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Restriction and Intersection Theorems in Real Analysis

This talk is about my Ph. D. thesis [Bu7] written in Hungarian in 1987. Although all of the theorems and proofs of this thesis are available in English [Bu1-6] I think that the abstract of this thesis might be interesting to an audience bigger than those who can read Hungarian.

The study of the level set of functions is among the basic tools of Real Analysis. A nice example for this method is due to N. Bary and D. Menchoff (1928, 1930) [S, Ch. IX, 8.]: in order that a continuous function f be representable as a superposition of two absolutely continuous functions, it is necessary and sufficient that almost every one of its values is assumed at most a finite number of times and $|f(H)| = 0$ if $|H| = 0$ where $|H|$ denotes the Lebesgue measure of H. Another result is due to E. Čech [Ce] (1931). He proved that if f is continuous on an interval and its every level set is finite then f is monotone on a subinterval.

Obviously the level set can be regarded as the projection to the $x-$axis of the intersection of the graphs of f and a constant function. The study of the intersection properties of wider function classes is a generalization of the level set technique. Here are some possible questions:

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are sets of functions.

(i) If the functions in $\mathcal{A}$ intersect every function in $\mathcal{B}$ in "nice" sets then what regularity properties will we have for the functions in $\mathcal{A}$?

(ii) Find properties $\mathcal{P}$ such that for every $f \in \mathcal{A}$ we can find a $g \in \mathcal{B}$ such that the intersection of the graphs of f and g has property $\mathcal{P}$.

(iii) For a given property $\mathcal{P}$ find some "pathological" functions in $\mathcal{A}$ such that for every $g \in \mathcal{B}$ the intersection of f and g has property $\mathcal{P}$.

The theorems of Bary-Menchoff and Čech are examples for the questions of type (i). An example for type (ii) is the following theorem of Filipczak [F] (1966): For every continuous function f there exists a non-empty perfect set P such that $f|_P$, the restriction of f to P, is monotone. Thus for every continuous function f there exists a monotone function g such that the intersection set of f and g contains a perfect set. This example shows why we speak about restriction and intersection theorems. In most of the cases if we can prove that the restriction of a function in $\mathcal{A}$ has "nice" properties, then using an extension theorem one can obtain an intersection theorem from the restriction theorem.

An example for type (iii) is due to Gillis [B, Ch. XIII. 3.]: There exists a continuous non-constant function such that its every level set is perfect.

A. Bruckner, J. Ceder and M. Weiss [BCW] (1969) and M. Laczkovich [L] (1984) have several results about differentiable restrictions of continuous functions. Namely, if f is defined on a non-empty perfect set P then there exists a non-empty $P' \subset P$ such that the restriction of f onto P' has a finite or infinite derivative everywhere. If $|P| > 0$ then $f|_{P'}$ can be differentiable.

Given a real function f we can color the couples of real numbers by blue (red) if $f|_{\{x,y\}}$ is increasing (resp. decreasing). Then Filipczak's theorem states that if f is continuous then there exists a non-empty perfect set P such that the couples taken from P are colored by the same color. This is a combinatorial "Ramsey type" interpretation of this "analytic" result. A. Blass [Bl] (1981) proved a combinatorial theorem which implies that every Borel measurable function defined on non-empty perfect sets has a monotone restriction on a non-empty perfect set.

Certain theorems about monotonicity extend to convexity and even more generally to $n-$convexity. Since Ramsey's theorem is valid for hypergraphs as well, the authors of [ABLP] asked whether we can generalize Filipczak's theorem. Suppose that f is continuous on an interval $[a, b]$ and n is a non-negative integer. Does there exist a non empty perfect set P on which f is either $n-$convex or $n-$concave? In [Bu2] we gave a positive answer to this question when $n = 2$, that is, in the case of ordinary convexity-concavity. This problem is still open for $n \geq 3$. If we increase n the character of this problem changes. While for $n = 1$ one can apply the theorem of Blass, for $n \geq 2$ the corresponding combinatorial results are not applicable. While for $n = 1$ the theorem remains valid for measurable functions defined on non-empty perfect sets, for $n = 2$ it is known [ABLP], that there exist continuous functions, defined on non-empty perfect sets such that they have no convex or concave restriction on any non-empty perfect set.

