Universality in inverse limit spaces of the logistic family occurs with positive measure

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Abstract

We prove that the trajectory of the turning point $c = \frac{1}{2}$ under $g_a = ax(1-x)$ at cutting times is dense in $[g_a^2(c), g_a(a)]$ for a set of parameters a of positive Lebesgue measure. Hence the inverse limit space (I, g_a) has a universality property for a positive measure set of parameters a of the logistic family.

1 Introduction

Inverse limit spaces of one-dimensional maps commonly appear as attractors in dynamical systems [1, 2, 3, 4, 5, 26]. In 1967, R.F. Williams [26] showed that hyperbolic one-dimensional attractors are inverse limits of maps on branched one-manifolds. Such attractors have the property of being everywhere locally homeomorphic with the product of a Cantor set and an arc. By contrast in [2], there is exhibited a residual full Lebesgue measure set of slopes a such that if T_a is a tent map with such a slope, then the inverse limit space (I, T_a) is nowhere locally homeomorphic to the product of a Cantor set and an arc. Moreover, every open set in (I, T_a) contains a homeomorphic copy of every inverse limit space (I, T_{a_i}) appearing in this family of inverse limit spaces (the slopes a_i are allowed to vary). Hence, there is a form of self-similarity and local recapitulation of the entire family of inverse limits of tent maps within (I, T_a). We say such inverse limit spaces have the *self-similarity* property; we precisely define this property

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below. The family of tent maps is given by: $T_a = \min\{ax, a(1-x)\}$ where $a \in [1,2]$ and $x \in [0,1]$.

When dealing with a one-parameter family of unimodal maps $\{f_a : I \to I\}_{a \in \Lambda}$ with common turning point c, each f_a has a kneading map $Q_a(k)$ and a sequence of cutting times $\{S_k(a)\}$. These notions are reviewed in the next section. If $f_a^2(c) < c < f_a(c)$, then $I_a \equiv [f_a^2(c), f_a(c)]$ is the smallest f_a-forward invariant interval and is referred to as the *core* of the map f_a. We let $\overline{U}$ denote the closure of a set U. To say that the *trajectory of the turning point c under f_a at cutting times is dense in the core,* means that

$$\overline{\{f_a^{S_k(a)}(c)\}}_{k \geq 0} = I_a. \tag{1}$$

For a sequence of parameters $\{a_i\}_{i \geq 1}$, the inverse limit space (I, f_{a_i}) is defined as $(I, f_{a_i}) = \{(x_0, x_1, x_2, ...) \mid f_{a_i}(x_i) = x_{i-1} \text{ for } i \geq 1\}$. For each parameter a let $\tilde{f}_a$ denote the map f_a restricted to its core I_a and rescaled such that $\tilde{f}_a$ maps I onto I. The inverse limit space (I, f_a) consists exactly of a homeomorphic copy of $(I, \tilde{f}_a)$ and an infinite ray entwined with $(I, \tilde{f}_a)$. Clearly any subcontinuum of $(I, \tilde{f}_a)$ is contained in (I, f_a) since $I_a \subset I$; the converse is not necessarily true. However for any proper non-trivial (not an arc or a point) subcontinuum $H \subset (I, f_a)$, $H \subset (I, \tilde{f}_a)$ holds [10]. Hence, one does not lose or gain nontrivial proper suncontinua when passing from (I, f_a) to $(I, \tilde{f}_a)$. We say an inverse limit space (I, f_a) has the *self-similarity* property provided that for every choice of parameters $\{a_i\}_{i \geq 1}$ and every open set $U \subset (I, \tilde{f}_a)$ there exists a subcontinuum $H \subset U$ such that H is homeomorphic to the inverse limit space $(I, \tilde{f}_{a_i})$. We call (I, f) a *unimodal inverse limit space* provided $f : I \to I$ is unimodal.

The one-parameter family of unimodal maps given by $g_a(x) = ax(1-x)$, for $a \in [0,4]$, is commonly referred to as the *logistic family.* There are other equivalent models, we use this model as it makes our calculations easiest. The logistic family is a full family, i.e., all possible combinatorics obtained by unimodal maps occurs within the logistic family [22, Chapter 2]. Hence, we say that a unimodal inverse limit space (I, f) has the *universality* property provided that for every choice of parameters $\{a_i\}_{i \geq 1}$, $a_i \in [0,4]$ for all i, and every open set $U \subset (I, \tilde{f})$ there exists a subcontinuum $H \subset U$ such that H is homeomorphic to the inverse limit space $(I, \tilde{g}_{a_i})$. Since not every one-parameter family of unimodal maps is full, universality is stronger than self-similarity. In this note, we prove that for a positive Lebesgue measure set of parameters a, the inverse limit space (I, g_a) has the universality property (and therfore also the self-similarity property). If the turning point $c = \frac{1}{2}$ is periodic or attracted to a periodic orbit under some g_a, then all proper subcontinua of (I, g_a) are points or arcs and hence (I, g_a) does not have the universality property. Thus, the set of parameters a for which (I, g_a) has the universality property cannot have full Lebesgue measure.

In [2], the tent maps which are shown to have the self-similarity property are such that the trajectory of the turning point c at cutting times is dense in the core. Moreover, if a tent map T_a has this property, i.e., (1) holds with $f_a = T_a$, then in fact the inverse limit space (I, T_a) has the self-similarity property; this follows from the constructions in the proofs of [2, Theorem 4, Theorem 5]. Set $\mathcal{A} = \{a \in [\sqrt{2}, 2] \mid \overline{\{T_a^{S_k(a)}(c)\}}_{k \geq 0} = [T_a^2(c), T_a(c)]\}$. In [2], it is shown that $\mathcal{A}$ is large topologically, namely it is a residual set, and it is remarked that $\mathcal{A}$ has full Lebesgue measure in $[\sqrt{2}, 2]$. In [9] it was shown that $\mathcal{D} = \{a \in [\sqrt{2}, 2] \mid \overline{\{T_a^n(c)\}}_{n \geq 0} = [T_a^2(c), T_a(c)]\}$ has full Lebesgue measure in $[\sqrt{2}, 2]$. Clearly, $\mathcal{A}$ is a proper subset of $\mathcal{D}$. Using the work of [9] it is not difficult to establish that $\mathcal{A}$ has full Lebesgue measure in $[\sqrt{2}, 2]$, see e.g. [14, Proposition 2]. Using standard renormalization techniques, one obtains similar results for $a \in [1, \sqrt{2})$. Although here, the 'core' consists of a finite number of pairwise disjoint nondegenerate closed intervals, see e.g. [25]. In related work [11], it is shown that $[\sqrt{2}, 2] \setminus \mathcal{D}$ is σ-porous.

