On series of translates of positive functions II

Zoltán Buczolich, Department of Analysis, Eötvös Loránd University, Kecskeméti u. 10-12, 1053 Budapest, Hungary
email: buczo@cs.elte.hu
www.cs.elte.hu/~buczo

R. Daniel Mauldin *, Department of Mathematics, Box 311430 University of North Texas, Denton, Texas 76203, USA
email: mauldin@unt.edu
www.math.unt.edu/~mauldin

February 13, 2001

Abstract

In this paper we continue our investigation of series of the form $\sum_{\lambda\in\Lambda} f(x+\lambda)$. Given a sequence of natural numbers $n_1 < n_2 < \dots$ we are interested in sets Λ of the form $\cup_{k=1}^{\infty}\alpha^k\mathbb{Z}\cap[n_k, n_{k+1})$. In case $\alpha = 1/q$, where $q > 1$ is an integer, there is a zero-one law showing that for every measurable $f : \mathbb{R} \to \mathbb{R}^+$ the above sum either converges almost everywhere or diverges almost everywhere. However, for any other value of $\alpha \in (0, 1)$ there is no such zero-one law.

1 Introduction

This paper is a continuation of papers [1] [2], and [3]. In [3], answering a question of Haight [6] and Weizsäcker [10] we showed that there is a measurable characteristic function $f : \mathbb{R} \to \mathbb{R}^+$ such that both of the sets $\{x : \sum_{n=1}^{\infty} f(nx) < +\infty\}$ and $\{x : \sum_{n=1}^{\infty} f(nx) = +\infty\}$ have positive measure. Earlier results related to convergence of $\sum f(nx)$ can be found in [7], [8] and [9]. In [1] and [2] we considered additive generalizations of this problem as follows:
Given Λ, an infinite discrete set of nonnegative numbers, and $f : \mathbb{R} \to \mathbb{R}$ a nonnegative Lebesgue measurable function we consider the sum of translates of

*Supported in part by NSF Grant DMS 9801583
2000 Mathematics Subject Classification: Primary 28A20; Secondary 11K31, 60F20
Keywords: sums of translates, Borel-Cantelli lemma, zero-one law

f:

$$s(x) = \sum_{\lambda\in\Lambda} f(x+\lambda) \tag{1}$$

and the sets of convergence and divergence $C(f,\lambda) = \{x : \sum_{\lambda\in\Lambda} f(x+\lambda) < +\infty\}$ and $D(f,\lambda) = \{x : \sum_{\lambda\in\Lambda} f(x+\lambda) = +\infty\}$.

A set Λ is of type 1 if either $C(f,\Lambda) = \mathbb{R}$ almost everywhere, or $C(f,\Lambda) = \emptyset$ a.e. (in this case a zero-one law holds for the convergence of (1)). Otherwise, Λ is of type 2. Using additive terminology, the result of [3] says that $\Lambda^{\log n} := \{\log n : n = 1, 2, ...\}$ is of type 2. In [1] and [2] several generalizations were considered. Given a sequence of natural numbers $n_1 < n_2 < ...$, it was shown that $\Lambda^{(\frac{1}{2})^k} := \cup_{k=1}^{\infty}\Lambda_k^{(\frac{1}{2})^k}$, where $\Lambda_k^{(\frac{1}{2})^k} := (\frac{1}{2})^k\mathbb{Z}\cap[n_k, n_{k+1})$ is of type 1. It was also shown that this type 1 set could be adjusted to become a type 2 set by making it become asymptotically dense at a faster rate as follows: there is a sequence $m(k) \to \infty$ such that $\Lambda^{(\frac{1}{2})^{m(k)}} = \cup_{k=1}^{\infty}\Lambda_k^{(\frac{1}{2})^{m(k)}}$, $\Lambda^{(\frac{1}{2})^{m(k)}} := (\frac{1}{2})^{m(k)}\mathbb{Z}\cap[n_k, n_{k+1})$ is of type 2.

A number $t > 0$ is called a translator of Λ if $(\Lambda+t)\setminus\Lambda$ is finite. Condition $(*)$ is said to be satisfied if $T(\Lambda)$, the countable additive semigroup of translators of Λ, is dense in $\mathbb{R}^+$. We showed that condition $(*)$ implies that $C(f,\Lambda)$ is either $\emptyset$, $\mathbb{R}$, or a closed left half-line modulo sets of measure zero. The two sets just described above both satisfy condition $(*)$. So condition $(*)$ is not enough to determine whether Λ is of type 1 or type 2.

In this paper for a given $\alpha \in (0,1)$, we are interested in the sets $\Lambda^{\alpha^k} := \cup_{k=1}^{\infty}\Lambda_k^{\alpha^k}$, $\Lambda_k^{\alpha^k} = \alpha^k\mathbb{Z}\cap[n_k, n_{k+1})$.

If $\alpha = \frac{1}{q}$ for some $q \in \{2, 3, ...\}$, then a slight modification of the proof of Theorem 1 of [1] shows that $\Lambda^{(\frac{1}{q})^k}$ is of type 1 and condition $(*)$ is satisfied. If $\alpha \notin \mathbb{Q}$, then one can apply Theorem 5 of [1] to show that Λ^{α^k} is of type 2.

