020
0222202022	0222202200	0222202202	0222202220	0222202222	2000020000
2000020002	2000020020	2000020022	2000020200	2000020202	2000020220
2000020222	2000022000	2000022002	2000022020	2000022022	2000022200
2000022202	2000022220	2000022222	2000220200	2002002022	2002002200
2002002220	2002002222	2002020000	2002020002	2002020020	2002020022
2002020200	2002020202	2002020220	2002020222	2002022000	2002022002
2002022020	2002022022	2002022202	2002022222	2002200000	2002200002
2002200020	2002200022	2002200200	2002200202	2002200220	2002200222
2002202000	2002202002	2002202020	2002202022	2020002000	2020002002
2020002020	2020002022	2020002200	2020002202	2020002220	2020002222
2020020000	2020020002	2020020222	2020200220	2020220200	2022002022
2022022000	2022200002	2022200020	2022200022	2022202002	2022202020
2022220020	2022220022	2022220200	2022220202	2022222000	2022222002
2022222200	2022222202	2022222220	2022222222	2220020222	2220220000
2220220200	2220222000	2220222002	2220222020	2220222022	2220222200
2220222202	2220222220	2220222222	2222002022	2222020000	2222020002
2222020020	2222020022	2222020200	2222020202	2222020220	2222020222
2222022000	2222022002	2222022020	2222022022	2222022200	2222022202
2222022220	2222022222	2222200000	2222200002	2222200020	2222200022
2222200200	2222200202	2222200220	2222200222	2222202000	2222202002
2222202020	2222202022	2222202200	2222202202	2222202220	2222202222
2222220000	2222220002	2222220020	2222220022	2222220200	2222220202
2222220220	2222220222	2222222000	2222222002	2222222020	2222222022
2222222200	2222222202	2222222220	2222222222		

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