Set

$$\mathcal{B} = \{a > 2 \mid \overline{\{g_a^{S_k(a)}(c)\}}_{k \geq 0} = [g_a^2(c), g_a(c)]\}. \tag{2}$$

If $a \in \mathcal{B}$, then g_a is topologically conjugate to a tent map, since any unimodal map without wandering intervals, restrictive intervals, or periodic attractors is conjugate to a tent map, see e.g. [12, page 16]. If two unimodal maps, $f : I \to I$ and $g : I \to I$, are topologically conjugate (meaning there is a homeomorphism $h : I \to I$, such that $h \circ f = g \circ h$), then (I, f) is homeomorphic to (I, g) as topological spaces. Hence, if $a \in \mathcal{B}$, then every open set $U \subset (I, g_a)$ contains a homeomorphic copy of any inverse limit space $(I, \tilde{T}_{a_i})$, since (1) is preserved by topological conjugacy and therefore (I, g_a) would be homeomorphic to some $(I, T_{\tilde{a}})$ with $\tilde{a} \in \mathcal{A}$. However, there exist (I, g_b) which are not homeomorphic to any (I, T_a) and hence this observation alone does not immediately give that (I, g_a) has the weaker self-similarity property for $a \in \mathcal{B}$. Using results of [4, 10] one can show that $a \in \mathcal{B}$ (or $a \in \mathcal{A}$) is indeed sufficient for (I, g_a) (or (I, T_a)) to have the universality property. Details are provided in the appendix of this note. Thus, in this note we prove the following.

Theorem 1 *The set $\mathcal{B}$, defined in (2), has positive Lebesgue measure, and thus universality in inverse limit spaces of the logistic family occurs with positive measure.*

As a first attempt at proving Theorem 1, one might consider the map $\varphi : a \mapsto h_{top}(g_a)$, where $h_{top}(g_a)$ denotes the topological entropy of g_a. The topological entropy of a tent map T_a is $\log(a)$ and topological entropy is preserved by topological conjugacy; thus if g_a is topologically conjugate to a tent map $T_{\tilde{a}}$, then $h_{top}(g_a) = \log(\tilde{a})$. Hence, if φ were nice, say absolutely continuous, then $\mathcal{A}$ having positive measure would give that $\mathcal{B}$ does also. However, it is unknown whether φ is absolutely continuous. It is known that φ^{-1} has very bad absolute continuity properties (even off the flat pieces) and it is conjectured that φ is not absolutely continuous. As is remarked in the introduction of [15, page 216], work of Bruin and Sands [15, 24] establish that for (Lebesgue) a.e. $\alpha \in [0, \log(2)]$, if f is an S-unimodal map with $h_{top}(f) = \alpha$, then f is Collet-Eckmann and the turning point is not typical for the absolutely continuous invariant probability measure (a.c.i.p.). On the other hand, Benedicks and Carleson [6, Theorem 3] show that there is a positive Lebesgue measure set of parameters a, which we denote by BC, for which the logistic map g_a is Collet-Eckmann and the turning point is typical for the a.c.i.p. Hence, $\{h_{top}(g_a) \mid a \in BC\}$ has zero Lebesgue measure but BC has positive Lebesgue measure; thus the bad absolutely continuous properties of φ^{-1}. It is precisely the set BC that we use in this note to prove Theorem 1.

As already remarked, it is known that for a positive Lebesgue measure set of parameters a, the map g_a has an a.c.i.p., see e.g. [6, 21, 22]. For a history and discussion of this and related results see [22, Chapter V]. We are not so interested in the a.c.i.p., but rather in the set $\mathcal{B}$ defined above. Our work follows that of Benedicks and Carleson [6, 7] as presented in [22, Theorem 6.1]. Although [22, Theorem 6.1] deals with a class of families of maps, of which the logisitic is one, we are interested in Theorem 6.1 applied to the logistic family. It is shown that there exists a set of parameters of positive Lebesgue measure, denoted here by BC, such that for $a \in BC$ the map g_a is Collet-Eckmann and hence has an a.c.i.p. [17]. We prove the following proposition, from which Theorem 1 is immediate.

Proposition 1 *Set $BC^* = \{a \in BC \mid \overline{\{g_a^{S_k}(c)\}}_{k \geq 0} = [g_a^2(c), g_a(c)]\}$. Then BC^* has positive Lebesgue measure. Notice that $BC^* = BC \cap \mathcal{B}$.*

One of the main ingredients in our proof of Proposition 1 is Lemma 2 of Section 3 of this note. Lemma 2 can be extracted (with some additional work) from the proof of [22, Theorem 6.1] and the work in [6, 7], and hence (in this sense) is not new. However, this lemma is not explicitly stated in these works. As we feel Lemma 2 is of independent interest, we explicitly state it in this note. The purpose of this note is to show explicitly how one can extract this cutting time result, i.e., Theorem 1, from the work of Benedicks and Carleson [6, 7], since cutting times are not considered in [6, 7].

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2 Preliminaries

A continuous map $f : [0,1] \to [0,1]$ is called *unimodal* if there exists a unique *turning* or *critical point*, c, such that $f|_{[0,c)}$ is increasing, $f|_{(c,1]}$ is decreasing, and $f(0) = f(1) = 0$. To avoid trivial cases, we assume that $f(c) > c > f(f(c))$. We denote the forward images of c by $c_i = f^i(c)$. Clearly the interval $[c_2, c_1]$ is forward invariant and f maps $[c_2, c_1]$ onto itself; the interval $[c_2, c_1]$ is called the *core* of the map f. The itinerary of a point $x \in [0,1]$ under the map f, denoted $I(x,f)$, is a one-sided infinite or finite string $(j \geq 0)$ given by:

$$I(x,f)_j = \begin{cases} 0 & \text{if } f^j(x) < c \\ * & \text{if } f^j(x) = c \\ 1 & \text{if } f^j(x) > c. \end{cases}$$

We take the convention that the itinerary terminates with the first $*$, past the $j = 0$ position, and hence in this case would be finite in length. The intinerary of c_1 is called the *kneading invariant* of the map f. For a basic introduction to the combinatorics of itineraries or kneading theory, see e.g. [16, Part II]. The next two remarks are included for Lemma 1.

Remark 1: There is a natural order on itineraries that corresponds to retaining or reversing orientation, depending on which branch of the unimodal map is used. For example, if f is increasing on $[0,c]$ and $I(x,f) = 101\dots$, then orientation is preserved in a neighborhood of x under f^3. This ordering has the property that $x < y$ implies $I(x,f) \leq I(y,f)$ [16, page 65]. Of course, we are assuming that f is a nice unimodal map, such as a tent or logistic map.

Remark 2: It is known that the map $a \mapsto I(c_1(a), g_a)$ is nondecreasing, this follows from work in [18] (again, there is a natural order on the collection of all kneading invariants for unimodal maps, see e.g. [16, Part II]). Thus if $a < b$, then for each $I(x, g_a)$ there exists $y \in [0,1]$ such that $I(y, g_b) = I(x, g_a)$ [16, Theorem II.3.8].

Let f^n be some iterate of f and let J be any maximal subinterval on which $f^n|_J$ is monotone. Then $f^n : J \to [0,1]$ is called a *branch* of f^n. A branch $f^n : J \to [0,1]$ is called a *central branch* if $c \in \partial J$. Hence there are always two central branches, the images of which are the same. An iterate n is called a *cutting time* if the image of the central branch of f^n contains c. The cutting times are denoted by $S_0, S_1, S_2, \dots$ ($S_0 = 1$ and $S_1 = 2$). If $f^{S_k} : J \to [0,1]$ is the left central branch of f^{S_k}, then there is a unique point $z_k \in J$ such that

$$f^{S_k}(z_k) = c. \tag{3}$$

By construction, z_k has the property that $\cup_{0<j\leq S_k} f^{-j}(c) \cap (z_k, c) = \emptyset$ and is therefore called a *closest precritical point*. The point $\hat{z}_k$, defined analogously for the right central branch of f^{S_k}, is also a closest precritical point. It can be proven that the difference of two consecutive cutting times is again a cutting time. Hence we can write

$$S_k - S_{k-1} = S_{Q(k)},$$

where $Q : \mathbf{N} \to \mathbf{N}$ is an integer function, called the *kneading map*. An equivalent statement is

$$c_{S_k} \in (z_{Q(k+1)-1}, z_{Q(k+1)}] \cup [\hat{z}_{Q(k+1)}, \hat{z}_{Q(k+1)-1}). \tag{4}$$

The kneading map was introduced by Hofbauer, see e.g. [19, 20]. The kneading map (or cutting times) determine the combinatorics of f completely. A survey of this tool can be found in [13]; our discussion follows [13].