The difficult case is when $\alpha = \frac{p}{q}$ with $(p,q) = 1$, $p, q > 1$, $p < q$. In this case we show that $\Lambda^{(\frac{p}{q})^k}$ is of type 2. The problem of showing this is not easy even in the special case when $p = 2$, $q = 3$. When working on this problem, for a while it seemed that we needed some information on the distribution of $\{(\frac{3}{2})^k\}$, (where $\{\cdot\}$ denotes the fractional part). To our surprize, and showing why the case $\Lambda^{(\frac{2}{3})^k}$ was difficult, it turned out, [4] that in 1980 it was not even known, whether $\{(\frac{3}{2})^k\}$ is uniformly distributed, or even dense in $[0,1]$. These questions were extensively studied previously (see further references in [4]). We could not find any reference in the literature showing that these questions were answered after 1980. Fortunately, we found a way avoiding any information about the distribution of $\{(\frac{3}{2})^k\}$, (or of $\{(\frac{q}{p})^k\}$) to determine that $\Lambda^{(\frac{2}{3})^k}$, (or $\Lambda^{(\frac{p}{q})^k}$) is of type 2. Another novelty of this paper is that in the cases $\Lambda^{(\frac{p}{q})^k}$, $(p > 1)$ condition $(*)$ is not satisfied and in Theorem 3 we also show that there exists a characteristic function f such that $C(f,\Lambda)$ does not equal $\emptyset$, $\mathbb{R}$, or a left half-line modulo sets of measure zero. This structure of $C(f,\Lambda)$ has not been seen before and casts a little more light on the question of what the possible structure could be.

In the whole paper we assume that $n_1 < n_2 < ...$ is a given arbitrary monotone increasing sequence of natural numbers. We use the notation $[x]$ for the integer part of x and $|A|$ for the Lebesgue measure of the set $A \subset \mathbb{R}$.

2 The cases when $\alpha \notin \mathbb{Q}$, or $\alpha = 1/q$

Recall first Theorem 5 of [1]:

Theorem 1. *Suppose that there exist three intervals I, J, K such that $J = K + I - I$ (algebraic sum), I is to the left of J, and dist$(I, J) \geq |I|$, and two sequences (y_j) and (N_j) tending to infinity ($y_j \in \mathbb{R}^+$, $N_j \in \mathbb{N}$) such that, for each j, $y_j - I$ contains a set of N_j points of Λ independent from $\Lambda \cap (y_j - J)$ in the sense that the additive groups generated by these sets have only 0 in common. Then Λ has type 2. Moreover, for some $f \in C_0^+(\mathbb{R})$, $D(f, \Lambda)$ contains I and $C(f, \Lambda)$ has full measure on K.*

If $\alpha \notin \mathbb{Q}$ then one can apply Theorem 1 above. Indeed, choose intervals $I = [0, \frac{1}{16}]$, $K = [\frac{1}{2} - \frac{1}{16}, \frac{1}{2} + \frac{1}{16}]$, and $J = [\frac{1}{2} - \frac{1}{8}, \frac{1}{2} + \frac{1}{8}]$. Set $y_j = (n_j + \frac{1}{8})$. Observe that $y_j - I \cap \Lambda^{\alpha^k} = \alpha^j \mathbb{Z} \cap (y_j - I)$, $y_j - J \cap \Lambda^{\alpha^k} = \alpha^{j-1} \mathbb{Z} \cap (y_j - J)$ and the additive groups $\alpha^j \mathbb{Z}$ and $\alpha^{j-1}\mathbb{Z}$ have only 0 in common. Setting $N_j = \#(y_j - I \cap \Lambda^{\alpha^k})$ we have $N_j \to \infty$ and hence all conditions of Theorem 1 are satisfied.

If $\alpha = 1/q$ one can repeat the proof of Theorem 1 in [1] by using the following modification of Lemma 2 of [1], we leave the details to the reader.

Lemma 2. *Let $\varphi_n : \mathbb{T} \to \mathbb{R}$ be a sequence of positive measurable functions, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. If $\sum_{n=1}^{\infty} \varphi_n(q^n t) < \infty$ a.e., it then follows that $\sum_{n=1}^{\infty} \varphi_n(q^n t + \frac{k}{q}) < \infty$ for $k = 1, ..., q-1$.*

3 The case $\alpha = \frac{p}{q}$

We assume that we work with fixed relative prime integers $p, q > 1$ and $p < q$. For ease of notation we denote $\Lambda^{(\frac{p}{q})^k}$ by Λ and $\Lambda_k^{(\frac{p}{q})^k}$ by Λ_k. Choose ν_k such that $n_k \leq \nu_k (\frac{p}{q})^k < n_k + (\frac{p}{q})^k$. Observe that all $\lambda \in \Lambda_k \cap [n_k, n_{k+1})$ are of the form $(\nu_k + l)(\frac{p}{q})^k$, where $l \in \{0, ..., [(n_{k+1} - n_k)(\frac{p}{q})^{-k}]\}$.

Assume the integer $r > 8$, $(r, p) = 1$, and $(r, q) = 1$.

The main result of our paper is the following:

Theorem 3. *The set Λ defined above is of type 2. Moreover, there exists a characteristic function $f : \mathbb{R} \to \mathbb{R}^+$ such that almost every point in $[\frac{1}{r}, \frac{2}{r}]$ belongs to $C(f, \Lambda)$, $|D(f, \Lambda) \cap [1 + \frac{1}{r}, 1 + \frac{2}{r}]| > \frac{1}{4r}$ and $|D(f, \Lambda) \cap [-1 + \frac{1}{r}, -1 + \frac{2}{r}]| > \frac{1}{4r}$.*

To prove Theorem 3 we need a lemma.