P. Komjáth asked the question about convex restrictions of arbitrary real functions. Is it true that for a real function defined on $[0, 1]$ one can always find a set of big cardinality on which the restriction of f is convex or concave? In [Bu6, Theorem 1.] we proved that there exists an upper semicontinuous function f such that f is not convex (or concave) on any uncountable set.

The combinatorial background of these problems motivated the following question of Paul Erdős. Suppose that a given real function $f : [0, 1] \to \mathbf{R}$ is not convex on any r element set. What can be said about f? We proved that if $r = 4$ then there exists an interval I on which f is strictly concave [Bu6, Theorem 2, Corollary 1]. If $r > 4$ and f is bounded from below then there exists an interval where the closure of the graph of f in the topology of the plane equals the graph of countably many concave functions [Bu6, Theorem 2.]. In [Bu6, Theorem 3] we proved that if $r = \aleph_0$ and f is bounded from below then there exists an interval on which the graph of f can be covered by countably many strictly concave functions.

Trying to solve for $n \geq 3$ the question about $n-$convex restrictions of continuous functions we rediscovered a generalized version of an existence theorem for Peano derivatives [Bu4]. Suppose that f is $k - 1$ times Peano differentiable at the points of a closed set $F \subset \mathbf{R}^m$. Denote by $P_{k-1}(f, x, h)$ the $(k - 1)'$th "Taylor polynomial" of f at x. Then if $f(x + h) = P_{k-1}(f, x, h) + O(||h||^k)$ then f is k times Peano differentiable at almost every point of F. For different versions of this statement see [D] (1935), [MZ] (1936), [M] (1936), [O] (1951), and [St, Ch. VIII.].

The authors of [ABLP] proved the following generalization of Čech's theorem [Ce]: If f is continuous and the graph of f intersects every polynomial of degree at most n in finitely many points then there exists a subinterval on which f is either $n + 1-$convex or $n + 1-$concave. The $n + 1-$convex, $(n + 1-$concave), functions are $n - 1$ times continuously differentiable. Thus if the graph of a continuous f intersects the graph of every polynomial of degree at most n in finitely many points then f is $n - 1$ times continuously differentiable on a dense open set. Therefore, if the graph of a continuous function f intersects the graph of every polynomial in finitely many points then f is infinitely differentiable on a dense G_δ set.

The authors of [ABLP] raised the following problem of type (i): Suppose that the graph of a continuous function f intersects the graph of every analytic function in finitely many points. What regularity properties does this condition imply? Is it true that f is infinitely differentiable on a subinterval? In [Bu5] we answered this question in the negative. Since every polynomial is analytic we obtain that if f intersects the graph of every polynomial in finitely many points then f is infinitely differentiable on a dense G_δ set. However in [Bu5] we proved that for every dense G_δ set H there exists a continuous function f such that f intersects every analytic function in finitely many points and f is infinitely differentiable exactly at the points of H.

Finally we present some results of type (ii) [Bu1] and of type (iii) [Bu3]. These results are related to the papers of B. S. Thomson [T], J. Hausserman [H], P. Humke and M. Laczkovich [HL], where intersection properties of typical continuous functions are studied.

P. Humke and M. Laczkovich proved that a typical continuous function intersects every monotone function in a bilaterally strongly porous set. It is a natural question what can we say about intersections of typical continuous functions with functions of bounded variation, BV. In [Bu1] we prove that for every Darboux (and hence for every continuous) function there is an absolutely continuous, and hence BV function g such that $\{x : f(x) = g(x)\}$ is not bilaterally strongly porous. In fact we have to study the following question. Given a monotone decreasing sequence $a_n \to 0$ and a function f we have to find an x and a sequence $x_n \in [x + a_{n+1}, x + a_n]$ such that the restriction of f on the set $\{x_n : n = 1, ...\}$ is of bounded variation. In [Bu1] we study the case $a_n = \delta^n$ with a $\delta < 1$. Using an obvious extension theorem one can find an absolutely continuous g such that $\{x : f(x) = g(x)\}$ is not bilaterally strongly porous.