Closely related to the kneading map is the *Hofbauer-tower* [19, 20]. Given a unimodal map f, the associated Hofbauer-tower is the disjoint union of intervals $\{D_n\}_{n\geq 1}$, where $D_1 = [0, c_1]$ and for $n \geq 2$,

$$D_{n+1} = \begin{cases} f(D_n) & \text{if } c \notin D_n \\ [c_{n+1}, c_1] & \text{if } c \in D_n. \end{cases}$$

Notice that the image of either central branch $f^n : J \to [0,1]$ is such that $f^n(J) = D_n$. Thus, n is a cutting time if and only if $c \in D_n$. Also, setting
$\nu = \min\{x < f(c) \mid f^{n-1}|[x, f(c)] \text{ is monotone}\}$, we have that

$$f^{n-1}([\nu, f(c)]) = D_n. \tag{5}$$

When more than one parameter value is being used, we may write $D_n(a)$ for the levels in the Hofbauer tower for g_a, $S_k(a)$ for the cutting times of g_a, or $c_n(a)$ for $g_a^n(c)$. Another notation for $c_n(a)$ is $\xi_n(a)$; the latter being used when one is interested in $g_a^n(c)$ as a function of the parameter a. The rationals are denoted by $\mathbf{Q}$. The Lebesgue measure of a set U is denoted $|U|$.

For a discussion of Lemma 1 see [24, Chapter 3]; we sketch a proof.

Lemma 1 *Fix $n \in \mathbf{N}$ and let U be a maximal open subinterval of monotonicity for the map $\xi_n(a) \equiv g_a^n(c)$ such that $\xi_n(U)$ contains the turning point $c = \frac{1}{2}$ in its interior. Let $a_* \in U$ be such that $\xi_n(a_*) = c$, set $W = (a_*, 4] \cap U$, and set $V = [0, a_*] \cap U$. Then n is a cutting time for each $a \in W$, and n is not a cutting time for each $a \in V$.*

Proof: Fix $b \in W$ and set $c_m = c_m(b) = g_b^m(c)$ for $m \in \mathbf{N}$. Without loss of generality, assume that $\xi_n(W) \subset (c, 1]$. As above, let $\nu = \min\{x < c_1 \mid g_b^{n-1}|[x, c_1] \text{ is monotone}\}$. It suffices to show that $c \in g_b^{n-1}((\nu, c_1)) = D_n(b)$, recall (5).

We have that $I(c_1(a), g_a)_{j=0,\dots,n} = I(c_1(b), g_b)_{j=0,\dots,n}$ for all $a \in W$ and $I(x, g_b)_{j=0,\dots,n-1} = I(c_1(b), g_b)_{j=0,\dots,n-1}$ for all $x \in (\nu, c_1(b)]$. Since $\xi_n(W) \subset (c, 1)$, we have $c_n(b) > c$. These two observations give that $(g_b^{n-1})' > 0$ on the maximal subinterval of monotonicity of g_b^{n-1} containing $c_1(b)$. Also, as $a_* < b$, there exists $\tilde{x} \in [0,1]$ with $I(\tilde{x}, g_b) = I(c_1(a_*), g_{a_*})$, see Remark 2. Combining these facts and using the monotonicity of itineraries (see Remark 1) leads to $\tilde{x} \in (\nu, c_1(b))$. Thus, $g_b^n(\tilde{x}) = c$ and hence $c \in D_n(b)$. Therefore n is a cutting time for g_b. Similarly one can argue that n is not a cutting time for any $a \in V$. □

3 Tools

Again, Lemma 2 can be extracted from the proof of [22, Theorem 6.1] and the work in [6, 7]. Providing a proof for Lemma 2 would require a reworking of most of the proof of [22, Theorem 6.1] and hence would be lengthy. In the interest of producing a short note, we do not include a proof but rather sketch how to extract the lemma from these works.

Lemma 2 *There exists a closed interval $I_0 \subset [0,1] \setminus \{c\}$ with $|I_0| > 0$ and such that for a set of parameters $BC_0 \subset BC$ we have for each $a_0 \in BC_0$ an infinite sequence $n_1 < n_2 < n_3 < \dots$ and subintervals $\omega_{n_i}^{a_0}$ such that*

1. $\omega_{n_i}^{a_0} \subset \omega_{n_i-1}^{a_0}$,

2. $\lim_{i\to\infty} |\omega_{n_i}^{a_0}| = 0$,
3. $a_0 \in \omega_{n_i}^{a_0}$,
4. $\xi_{n_i}(\omega_{n_i}^{a_0}) \supseteq I_0$,
5. $\xi_{n_i}|\omega_{n_i}^{a_0}$ *is monotone,*
6. *there exists* $C > 0$ *such that* $\frac{|\xi'_{n_i}(a)|}{|\xi'_{n_i}(b)|} < C$ *for all* $a, b \in \omega_{n_i}^{a_0}$*, and*
7. $|BC_0 \cap (4-\epsilon, 4]| > 0$ *for all* $\epsilon > 0$.

In fact, C *is independent of* a_0 *and* n_i*; note that* $C > 1$.

Here we give a short outline to explain why Lemma 2 is true and how it relates to the proof of the Benedicks and Carleson result as proved in [22, pages 402-433]. Recall that BC denotes the set of parameters constructed in the proof of [22, Theorem 6.1], explicitly $BC = E$ of [22, Theorem 6.1]. To each $a \in BC$ there are intervals ω_n^a associated. These intervals are nested, their lengths tend to zero, and ω_0^a is a small left neighborhood of $4 = a_*$. A neighborhood $U_\Delta \equiv (c - e^{-\Delta}, c + e^{-\Delta})$ of the turning point is considered, where Δ is some large integer. Assume now that $a_0 \in BC$ is fixed and for ease of notation we will write ω_n for $\omega_n^{a_0}$. In [22, pages 414-416] free returns of ω_n's to U_Δ are defined. If for some $n \in \mathbf{N}$ we have a free return, then $\xi_n(\omega_{n-1}) \cap U_\Delta \neq \emptyset$ and ω_n is defined as a suitable subinterval of ω_{n-1} (this subinterval might equal ω_{n-1}). If for n we do not have a free return, then $\omega_n = \omega_{n-1}$.