Lemma 4. *Given positive integers M, K_{M-1} we can choose the sets $L^M \subset [\frac{1}{r}, \frac{2}{r}]$, $H^{1,M} \subset [1+\frac{1}{r}, 1+\frac{2}{r}]$, $H^{-1,M} \subset [-1+\frac{1}{r}, -1+\frac{2}{r}]$ and an integer K_M such that*

$$|L^M| < \frac{2^{-M}}{r}, \tag{2}$$

$$|H^{1,M}| \geq \frac{1}{4r}, \quad |H^{-1,M}| \geq \frac{1}{4r} \tag{3}$$

and a characteristic function $f_M : \mathbb{R} \to \mathbb{R}^+$, such that letting

$$\Lambda_{K_{M-1},K_M} = \cup_{k=K_{M-1}+1}^{K_M} \Lambda_k$$

and

$$s_{M-1,M}(x) = \sum_{\lambda \in \Lambda_{K_{M-1},K_M}} f_M(x+\lambda)$$

we have

$$s_{M-1,M}(x) = 0, \text{ for } x \in \left[\frac{1}{r}, \frac{2}{r}\right) \setminus L^M, \tag{4}$$

$$s_{M-1,M}(x) \geq 1, \text{ for } x \in H^{1,M} \cup H^{-1,M} \tag{5}$$

and $f_M = 0$ on $(-\infty, n_{K_{M-1}+2}) \cup (n_{K_M-2}, +\infty) \supset (-\infty, n_{K_{M-1}}+2) \cup (n_{K_M} - 2, +\infty)$.

Proof of Theorem 3. Set $M = 1$, $K_0 = 1$. Using induction and repeated application of Lemma 4 choose the sets L^M, $H^{1,M}$, $H^{-1,M}$, the functions f_M and the constants K_M for $M = 1, 2, \ldots$. Set $f(x) = \sum_{M=1}^\infty f_M(x)$ and observe that $s(x) = \sum_{M=1}^\infty s_{M-1,M}(x)$ on $[-1, 2]$ and hence by the Borel-Cantelli lemma, (2) and (4) imply that $s(x) < \infty$ for almost every point of $[\frac{1}{r}, \frac{2}{r}]$. On the other hand, (5) implies that $s(x) = \infty$ for $\cap_{N=1}^\infty \cup_{M=N}^\infty H^{1,M} \cup H^{-1,M}$. Using (3), we have

$$\left| \bigcap_{N=1}^\infty \bigcup_{M=N}^\infty H^{1,M} \cap \left[1+\frac{1}{r}, 1+\frac{2}{r}\right] \right| \geq \frac{1}{4r},$$

and

$$\left| \bigcap_{N=1}^\infty \bigcup_{M=N}^\infty H^{-1,M} \cap \left[-1+\frac{1}{r}, -1+\frac{2}{r}\right] \right| \geq \frac{1}{4r}.$$

□

Proof of Lemma 4. Set $m = 2^{M+2}$ and choose $k_1 > K_{M-1} + 2$ such that $(\frac{p}{q})^{-k_1} < \frac{1}{4r}$. Assuming k_{j-1} is defined choose $k_j > k_{j-1}$ such that

$$q^{k_j - k_{j-1}} > \frac{10p(n_{k_j+1} - n_{k_j})}{(\frac{p}{q})^{k_j}}. \tag{6}$$

Do this for $j = 2, ..., m$ and put $K_M = k_m + 4$.

Set

$$N_j = \left[\frac{3(n_{k_j+1} - n_{k_j})}{(\frac{p}{q})^{k_j}} \right] \tag{7}$$

and assume $\tau \in [0, \frac{1}{r})$ is fixed. Let

$$\Phi(\tau) = \{\tau + l_1(\frac{p}{q})^{k_1} + ... + l_m(\frac{p}{q})^{k_m} + \frac{1}{r}t : 1 \le l_j \le N_j, t \in \mathbb{Z}\}. \tag{8}$$

Note that $\Phi(\tau)$ is periodic by $\frac{1}{r}$, and next we show that its elements have a certain "independence" property.

Assume $x \in \Phi(\tau)$ has the representations:

$$x = \tau + l_1(\frac{p}{q})^{k_1} + ... + l_m(\frac{p}{q})^{k_m} + \frac{1}{r}t = \tau + l_1'(\frac{p}{q})^{k_1} + ... + l_m'(\frac{p}{q})^{k_m} + \frac{1}{r}t'. \tag{9}$$

Then, after multiplying by $r \cdot q^{k_m}$ and rearranging we obtain $q^{k_m}(t - t') = A_1 r$, where A_1 is an integer. Since $(q, r) = 1$ we have $r|t - t'$ and hence $\frac{1}{r}t - \frac{1}{r}t' \in \mathbb{Z}$.

Now, assume $j \le m$ and we have already verified that $l_i = l_i'$ for $i > j$. Then, multiplying (9) by q^{k_j} and rearranging we obtain that $(l_j - l_j')p^{k_j} = A_2 \cdot q^{k_j - k_{j-1}}$, where $A_2 \in \mathbb{Z}$. Hence, $q^{k_j - k_{j-1}}|l_j - l_j'$, but by (6), (7), and (8) we have $|l_j - l_j'| < q^{k_j - k_{j-1}}$. Therefore, $l_j = l_j'$. Thus each x in $\Phi(\tau)$ has a unique representation given by (9). This implies

$$\#\left(\Phi(\tau) \cap \left[\alpha, \alpha + \frac{1}{r}\right)\right) = N_1 \cdots N_m \text{ for arbitrary } \alpha \in \mathbb{R}. \tag{10}$$

The set $\Phi(\tau)$ will be used to define those points in $[\frac{1}{r}, \frac{2}{r})$ where $s_{M-1,M}(x)$ can take values different from 0. In the sequel it will be enlarged to obtain a set of positive measure. Set $S_j = [(n_{k_j+1} - n_{k_j})(\frac{p}{q})^{-k_j}] + 1$,

$$\begin{aligned} \Theta_j(\tau) = \{ &\tau + l_1(\frac{p}{q})^{k_1} + ... + (\nu_{k_j} + l_j)(\frac{p}{q})^{k_j} + ... + l_m(\frac{p}{q})^{k_m} + \frac{1}{r}t : \\ &l_i \in \{1, ..., N_i\} \text{ for } i \ne j, \ l_j \in \{S_j, ..., N_j\}, \ t \in \mathbb{Z}\}. \end{aligned} \tag{11}$$

In the sequel we will choose certain values of τ and fatten up to be $\Theta_j(\tau) \cap [n_{k_j} + \frac{1}{2}, n_{k_j+1})$ the collection of those points where $f_M = 1$ in $[n_{k_j}, n_{k_j+1})$.

If $x \in (0, \frac{1}{2})$ and $x + \lambda \in [n_{k_j} + \frac{1}{2}, n_{k_j+1})$ for some $\lambda \in \Lambda$, then $n_{k_j} < \lambda < n_{k_j+1}$ and hence

$$\lambda = (\nu_{k_j} + \bar{l}_j)(\frac{p}{q})^{k_j} \text{ for some } \bar{l}_j \in \{0, ..., S_j - 1\}. \tag{12}$$

Thus, $x+\lambda \in \Theta_j(\tau)$ (that is, $f_M(x)=1$) would imply

$$x+(\nu_{k_j}+\bar{l}_j)(\frac{p}{q})^{k_j} = \tau + l_1(\frac{p}{q})^{k_1} + ... + (\nu_{k_j}+l_j)(\frac{p}{q})^{k_j} + ... + l_m(\frac{p}{q})^{k_m} + \frac{1}{r}t.$$

Therefore,

$$x = \tau + l_1(\frac{p}{q})^{k_1} + ... + (l_j-\bar{l}_j)(\frac{p}{q})^{k_j} + ... + l_m(\frac{p}{q})^{k_m} + \frac{1}{r}t.$$

By (11) and (12), $l_j-\bar{l}_j \in \{1,...,N_j\}$ and hence

$$x \in (0,\frac{1}{2}),\ x+\lambda \in \Theta_j(\tau)\cap[n_{k_j}+\frac{1}{2}, n_{k_j+1}) \text{ implies } x \in \Phi(\tau). \tag{13}$$

Next we fix the values of $\bar{\bar{l}}_{j,1}$ and $\bar{\bar{l}}_{j,-1}$ so that

$$\begin{aligned} &\text{i) } p \not| \nu_{k_j-1}+\bar{\bar{l}}_{j,1} \text{ and } q \not| \nu_{k_j+1}+\bar{\bar{l}}_{j,-1}, \\ &\text{ii) } \lambda_{j,1} := (\nu_{k_j-1}+\bar{\bar{l}}_{j,1})\left(\frac{p}{q}\right)^{k_j-1} \in \Lambda_{k_j-1}\cap[n_{k_j}-1, n_{k_j}), \\ &\quad\ \lambda_{j,-1} := (\nu_{k_j+1}+\bar{\bar{l}}_{j,-1})\left(\frac{p}{q}\right)^{k_j+1} \in \Lambda_{k_j+1}\cap[n_{k_j+1}, n_{k_j+1}+1), \\ &\text{iii) } \left[1+\frac{1}{r}, 1+\frac{2}{r}\right]+\lambda_{j,1} \subset \left[n_{k_j}+1-\frac{4}{r}, n_{k_j}+1-\frac{2}{r}\right], \\ &\text{iv) } \left[-1+\frac{1}{r}, -1+\frac{2}{r}\right]+\lambda_{j,-1}\left(\frac{p}{q}\right)^{k_j+1} \subset \left[n_{k_j+1}-\frac{4}{r}, n_{k_j+1}-\frac{2}{r}\right]. \end{aligned} \tag{14}$$

Set

$$\Psi_{j,1}(\tau) = (\Theta_j(\tau)-\lambda_{j,1})\cap[1+\frac{1}{r}, 1+\frac{2}{r}),$$

and

$$\Psi_{j,-1}(\tau) = (\Theta_j(\tau)-\lambda_{j,-1})\cap[-1+\frac{1}{r}, -1+\frac{2}{r}).$$

These sets will be enlarged by adding a small interval to define $H^{1,M}$ and $H^{-1,M}$ and will give points $x \in [-1+\frac{1}{r}, -1+\frac{2}{r}] \cup [1+\frac{1}{r}, 1+\frac{2}{r})$ for which $f_M(x+\lambda_{j,-1})=1$ or $f_M(x+\lambda_{j,1})=1$.