There are some free returns for which there is a substantial escape [22, page 415]; at these substantial escapes $|\xi_n(\omega_{n-1})| \geq \sqrt{|U_\Delta|} >> |U_\Delta|$. For almost every parameter value in BC there are infinitely many substantial escapes. This can be verified by scrutinizing the proofs of Lemmas 6.7-9 in [22], or by following Section 2.2 of [7, pages 81-86]. (The notation of [7] differs slightly from that of [22]; for example, escape positions are discussed instead of substantial escapes, etc..) We denote by BC' those parameter values in BC for which there are infinitely many substantial escapes. Assume now that $a_0 \in BC'$.

Given $\epsilon_1 > 0$ and using that $|\omega_n| \to 0$, choose and fix a large n' such that there is a substantial escape at n' and $|\omega_{n'}| < \epsilon_1$. For ease of notation, set $\omega = \omega_{n'}$. It is not difficult to see that there is a constant ζ_1 (not depending on n' or on a_0) such that $|\xi_n(\omega)| < \zeta_1 |\xi_{n-1}(\omega)|$ for all $n \leq n'$. Let $k_0 \leq n'$ be maximal such that $|\xi_{k_0}(\omega)| \leq |U_\Delta|$. Then $|\xi_{k_0}(\omega)| > \frac{|U_\Delta|}{\zeta_1} = \mu$.

Using the argument of the Main Distortion Lemma [22, pages 421-424] and knowing that for parameters in BC the parameter and space derivatives are comparable [22, Prop. 6.2, page 406], one can infer that $\frac{|\xi'_{k_0}(a)|}{|\xi'_{k_0}(b)|} < C$ for $a, b \in \omega$. This distortion estimate clearly implies that ξ_{k_0} is monotone on ω.

By Corollary (Jakobson) [22, page 403], we have that $|BC \cap (4-\epsilon, 4])| > 0$ for all $\epsilon > 0$. Now choose finitely many subintervals I^l such that each is of length $\frac{\mu}{3}$ and their union covers $[0,1]$. Then clearly there exists l_0 such that $I^{l_0} \subset \xi_{k_0}(\omega)$.

Since there are only finitely many intervals I^l, one can choose one of these intervals as I_0 and choose a measurable subset BC_0 of $BC' \subset BC$ such that conditions 1-7 of Lemma 2 hold for any $a_0 \in BC_0$, for the intervals ω_{n_i} defined by using the above substantial escape intervals and a sequence of $\epsilon_{1,i}$'s tending to zero.

Lemma 3 *Let* I_0 *be as in Lemma 2. Fix* $d_l \in (0, c)$ *and* $d_r \in (c, 1)$ *such that* $|c - d_l| = |c - d_r| \leq \frac{1}{100^{100}}$*. Choose* $k_1 \in \mathbf{N}$ *and* $I_1 \subset I_0$ *such that* $g_4^{k_1}|I_1$ *is monotone and* $g_4^{k_1}(I_1) = [0,1]$*. Set* $k_2 = \min\{k \mid g_4^k((c, d_l)) \ni c\}$*. Then there exists* $\epsilon > 0$ *such that for every* $a \in [4-\epsilon, 4]$ *we have the following:*

1. *there exists $I_a \subset I_1$ such that $g_a^{k_1}(I_a) \supset [d_l, d_r]$ and $g_a^{k_1}|I_a$ is monotone,*
2. *$g_a^{k_2+i}(c) < c$ for $0 \leq i \leq 2$,*
3. *$g_a^{k_2}((d_l, c)) \ni c$, and*
4. *$g_a^{k_2}|[d_l, c]$ is monotone.*

Each of I_1 and I_a are chosen to be closed intervals. As logistic maps are symmetric about the turning point c, from (3) and (4) we also have that $g_a^{k_2}((c, d_r)) \ni c$, and $g_a^{k_2}|[c, d_r]$ is monotone.

Proof: The lemma follows from the smoothness in both variables of the function $g(a, x) = ax(1-x)$, $g_4(c) = 1$, and $g_4^n(c) = 0$ for all $n > 1$. $\square$

Lemma 4 *Let ϵ be as in Lemma 3. Fix $a_0 \in (4-\epsilon, 4)$ and p, $q \in \mathbf{Q}$ with $(p - \frac{p+q}{4}, q + \frac{p+q}{4}) \subset (c_2(a_0), c_1(a_0)) \setminus \{c\}$. Let I_{a_0} be chosen by item (1) of Lemma 3. Then, we can define nondegenerate closed intervals $I_l \equiv I_l(a_0, p, q)$, $I_r \equiv I_r(a_0, p, q) \subset I_{a_0}$ and $\delta_{a_0} > 0$ with the following properties:*

1. *either $(p, q) \subset (c, c_1(a))$ for every $a \in (a_0 - \delta_{a_0}, a_0 + \delta_{a_0})$, or $(p, q) \subset (c_2(a), c)$ for every $a \in (a_0 - \delta_{a_0}, a_0 + \delta_{a_0})$,*
2. *$g_{a_0}^{k_1}(I_l) \subset (d_l, c)$ and $g_{a_0}^{k_1}(I_r) \subset (c, d_r)$,*
3. *if $a \in (a_0 - \delta_{a_0}, a_0 + \delta_{a_0})$, then $g_a^{k_1}(I_l) \subset (d_l, c)$, $g_a^{k_1}(I_r) \subset (c, d_r)$, and $g_a^{k_1}|J_{a_0}$ is monotone, where J_{a_0} is the shortest closed interval containing both I_l and I_r,*
4. *if $a \in (a_0 - \delta_{a_0}, a_0 + \delta_{a_0})$, then $g_a^{k_1+k}(I_l)$, $g_a^{k_1+k}(I_r) \subset (0, c)$ for $k = 1, \ldots, k_2$; if additionally $(p, q) \subset (c_2(a_0), c)$, then $g_a^{k_1+k_2+1}(I_l)$, $g_a^{k_1+k_2+1}(I_r) \subset (c, 1)$, and*
5. *if $a \in (a_0 - \delta_{a_0}, a_0 + \delta_{a_0})$, then $g_a^{k_1+k_2+i}(I_l)$, $g_a^{k_1+k_2+i}(I_r) \subset (p, q)$, where $i = 1$ if $(p, q) \subset (c, c_1(a))$ and $i = 2$ if $(p, q) \subset (c_2(a), c)$.*

Notice that each of $g_{a_0}^{k_1}|I_*$, $g_{a_0}^{k_1+k_2}|I_*$, and $g_{a_0}^{k_1+k_2+i}|I_*$ are monotone, where $* = r$ or l. Also, as $I_{a_0} \subset I_1 \subset I_0$, we have that I_r, $I_l \subset I_0$.

Proof: Again, consider the function of two variables, $g(a, x) = ax(1-x)$, which is continuous in both variables. The continuity of $g(a, x)$ gives (1). Item (2) follows from (1) of Lemma 3. Item (1) of Lemma 3 gives (3). Items (1)-(4) of Lemma 3 give (4) holding for $a = a_0$; then use the continuity of $g(a, x)$ to obtain (4) holding for $a \in (a_0 - \delta_0, a_0 + \delta_0)$. It remains to prove (5).