Arguing as at (10), using (11) and (7) one can show that

$$\#\left(\Theta_j(\tau)\cap\left[\alpha, \alpha+\frac{1}{r}\right)\right) > \frac{N_1\cdots N_m}{2}$$

holds for all $\alpha \in \mathbb{R}$. This implies

$$\begin{aligned} &\#\Psi_{j,1}(\tau) > \frac{N_1\cdots N_m}{2} = \frac{\#(\Phi(\tau)\cap[\alpha,\alpha+\frac{1}{r}))}{2}, \text{ and} \\ &\#\Psi_{j,-1}(\tau) > \frac{N_1\cdots N_m}{2} = \frac{\#(\Phi(\tau)\cap[\alpha,\alpha+\frac{1}{r}))}{2}, \text{ for all } \alpha \in \mathbb{R}. \end{aligned} \tag{15}$$

Next we want to show that for $j' < j$

$$\Psi_{j,1}(\tau) \cap \Psi_{j',1}(\tau) = \emptyset, \text{ and} \tag{16}$$

$$\Psi_{j,-1}(\tau) \cap \Psi_{j',-1}(\tau) = \emptyset. \tag{17}$$

This will be the "independence" property of the sets $\Psi_{j,\pm 1}$. Since they are disjoint we will obtain more than $m \cdot N_1 \cdots N_m/2 = (m/2)\#(\Phi(\tau) \cap [\frac{1}{r}, \frac{2}{r}))$ points in $[-1 + \frac{1}{r}, -1 + \frac{2}{r})$ or in $[1 + \frac{1}{r}, 1 + \frac{2}{r})$, where $\sum_{\lambda \in \Lambda_{M-1,M}} f(x + \lambda) \geq 1$ will hold.

First we argue to verify (16). Proceeding towards a contradiction assume $x \in \Psi_{j,1}(\tau) \cap \Psi_{j',1}(\tau)$. Then there exist

$$t, t' \in \mathbb{Z},\ l_i \in \{1, ..., N_i\} \text{ for } i \neq j, l'_i \in \{1, ..., N_i\} \text{ for } i \neq j', \tag{18}$$

$$l_j \in \{S_j, ..., N_j\}, \quad l'_{j'} \in \{S_{j'}, ..., N_{j'}\}, \text{ such that} \tag{19}$$

$$\tau + l_1(\frac{p}{q})^{k_1} + ... + (\nu_{k_j} + l_j - (\nu_{k_j-1} + \bar{\bar{l}}_{j,1})\frac{q}{p})(\frac{p}{q})^{k_j} + ... + l_m(\frac{p}{q})^{k_m} + \frac{1}{r}t =$$

$$\tag{20}$$

$$\tau + l'_1(\frac{p}{q})^{k_1} + ... + (\nu_{k_{j'}} + l'_{j'} - (\nu_{k_{j'}-1} + \bar{\bar{l}}_{j',1})\frac{q}{p})(\frac{p}{q})^{k_j} + ... + l'_m(\frac{p}{q})^{k_m} + \frac{1}{r}t'.$$

Again, as we argued earlier, after multiplying by $r \cdot q^{k_m}$ and rearranging we obtain $q^{k_m}(t - t') = A_3 \cdot r$ where $A_3 \in \mathbb{Z}$. Using $(q, r) = 1$ we have $r | t - t'$. Therefore, $\frac{1}{r}t - \frac{1}{r}t' \in \mathbb{Z}$.

Now assume $i \in \{j+1, ..., m\}$ and we have shown $l_\iota = l'_\iota$ for $\iota > i$. Multiplying (20) by q^{k_i} and rearranging we infer $(l_i - l'_i)p^{k_i} = A_4 q^{k_i - k_{i-1}}$ where $A_4 \in \mathbb{Z}$.

Since $(p, q) = 1$ we again find that $q^{k_i - k_{i-1}} | l_i - l'_i$. But by (18) and (6) we have $|l_i - l'_i| < q^{k_i - k_{i-1}}$, hence $l_i = l'_i$.

Now we turn to the case when $i = j$. This time after multiplying (20) by q^{k_j} and rearranging we have

$$(\nu_{k_j} + l_j - (\nu_{k_j-1} + \bar{\bar{l}}_{j,1})\frac{q}{p} - l'_{j'})p^{k_j} = q^{k_j - k_{j-1}} A_5 \tag{21}$$

where $A_5 \in \mathbb{Z}$.