From (2)-(4) of Lemma 3 we have that $g_{a_0}^{k_2}|[d_l, c]$ is monotone, $g_{a_0}^{k_2}(c) < c$, and $[g_{a_0}^{k_2}(c), c] \subset g_{a_0}^{k_2}([d_l, c])$. Let z^* be the unique point in (d_l, c) such that $g_{a_0}^{k_2}(z^*) = c$. If $(p, q) \subset (c, c_1(a_0))$, then $g_{a_0}^{k_2+1}([z^*, c]) \supset (p, q)$ since (by (2) of Lemma 3) $g_{a_0}^{k_2+1}(c) < c$. If $(p, q) \subset (c_2(a_0), c)$, let z^{**} be the unique point in (z^*, c) such that $g_{a_0}^{k_2+1}(z^{**}) = c$; such a z^{**} exists by (2) of Lemma 3. Then, $g_{a_0}^{k_2+2}([z^*, z^{**}]) = (c_2(a_0), c_1(a_0)) \supset (p, q)$. Thus, (5) holds for $a = a_0$. Again, use the continuity of $g(a, x)$ to get that (5) holds for all $a \in (a - \delta_0, a + \delta_0)$. $\square$

Lemma 5 *Let ϵ be as in Lemma 3. Fix $a_0 \in (4-\epsilon, 4) \cap BC_0$, let $\{n_i\}$ be as in Lemma 2 for the fixed a_0, and let δ_0 be as in Lemma 4. Let $M \in \mathbf{N}$ be such that $i \geq M$ implies that $\omega_{n_i}^{a_0} \subset (4-\epsilon, 4) \cap (a_0 - \delta_{a_0}, a_0 + \delta_{a_0})$. Fix $i \geq M$. Set $\omega_l = \xi_{n_i}^{-1}(I_l) \cap \omega_{n_i}^{a_0}$ and $\omega_r = \xi_{n_i}^{-1}(I_r) \cap \omega_{n_i}^{a_0}$; I_r and I_l are as in Lemma 4. Then,*

1. *$n_i + k_1$ is a cutting time either for all $a \in \omega_l$ or for all $a \in \omega_r$,*

2. *if $n_i + k_1$ is a cutting time for all $a \in \omega_l$, then $n_i + k_1 + k_2 + i$ is a cutting time for all $a \in \omega_l$, where $i = 1$ if $(p, q) \subset (c, c_1(a_0))$ and $i = 2$ if $(p, q) \subset (c_2(a_0), c)$,*

3. *if $n_i + k_1$ is a cutting time for all $a \in \omega_r$, then $n_i + k_1 + k_2 + i$ is a cutting time for all $a \in \omega_r$, where $i = 1$ if $(p, q) \subset (c, c_1(a_0))$ and $i = 2$ if $(p, q) \subset (c_2(a_0), c)$.*

Proof: We have that $\xi_{n_i}|\omega_{n_i}^{a_0}$ is monotone, $I_0 \subset \xi_{n_i}(\omega_{n_i}^{a_0})$ (follows from Lemma 2), $I_0 \supset I_l,\ I_r$ (follows from Lemmas 3 and 4), and thus $\omega_l,\ \omega_r \subset \omega_{n_i}^{a_0}$.

Item (3) of Lemma 4 gives that $\xi_{n_i+k_1}(\omega_l) \subset (d_l, c)$ and $\xi_{n_i+k_1}(\omega_r) \subset (c, d_r)$. Let Γ_{a_0} be the shortest closed interval containing both ω_l and ω_r. Thus, $c \in \xi_{n_i+k_1}(\Gamma_{a_0}) \setminus (\xi_{n_i+k_1}(\omega_l) \cup \xi_{n_i+k_1}(\omega_r))$. Moreover, $\xi_{n_i+k_1}|\Gamma_{a_0}$ is monotone; for if not, then there exists $a \in (a_0 - \delta_{a_0}, a_0 + \delta_{a_0}) \cap \Gamma_{a_0}$ and $0 < j < k_1$ such that $\xi_{n_i+j}(a) = c$. But then, $x \equiv \xi_{n_i}(a) \in J_{a_0}$ (J_{a_0} from Lemma 4-3) and $g_a^j(x) = g_a^{n_i+j}(c) = \xi_{n_i+j}(a) = c$, contradicting item (1) of Lemma 3. Summing up, we have that $\xi_{n_i+k_1}|\Gamma_{a_0}$ is monotone, $\Gamma_{a_0} \supset \omega_l,\ \omega_r$, and $c \in \xi_{n_i+k_1}(\Gamma_{a_0}) \setminus (\xi_{n_i+k_1}(\omega_l) \cup \xi_{n_i+k_1}(\omega_r))$. Thus from Lemma 1 we have that $n_i + k_1$ is a cutting time for all $a \in \omega_l$, or for all $a \in \omega_r$, i.e., (1) holds.

We next prove (2) as the proof of (3) is similar. Assume that $n_i + k_1$ is a cutting time for all $a \in \omega_l$ and that $(p, q) \subset (c, c_1(a))$ for every $a \in (a_0 - \delta_{a_0}, a_0 + \delta_{a_0})$; the case for $(p, q) \subset (c_2(a), c)$ being similar.

From the above, we have that $\xi_{n_i+k_1}|\Gamma_{a_0}$ is monotone and that there is a unique $a_* \in \Gamma_{a_0}$ with $\xi_{n_i+k_1}(a_*) = c$. Lemma 1 implies that a_* is to the left of ω_l. Let E denote the shortest closed interval containing both a_* and ω_l; then $E \subset \Gamma_{a_0}$. Notice that $\xi_{n_i+k_1}(E) \subset [d_l, c]$, since $\xi_{n_i+k_1}(\omega_l) \subset (d_l, c)$, $\xi_{n_i+k_1}(a_*) = c$, and $\xi_{n_i+k_1}|E$ is monotone. See Figure 1-a.

We have that $\xi_{n_i+k_1+k_2}|E$ is monotone; for else, there exists $a \in E \subset \Gamma_{a_0}$ and $0 < j < k_2$ such that $\xi_{n_i+k_1+j}(a) = c$. Set $x = \xi_{n_i+k_1}(a) \in (d_l, c)$. Then, $g_a^j(x) = \xi_{n_i+k_1+j}(a) = c$, contradicting (4) of Lemma 3.

Next, $\xi_{n_i+k_1}(a_*) = c$ and (2) of Lemma 3 imply that $g_{a_*}^{k_2}(c) = \xi_{n_i+k_1+k_2}(a_*) < c$. Also, (4) of Lemma 4 implies that $\xi_{n_i+k_1+k_2}(\omega_l) \subset (0, c)$. These two observations and $\xi_{n_i+k_1+k_2}|E$ monotone, imply that $\xi_{n_i+k_1+k_2}(E) \subset (0, c)$ and hence that $\xi_{n_i+k_1+k_2+1}|E$ is monotone. Lastly, note that $\xi_{n_i+k_1+k_2+1}(\omega_l) \subset (p, q) \subset (c, c_1(a_0))$ by (5) of Lemma 4 and that $\xi_{n_i+k_1+k_2+1}(a_*) < c$ from (2) of Lemma 3. Thus, recall Lemma 1, we have (2) holding. See Figure 1-b. $\square$