Hence,

$$q^{k_j - k_{j-1}} | (p(\nu_{k_j} + l_j - l'_j) - q(\nu_{k_j-1} + \bar{\bar{l}}_{j,1}))p^{k_j-1}. \tag{22}$$

Since $(p, q) = 1$, we obtain

$$q^{k_j - k_{j-1}} | p(\nu_{k_j} + l_j - l'_j) - q(\nu_{k_j-1} + \bar{\bar{l}}_{j,1}) := B_1. \tag{23}$$

Recall that

$$n_{k_j} \leq \nu_{k_j} \left(\frac{p}{q}\right)^{k_j} < n_{k_j} + (\frac{p}{q})^{k_j} \text{ and } n_{k_j} - 1 \leq (\nu_{k_j-1} + \bar{\bar{l}}_{j,1}) \left(\frac{p}{q}\right)^{k_j-1} < n_{k_j}.$$

Thus,

$$0 < \nu_{k_j}\left(\frac{p}{q}\right)^{k_j} - (\nu_{k_j-1} + \bar{\bar{l}}_{j,1})\left(\frac{p}{q}\right)^{k_j-1} < 2$$

and

$$0 < p\nu_{k_j} - (\nu_{k_j-1} + \bar{\bar{l}}_{j,1})q < \frac{2 \cdot q^{k_j}}{p^{k_j-1}} = \frac{2p}{(\frac{p}{q})^{k_j}}.$$

Using (19), (7) and (6) we infer $|B_1| < q^{k_j - k_{j-1}}$ and (23) implies $B_1 = 0$, but this is impossible since by (14) i), we assumed $p \not|(\nu_{k_j-1} + \bar{\bar{l}}_{j,1})$. This proves (16).

To verify (17) we use an argument which is simpler and similar to the case (16). We just outline the differences of the arguments. So, (20) is replaced by

$$\tau + l_1(\frac{p}{q})^{k_1} + ... + (\nu_{k_j} + l_j - (\nu_{k_j+1} + \bar{\bar{l}}_{j,-1})\frac{p}{q})(\frac{p}{q})^{k_j} + ... + l_m(\frac{p}{q})^{k_m} + \frac{1}{r}t =$$

$$\tau + l_1'(\frac{p}{q})^{k_1} + ... + (\nu_{k_{j'}} + l_{j'}' - (\nu_{k_{j'}+1} + \bar{\bar{l}}_{j',-1})\frac{p}{q})(\frac{p}{q})^{k_{j'}} + ... + l_m'(\frac{p}{q})^{k_m} + \frac{1}{r}t'. \tag{24}$$

One can show again that $\frac{1}{r}t - \frac{1}{r}t' \in \mathbb{Z}$ and for $i \in \{j+1, ..., m\}$, $l_i = l_i'$. When $i = j$ we need to replace (22) by

$$q^{k_j - k_{j-1}}|(\nu_{k_j} + l_j - l_j')p^{k_j} - (\nu_{k_j+1} + \bar{\bar{l}}_{j,-1})\frac{p^{k_j+1}}{q} \tag{25}$$

which contradicts (14) i), since by this assumption the right hand side of the above formula is not an integer.

Now we introduce
$\widetilde{\Psi}_{j,1}(\tau) = \Psi_{j,1}(\tau) + \frac{1}{r}t$, $t \in \mathbb{Z}$ and $\widetilde{\Psi}_{j,-1}(\tau) = \Psi_{j,1}(\tau) + \frac{1}{r}t$, $t \in \mathbb{Z}$, which are the $\frac{1}{r}$ periodic extensions of $\Psi_{j,1}(t)$ and $\Psi_{j,-1}(t)$.

Our aim is to show that if $\epsilon > 0$ is sufficiently small and we add an interval I_ϵ of length ϵ to these discrete periodic sets then we have the required function. For $\epsilon > 0$ set $I_\epsilon = [0, \epsilon]$ and put
$\Phi'(\tau) = (\Phi(\tau) + I_\epsilon) \cap [\frac{1}{r}, \frac{2}{r})$,
$\Psi_1'(\tau) = (\cup_{j=1}^m(\widetilde{\Psi}_{j,1}(\tau) + I_\epsilon)) \cap [1 + \frac{1}{r}, 1 + \frac{2}{r})$,
$\Psi'_{-1}(\tau) = (\cup_{j=1}^m(\widetilde{\Psi}_{j,-1}(\tau) + I_\epsilon)) \cap [-1 + \frac{1}{r}, -1 + \frac{2}{r})$.
Now, next we want to take advantage of the estimates (10) and (15), to obtain measure estimate (26) below. In our notation usually we denote by $'$ those sets which are obtained from discrete point sets after the addition of I_ϵ.

Indeed, using (10), (15), (16) and (17) one can easily see that if $\epsilon > 0$ is chosen sufficiently small then we have

$$|\Psi_1'(\tau)| > \frac{m}{2}|\Phi'(\tau)| \text{ and } |\Psi'_{-1}(\tau)| > \frac{m}{2}|\Phi'(\tau)|. \tag{26}$$

We now fix such a small $\epsilon > 0$ and set
$\Theta'(\tau) = \cup_{j=1}^m(\Theta_j(\tau) + I_\epsilon \cap [n_{k_j} + \frac{1}{2}, n_{k_j+1}))$.

Observe that from (13) it follows that

$$\text{if } x \in \left[\frac{1}{r}, \frac{2}{r}\right), \text{ and } 1\!\!1_{\Theta'(\tau)}(x+\lambda) = 1 \text{ then } x \in \Phi'(\tau). \tag{27}$$

It is also clear that from $x \in \Psi'_1(\tau)$ or from $x \in \Psi'_{-1}(\tau)$, it follows that

$$\sum_{j=1}^{m} 1\!\!1_{\Theta'(\tau)}(x+\lambda_{j,1}) \geq 1 \text{ or } \sum_{j=1}^{m} 1\!\!1_{\Theta'(\tau)}(x+\lambda_{j,-1}) \geq 1.$$

Hence

$$\sum_{\lambda \in \Lambda_{K_{M-1}, K_M}} 1\!\!1_{\Theta'(\tau)}(x+\lambda) \geq 1 \tag{28}$$

for these $x \in \Psi'_1(\tau) \cup \Psi'_{-1}(\tau)$.

Next we choose translations $0 \leq \tau_{i,1}, \tau_{i,-1} < \frac{1}{r}$, $i = 1, ..., \gamma$, where the number γ will be determined later. Set $\tau_{1,1} = \tau_{1,-1} = 0$. Assume that we have already defined $\tau_{i,1}$, and $\tau_{i,-1}$ for $i = 1, ..., j$ for some $j \geq 1$ so that if $H'_{j,1} = \cup_{i=1}^{j} \Psi'_1(\tau_{i,1})$, and $H'_{j,-1} = \cup_{i=1}^{j} \Psi'_{-1}(\tau_{i,-1})$, then

$$\begin{aligned} &\text{from } |H'_{j,1}| < \frac{1}{4}\frac{1}{r} \text{ it follows that } |H'_{j,1}| \geq \frac{jm}{4}|\Phi'(0)|, \text{ and} \\ &\text{from } |H'_{j,-1}| < \frac{1}{4}\frac{1}{r} \text{ it follows that } |H'_{j,-1}| \geq \frac{jm}{4}|\Phi'(0)|. \end{aligned} \tag{29}$$

By choosing $\epsilon > 0$ sufficiently small we can assume $|H'_{1,\pm 1}| < \frac{1}{4}\frac{1}{r}$.

We also put $L'_{j,1} = \cup_{i=1}^{j} \Phi'(\tau_{i,1})$, $L'_{j,-1} = \cup_{i=1}^{j} \Phi'(\tau_{i,-1})$.

Notice that $|\Phi'(\tau)| = |\Phi'(0)|$, $|\Psi'_1(\tau)| = |\Psi'_1(0)|$, and $|\Psi'_{-1}(\tau)| = |\Psi'_{-1}(0)|$ for all $\tau \in [0, \frac{1}{r})$. It is clear that (29) holds when $j = 1$.

The definitions of $L'_{j,1}$ and $L'_{j,-1}$ imply that $|L'_{j,1}| \leq j|\Phi'(0)|$, and $|L'_{j,-1}| \leq j|\Phi'(0)|$.

If $|H'_{j,1}| < \frac{1}{4}\frac{1}{r}$ then $\frac{1}{4}\frac{1}{r} > |H'_{j,1}| \geq \frac{jm}{4}|\Phi'(0)| \geq \frac{m}{4}|L'_{j,1}|$. Similarly, if $|H'_{j,-1}| < \frac{1}{4}\frac{1}{r}$ then $\frac{1}{4}\frac{1}{r} > |H'_{j,-1}| \geq \frac{jm}{4}|\Phi'(0)| \geq \frac{m}{4}|L_{j,-1}|$. This implies

$$2^{-M-2} \cdot \frac{1}{r} > |L'_{j,1}|, \text{ or } 2^{-M-2} \cdot \frac{1}{r} > |L'_{j,-1}|. \tag{30}$$

Next we show how to choose $\tau_{j+1,1}$ and $\tau_{j+1,-1}$.

Assume $|H'_{j,1}| < \frac{1}{4}\frac{1}{r}$. We claim that there exists $\tau_{j+1,1} \in [0, \frac{1}{r})$ such that

$$|H'_{j,1} \cap \Psi'_1(\tau_{j+1,1})| < \frac{1}{2}|\Psi'_1(\tau_{j+1,1})|. \tag{31}$$

In the sequel $\sigma +_r \tau$ denotes the sum of σ and τ modulo $\frac{1}{r}$. We have

$$r\int_0^{\frac{1}{r}} |H'_{j,1} \cap \Psi'_1(\tau)| d\tau = r\int_0^{\frac{1}{r}} \int_0^{\frac{1}{r}} 1\!\!1_{H'_{j,1}}(\sigma) \cdot 1\!\!1_{\Psi'_1(0)+_r\tau}(\sigma) d\sigma d\tau =$$

$$r\int_0^{\frac{1}{r}}\int_0^{\frac{1}{r}} \mathbb{1}_{H'_{j,1}}(\sigma)\cdot \mathbb{1}_{\Psi'_1(0)}(\sigma +_r (-\tau))d\sigma d\tau =$$

$$r\int_0^{\frac{1}{r}}\int_0^{\frac{1}{r}} \mathbb{1}_{H'_{j,1}}(\sigma' +_r \tau)\cdot \mathbb{1}_{\Psi'_1(0)}(\sigma')d\sigma' d\tau = r|H'_{j,1}|\cdot|\Psi'_1(0)| < \frac{1}{4}|\Psi'_1(0)|.$$

This implies that there exists $\tau_{j+1,1}$ for which (31) holds.

Now,

$$|H'_{j+1,1}| = |H'_{j,1}\cup(\Psi'_1(\tau_{j+1,1})\setminus H'_{j,1})| >$$

$$\frac{jm}{4}|\Phi'(0)| + \frac{1}{2}|\Psi'_1(\tau_{j+1,1})| >$$

$$\frac{jm}{4}|\Phi'(0)| + \frac{1}{2}\frac{m}{2}|\Phi'(0)| = (j+1)\frac{m}{4}|\Phi'(0)|,$$

where we also used (26).