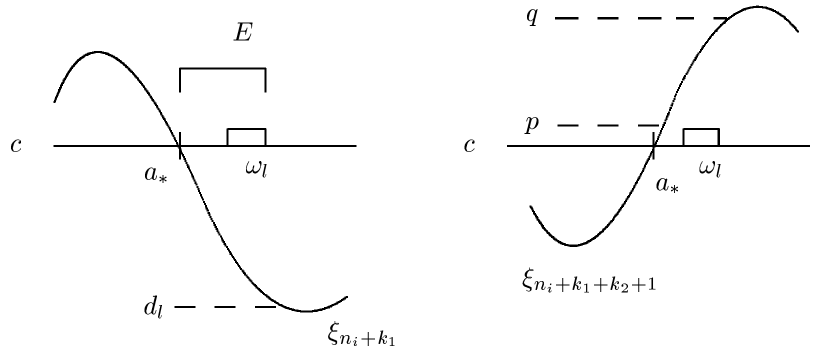

Figure 1-a: Piece of $\xi_{n_i+k_1}$ Figure 1-b: Piece of $\xi_{n_i+k_1+k_2+1}$

4 Gathering

Let $p < q$ be rational numbers in $[0,1]$ with $c = \frac{1}{2} \notin (p,q)$. Let $\epsilon > 0$ be as in Lemma 3. Set

$$\begin{aligned} \Omega_{p,q} &= \{a \in BC_0 \mid \text{ there exists } S_k(a) \text{ such that } c_{S_k(a)} \in (p,q)\}, \text{ and} \\ \Omega^0_{p,q} &= \{a \in BC_0 \cap (4-\epsilon, 4] \mid (p,q) \subset (\xi_2(a), \xi_1(a))\}. \end{aligned} \tag{6}$$

Note that both $\Omega_{p,q}$ and $\Omega^0_{p,q}$ are Lebesgue measurable sets.

Lemma 6 *Let $a_0 \in (4-\epsilon, 4) \cap BC_0$ be a Lebesgue density point of $\Omega^0_{p,q}$. Then the upper density of $\Omega_{p,q}$ at a_0 is positive.*

Proof: Choose $i \geq M$ from Lemma 5 and $\omega^{a_0}_{n_i}$ from Lemma 2. Let I_l, I_r be as in Lemma 4 (for the fixed a_0), ω_l and ω_r be as in Lemma 5, and C be as in Lemma 2. Then, from 5-6 of Lemma 2, we have for a suitable constant K (in fact, $K = \frac{|I_0|}{C^2}$ works) that

$$\frac{|\omega_l|}{|\omega^{a_0}_{n_i}|} \geq K \frac{|I_l|}{|I_0|} \equiv \eta > 0,$$

and

$$\frac{|\omega_r|}{|\omega^{a_0}_{n_i}|} \geq K \frac{|I_r|}{|I_0|} \equiv \eta > 0.$$

We have, by Lemma 5, either

$$\omega_l \cap BC_0 \subset \Omega_{p,q}$$

or

$$\omega_r \cap BC_0 \subset \Omega_{p,q}.$$

Hence, since a_0 is a density point of BC_0 and $a_0 \in \omega^{a_0}_{n_i}$, the upper density of $\Omega_{p,q}$ at a_0 is positive. □

Proof of Proposition 1: Observe that almost every point of $\Omega^0_{p,q}$ is a Lebesgue density point of $\Omega^0_{p,q}$ and hence, by Lemma 6, almost all of these points are upper density points of $\Omega_{p,q}$. Since $\Omega_{p,q}$ is Lebesgue measurable, almost every of its upper density points belong to $\Omega_{p,q}$. Thus, almost every point of $\Omega^0_{p,q}$ is in $\Omega_{p,q}$. Therefore, setting $\Omega^1_{p,q} = \{a \in \Omega^0_{p,q} \mid a \notin \Omega_{p,q}\}$, we have that $|\Omega^1_{p,q}| = 0$. Set

$$\Omega^1 = \bigcup_{\substack{p<q,\ c\notin(p,q) \\ p,q\in\mathbf{Q}}} \Omega^1_{p,q}.$$

Then $|\Omega^1| = 0$.

Assume that $a \in (BC_0 \cap (4-\epsilon, 4]) \setminus \Omega^1$. If $(p,q) \subset (\xi_2(a), \xi_1(a))$ and $c \notin (p,q)$, then $a \in \Omega^0_{p,q}$. As $a \notin \Omega^1$, we have that $a \notin \Omega^1_{p,q}$. Hence, $a \in \Omega_{p,q}$ and thus there exists $S_k(a)$ such that $c_{S_k(a)} \in (p,q)$. This implies $a \in BC^*$. □

5 Appendix

The purpose of this appendix is to prove Proposition 2 below. One can prove this proposition using Lemma 8 below and either [10, Proposition 4] or [4, Lemma 1.2]. We use [10, Proposition 4]. Using arguments similar to those here, one can prove that $a \in \mathcal{A}$ guarantees that (I, T_a) has the universality property.

Proposition 2 *Let $a_0 \in \mathcal{B}$. Then (I, g_{a_0}) has the universality property, i.e., for every choice of parameters $\{a_i\}_{i\geq 1}$, with $a_i \in [2,4]$ for all i, and open set $U \subset (I, \tilde{g}_{a_0})$ there exists a subcontinuum $H \subset U$ such that H is homeomorphic to $(I, \tilde{g}_{a_i})$.*

First some preliminaries needed to state [10, Proposition 4], which appears here as Proposition 3. We need to generalize and slightly modify some of our earlier definitions and state some new ones.

Given continuous maps $f_i : I_i \to I_{i-1}$, the associated *inverse limit space* with *bonding maps* f_i is

$$(I_i, f_i) = \{(x_1, x_2, x_3, \dots) \mid x_{i-1} = f_i(x_i) \text{ for } i \geq 2\}.$$

It has the metric $d(x,y) = \sum_i \frac{|x_i - y_i|}{2^i}$. The i-th projection is denoted as $\pi_i : (I_i, f_i) \to I_i$. In general, (I_i, f_i) is a *continuum*, i.e. a compact connected metric space. It is one-dimensional and chainable, see [23]. For $i > j$ set $f_{i,j} = f_{j+1} \circ \cdots \circ f_i : I_i \to I_j$; $f_{i,i}$ denotes the identity map.

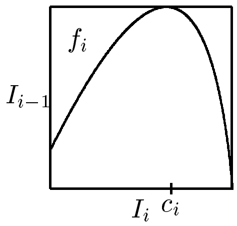

Figure 2

For any (I_i, f_i) we assume each $f_i : I_i \to I_{i-1}$ is unimodal and onto. For convenience, let c represent the turning point of each f_i; allowing c to vary with i. Here we relax the definition of unimodal maps by not assuming that $f_i(\partial I_i) \subset \partial I_{i-1}$.

Let (I_i, f_i) be given and for $i \geq 1$ and $z \in I_i$ set,

$$\hat{\nu}_i(z) = \begin{cases} 1 & \text{if } z \in (c, f_{i+1}(c)], \\ * & \text{if } z = c, \\ 0 & \text{if } z \notin [c, f_{i+1}(c)]. \end{cases}$$

For $i \geq 1$ and $z \in I_i$ let $\hat{\nu}|_i(z) = \hat{\nu}_1(f_{i,1}(z)), \dots, \hat{\nu}_{i-1}(f_i(z)), \hat{\nu}_i(z)$ for $z \in I_i$. Set $C_1 = \partial I_1 \cup \{c\}$ and for $i \geq 2$ set

$$C_i = \partial I_i \cup \bigcup_{1 \leq j \leq i} f_{i,j}^{-1}(c) \subset I_i.$$

Each C_i defines a partition of I_i, which we denote by $\mathcal{P}_i$. When more than one inverse limit space is being discussed, we use C_i^f and C_i^g etc.

Definition 1 *We say (I_i, f_i) is* combinatorially equivalent to *(J_i, g_i), written $(I_i, f_i) \approx (J_i, g_i)$, provided that for each i there exists a monotone bijection $h_i : C_i^f \to C_i^g$ such that $\hat{\nu}|_i(\alpha) = \hat{\nu}|_i(h_i(\alpha))$ for each $\alpha \in C_i^f$ and such that $h_i(c^f) = c^g$.*

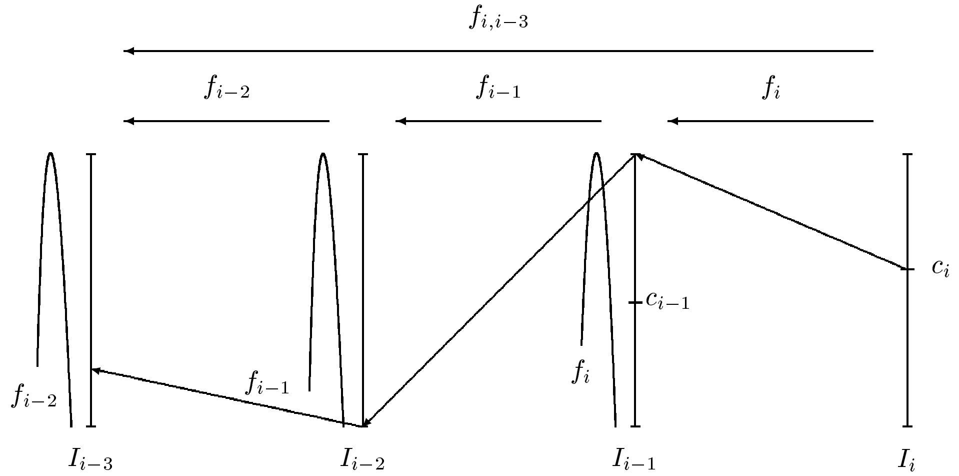

Figure 3

For ease of notation, in the statement of Proposition 3 it is assumed that $I_i = [f_{i+2,i}(c), f_{i+1,i}(c)]$ and $J_i = [g_{i+2,i}(c), g_{i+1,i}(c)]$ for all i. Assuming no f_i nor g_i is the full unimodal map, we have that $f_{i+3,i}(c)$ and $g_{i+3,i}(c)$ are in the interior of I_i respectively J_i. Using results of [8] there is no loss of generality by assuming that no f_i nor g_i is a full unimodal map (we say $f : I \to I$ is a full unimodal map provided that f maps the closure of each component of $I \setminus \{c\}$ onto I).

Proposition 3 *[10, Proposition 4] Let (I_i, f_i) and (J_i, g_i) be given such that*

$$(I_i, f_i) \approx (J_i, g_i). \tag{7}$$

Suppose that $a^f_{i-3}, b^f_{i-3} \in C^f_{i-3}$ are such that $f_{i,i-3}(c) \in [a^f_{i-3}, b^f_{i-3}] = cl P^f_{i-3}$ for some $P^f_{i-3} \in \mathcal{P}^f_{i-3}$. Write $A^f_{i-3} = f_{i-3,1}(a^f_{i-3})$ and $B^f_{i-3} = f_{i-3,1}(b^f_{i-3})$. Let a^g_{i-3}, b^g_{i-3}, A^g_{i-3} and B^g_{i-3} be the corresponding points for (J_i, g_i). If for all i sufficiently large

$$(1 - \frac{1}{i^2}) \frac{|B^f_{i-3} - f_{i,1}(c)|}{|B^f_{i-3} - A^f_{i-3}|} \le \frac{|B^g_{i-3} - g_{i,1}(c)|}{|B^g_{i-3} - A^g_{i-3}|} \le (1 + \frac{1}{i^2}) \frac{|B^f_{i-3} - f_{i,1}(c)|}{|B^f_{i-3} - A^f_{i-3}|}, \tag{8}$$

then (I_i, f_i) is homeomorphic to (J_i, g_i).

Lemma 7 *Let $a \in \mathcal{B}$, Then* $\liminf Q_a(k) < 2$.

Lemma 7 is immediate from [12, Lemma 3.5].

Remark 1 *Let $a \in \mathcal{B}$. Then $\lim_{n\to\infty} z_n(a) = \frac{1}{2}$ and $\lim_{n\to\infty} \hat{z}_n(a) = \frac{1}{2}$. This follows from the fact that any logistic map has no wandering interval. For details see [22].*

Lemma 8 *Fix $a_0 \in \mathcal{B}$. There exists $\Gamma \in (c_2(a_0), c)$ such that for every $z \in [c_2(a_0), \Gamma]$ and $\epsilon > 0$ there exist positive integers $m_1 < m_2 < m_3 < \cdots$ and real numbers d_i, e_i, $i \ge 1$, such that for all i:*

1. *$c_2(a_0) < d_i < e_i$,*
2. *$g_{a_0}^{S_{m_i}+1}(c_2(a_0)) \in (z-\epsilon, z+\epsilon)$,*
3. *$g_{a_0}^{S_{m_i}+1}(d_i) = c_2$ and $g_{a_0}^{S_{m_i}+1}(e_i) = c$,*
4. *$g_{a_0}^{S_{m_i}+1}|[c_2(a_0), d_i]$ is strictly decreasing,*
5. *$g_{a_0}^{S_{m_i}+1}|[d_i, e_i]$ is strictly increasing, and*
6. *$\lim_{i\to\infty} d_i = \lim_{i\to\infty} e_i = c_2(a_0)$.*

See Figure 4.

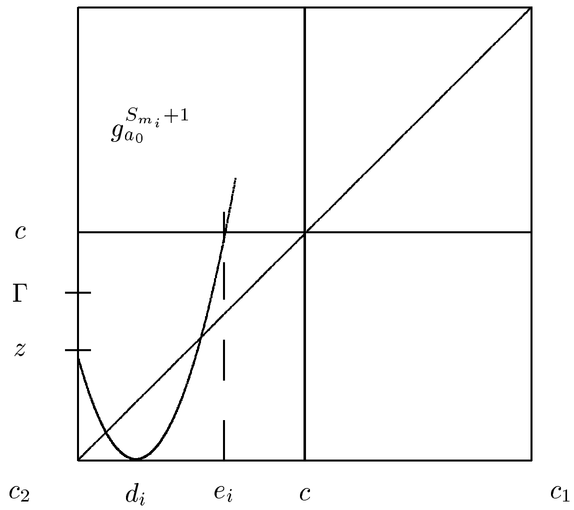

Figure 4: Piece of $g_{a_0}^{S_{m_i}+1}$

Proof: Let $W \ni \frac{1}{2}$ be an open interval such that $W \subset (z_1(a), \hat{z}_1(a))$ for all $a \in [2,4]$; since $S_1(a) = 2$ for all a, we have $z_1(a) \in g_a^{-2}(c)$, recall display (3). We let $c_n = c_n(a_0)$, $Q(k) = Q_{a_0}(k)$, $S_k = S_k(a_0)$, and $z_n = z_n(a_0)$ for each n, k. Fix $p \in \mathbf{N}$ such that $Q(p+1) \in \{0,1\}$ and $(z_{p-1}, \hat{z}_{p-1}) \subset W$; we can choose such a p due to Remark 1 and Lemma 7. Set $\Delta = \overline{W} \cap [c_{S_{Q(p)}}, c]$. Notice that $g_{a_0}^2|\Delta$ is strictly monotone. Let Γ be such that

$$g_{a_0}^2(\Delta) = [c_2, \Gamma]. \tag{9}$$

By the choice of p, if for some m we have $c_{S_m} \in (z_{p-1}, z_p)$, then (use (4)) $Q(m+1) = p$ and therefore $Q(Q(m+1)+1) \in \{0,1\}$. Thus, $c_{S_m} \in (z_{p-1}, z_p)$ implies that

$$c_{S_{m+1}} \in (c, c_{S_{Q(p)}}) \quad \textit{and} \quad c_{S_{Q(m+1)}} \in (0, z_2) \cup (\hat{z}_2, c_1). \tag{10}$$