This will yield (29) with j replaced by $j+1$ when $|H'_{j+1,1}| < \frac{1}{4}\frac{1}{r}$. Also observe that if $|H'_{j+1,1}| \geq \frac{1}{4}\frac{1}{r}$ then, recalling (30),
$|\Phi'(\tau_{j+1,1})| = |\Phi'(\tau_{j,1})| \leq |L'_{j,1}| < 2^{-M-2}\cdot\frac{1}{r}$ implies

$$|L'_{j+1,1}| = |L'_{j,1}\cup\Phi'(\tau_{j+1,1})| < 2^{-M-1}\cdot\frac{1}{r}. \tag{32}$$

From (29) it follows that we reach a step j when $|H'_{j+1,1}| \geq \frac{1}{4}\frac{1}{r}$ but $|H'_{j,1}| < \frac{1}{4}\frac{1}{r}$. For this value of j set $\gamma_1 = j+1$. By (32) we have $|L'_{\gamma_1,1}| < 2^{-M-1}\cdot\frac{1}{r}$.

Argue similarly for choosing $\tau_{j+1,-1}$ and choose γ_{-1} as γ_1 was chosen.

Set $\gamma = \max(\gamma_1,\gamma_{-1})$. If $\gamma > \gamma_{-1}$ then for $\gamma_{-1} < j \leq \gamma$ set $\tau_{j,-1} = \tau_{1,-1} = 0$. This implies that $L'_{j,-1} = L'_{\gamma_{-1},-1}$ for $\gamma_{-1} \leq j \leq \gamma$. Similarly if $\gamma > \gamma_1$ then for $\gamma_1 < j \leq \gamma$ set $\tau_{j,1} = \tau_{1,1} = 0$. This implies that $L'_{j,1} = L'_{\gamma_1,1}$ for $\gamma_1 \leq j \leq \gamma$.

Set $L^M = L'_{\gamma,1}\cup L'_{\gamma,-1}$. We have
$|L^M| \leq |L'_{\gamma_1,1}| + |L'_{\gamma_{-1},-1}| < 2^{-M}\cdot\frac{1}{r}$.

Put $H^{1,M} = H'_{\gamma,1} = H'_{\gamma_1,1}$, and $H^{-1,M} = H'_{\gamma,-1} = H'_{\gamma_{-1},-1}$. Hence $|H^{1,M}| \geq \frac{1}{4}\cdot\frac{1}{r}$, $|H^{-1,M}| \geq \frac{1}{4}\cdot\frac{1}{r}$.

Finally, set $f_M = \sum_{j=1}^{\gamma}(\mathbb{1}_{\Theta'(\tau_{j,1})} + \mathbb{1}_{\Theta'(\tau_{j,-1})})$. Now (27) implies that if $x\in[\frac{1}{r},\frac{2}{r})$ and $f_M(x+\lambda) = 1$ then $x\in\cup_{j=1}^{\gamma}(\Phi'(\tau_{j,1})\cup\Phi'(\tau_{j,-1})) = L'_{\gamma,1}\cup L'_{\gamma,-1} = L^M$. Hence $s_{M-1,M}(x) = 0$ for $x\in[\frac{1}{r},\frac{2}{r})\setminus L^M$.

By using (28) we obtain that if
$x\in\cup_{j=1}^{\gamma}(\Psi'_1(\tau_{j,1})\cup\Psi'_{-1}(\tau_{j,-1})) = H^{1,M}\cup H^{-1,M}$ then $s_{M-1,M}(x)\geq 1$. This completes the proof of the lemma. □

References

[1] Z. Buczolich, J-P. Kahane and R.D. Mauldin, On series of translates of positive functions, (to appear).

[2] Z. Buczolich, J-P. Kahane and R.D. Mauldin, Sur les séries de fonctions positives, *C. R. Acad. Sci. Paris Sér. I. Math.* **329** (1999), no. 4, 261-264.

[3] Z. Buczolich and R.D. Mauldin, On the convergence of $\sum f(nx)$ for measurable functions, *Mathematika,* (to appear).

[4] Choquet, G., Construction effective de suites $(k(3/2)^n)$. Études des mesures (3/2)-stables, *C. R. Acad Sci. Paris Sér. A-B* **291** (1980), no. 2, A69-A74.

[5] J.A. Haight, A linear set of infinite measure with no two points having integral ratio, *Mathematika* **17** (1970), 133-138.

[6] J.A. Haight, A set of infinite measure whose ratio set does not contain a given sequence, *Mathematika* **22** (1975), 195-201.

[7] A. Khinchin, Ein Satz über Kettenbruche mit arithmetischen Anwendungen, *Math. Zeit.,* **18** (1923) 289-306.

[8] C. G. Lekkerkerker, Lattice points in unbounded point sets, I. *Indag. Math.,* **20** (1958) 197-205.

[9] J. M. Marstrand, On Khinchin's conjecture about strong uniform distribution, *Proc. London Math. Soc.* **3** 21 (1970) 540-56.

[10] H. v. Weizsäcker, *Zum Konvergenzverhalten der Reihe* $\Sigma_{n=1}^{\infty} f(nt)$ *für* λ-*messbare Funktionen* $f : \mathbb{R}^+ \to \mathbb{R}^+$, Diplomarbeit, Universität München, 1970.