It follows from $a_0 \in \mathcal{B}$ and (10) that if $w \in \Delta$ and $\delta > 0$, then there exist positive integers $m_1 < m_2 < m_3 < \cdots$ such that for each i we have:

$$\begin{aligned} c_{S_{m_i}} &\in (z_{p-1}, z_p), \\ c_{S_{m_i+1}} &= g_{a_0}^{S_p}(c_{S_{m_i}}) \in (w-\delta, w+\delta), \quad \textit{and} \\ c_{S_{Q(m_i+1)}} &\in (0, z_2) \cup (\hat{z}_2, c_1). \end{aligned} \tag{11}$$

Let $z \in [c_2, \Gamma]$ and $\epsilon > 0$. Choose $w \in \Delta$ and $\delta > 0$ such that $g_{a_0}^2((w-\delta, w+\delta)) \subset (z-\epsilon, z+\epsilon)$. Let $m_1 < m_2 < m_3 < \cdots$ be as in (11) for this choice of w and δ. Then for each i there exists $\alpha_i < z_{m_i+1}$ such that:

$$\begin{aligned} g_{a_0}^{S_{m_i+1}+2}(\alpha_i) &= c \ \ \text{and} \ \ g_{a_0}^{S_{m_i+1}+2} : [\alpha_i, z_{m_i+1}] \to [c_2, c], \\ g_{a_0}^{S_{m_i+1}+2}(c) &\in (z-\epsilon, z+\epsilon) \ \ \text{and} \ \ g_{a_0}^{S_{m_i+1}+2} : [z_{m_i+1}, c] \to [c_2, c_{S_{m_i+1}+2}]. \end{aligned} \tag{12}$$

Moreover, $g_{a_0}^{S_{m_i+1}+2}|[\alpha_i, z_{m_i+1}]$ is strictly decreasing and $g_{a_0}^{S_{m_i+1}+2}|[z_{m_i+1}, c]$ is strictly increasing for each i. Of course, $g_{a_0}^{S_{m_i+1}+2}(z_{m_i+1}) = c_2$ for each i.

For each i set $d_i = g_{a_0}^2(z_{m_i+1})$ and $e_i = g_{a_0}^2(\alpha_i)$. Properties 1-4 now easily follow from (12) and $(z_{p-1}, \hat{z}_{p-1}) \subset W$. Property 6 follows from Remark 1 and $\lim_{i\to\infty} m_i = \infty$. □

Proof of Proposition 2: Fix $a_0 \in \mathcal{B}$ and choose a sequence of parameters $\{a_i\}_{i\geq 1}$ with $a_i \in [2,4]$ for all i. If $a_i = 4$ for some i's, then replacing such a_i's by sufficiently close $\tilde{a}_i$'s results in an inverse limit space that is homeomorphic to $(I, \tilde{g}_{a_i})$ [8]. Thus we may assume that $a_i \neq 4$ for all i.

Repeated use of Lemma 8 allows us to choose positive integers $n_1 < n_2 < n_3 < \cdots$ and real numbers $b_1 > b_2 > b_3 > \cdots$ such that the following hold.

1. For each i, $g_{a_0}^{S_{n_i}}$ maps $[c_2(a_0), b_{i+1}]$ onto $[c_2(a_0), b_i]$. Set $I_i = [c_2(a_0), b_i]$ and $h_i = g_{a_0}^{S_{n_i}}|I_i$ for all i. Set $m_i = (c_2(a_0) + b_i)/2$, and denote by τ_i the reflection of I_i about m_i, that is, $\tau_i(x) = 2m_i - x$. Set $f_i = \tau_i \circ h_i \circ \tau_{i+1}$. It is easy to see that the inverse limit spaces (I_{i+1}, f_i) and (I_{i+1}, h_i) are homeomorphic.

2. For each i there exists a unique turning point $c_i \in I_i$ such that $g_{a_0}^{S_{n_i}}(c_{i+1}) = c_2(a_0)$, $g_{a_0}^{S_{n_i}}|[c_2(a_0), c_{i+1}]$ is strictly decreasing, and $g_{a_0}^{S_{n_i}}|[c_{i+1}, b_{i+1}]$ is strictly increasing. Recall Figure 4. Hence $f_i : I_{i+1} \to I_i$ is unimodal and onto.

3. Setting $g_i = g_{a_i}$ for $i \geq 1$, we have the hypotheses of Proposition 3 holding, i.e, (I, g_i) is combinatorially equivalent to (I_{i+1}, f_i) and (8) of Proposition 3 holds.

Thus (I, g_i) is homeomorphic to (I_{i+1}, f_i). Let X be the subcontinuum of $(I, \tilde{g}_{a_0})$ such that

$$x \in X \iff \pi_{N_i}(x) \in [c_2(a_0), b_{i+1}],$$

for $i = 0, 1, 2, \cdots$ with $N_0 = 0$ and $N_i = \sum_{k=1}^{i} n_k$. Then X is homeomorphic to (I_{i+1}, f_i) and hence to (I, g_i).

This paragraph follows [2, Corollary 6]. Let $\hat{g}_{a_0}$ be the induced homeomorphism on $(I, \tilde{g}_{a_0})$ given by $\hat{g}_{a_0}((x_1, x_2, x_3, \cdots)) = (g_{a_0}(x_1), x_1, x_2, x_3, \cdots)$. Let $U \subset (I, \tilde{g}_{a_0})$ be open. Without loss of generality assume that U is of the form $[U_1 \times U_2 \times \cdots \times U_n \times I \times \cdots] \cap (I, \tilde{g}_{a_0})$. Due to 3 above, to show that there exists $H \subset U$ with H homeomorphic to $(I, \tilde{g}_{a_i})$ it suffices to show there exists a subcontinuum D of $(I, \tilde{g}_{a_0})$ such that D is homeomorphic to $(I, \tilde{g}_{a_0})$ and $\pi_1(D) \subset U_n$, since then $\pi_n(\hat{g}_{a_0}^{n-1}(D)) \subset U_n$ and thus $\hat{g}_{a_0}^{n-1}(D) \subset U$. Repeating the construction given in 1-3 above and choosing n_1 appropriately, we can guarantee that $X \equiv D$ is such that $\pi_1(D) \subset U_n$. □

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