In [Bu3, Theorem 1] we prove that if $\lim_{n\to\infty} a_{n+1}/a_n = 1$ then for the typical continuous function for every x and sequence x_n defined above $f|_{\{x_n:n=1,...\}}$ is not of bounded variation. In [Bu3, Theorem 2] we prove that for any monotone decreasing $a_n \to 0$ if we allow only the sequences $x_n = x + a_n$ then for the typical continuous function $f|_{\{x_n:n=1,...\}}$ is not of bounded variation.

REFERENCES

[ABLP] S.Agronsky, A.M.Bruckner, M.Laczkovich, and D.Preiss, Convexity conditions and intersections with smooth functions, *Trans. Amer. Math. Soc.* 289 (2), (1985), 659-677.

[Bl] A. Blass, A partition theorem for perfect sets, *Proc. Amer. Math. Soc.* 82, (1981), 271-277.

[Bu1] Z. Buczolich, For every continuous f there is an absolutely continuous g such that $\{x : f(x) = g(x)\}$ is not bilaterally strongly porous, *Proc. Amer. Math. Soc.*, 100(3)(1987), 485-488.

[Bu2] -, Sets of convexity of continuous functions, *Acta Math. Hung.,* 53(3-4)(1988), 291-303.

[Bu3] -, Continuous functions with everywhere infinite variation with respect to sequences, *Proc. Amer. Math. Soc.* 103(2)(1988), 497- 502.

[Bu4] -, An existence theorem for higher Peano derivatives in $\mathbf{R}^m$, *Real Analysis Exchange,* 13(1)(1987-88), 245-252.

[Bu5] Functions with finite intersections with analytic functions, *Proc. Royal Soc. Edinburgh,* (accepted to appear).

[Bu6] Ramsey type theorems for real functions, *Mathematika* London, 36, (1989), 131-141 (accepted to appear).

[Bu7] *Megszorítási és metszettételek a valós analízisben,* Kandidátusi Értekezés, (*Restriction and Intersection Theorems in Real Analysis,* Ph. D. Dissertation,) 1987, Hungarian Academy of Sciences.

[B] A. Bruckner, *Differentiation of real functions,* Springer Verlag, 1978.

[BCW] A. Bruckner, J. Ceder, and M. Weiss, On the differentiability structure of real functions, *Trans. Amer. Math. Soc.,* 142, (1969), 1-13.

[Ce] E. Čech, Sur les fonctions continues qui prennent chaque leur valeur un nombre fini de fois, *Fund. Math.* 17, (1931), 32-39.

[F] F. M. Filipczak, Sur les fonctions continues relativement monotones, *Fund. Math.* 58, (1966), 75-78.

[H] J. Hausserman,*Generalized porosity characteristics of a residual set of continuous functions,* Ph. D. dissertation, University of California, Santa Barbara, 1984.

[HL] P. Humke, and M. Laczkovich, Typical continuous functions are virtually nonmonotone, *Proc. Amer. Math. Soc.* 94, (2), (1985), 244-248.

[L] M. Laczkovich, Differentiable restrictions of continuous functions, *Acta. Math. Hung.* 44, (1984), 353- 360.

[M] J. Marcinkiewicz, Sur les series de Fourier, *Fund. Math.* 27, (1936), 38-69.

[MZ] J. Marcinkiewicz, A. Zygmund, On the differentiability and summability of trigonometrical series, *Fund. Math.*, 26, (1936), 1-43.

[O] W. H. Oliver, *Differential properties of real functions,* Univ. of Chicago, (1951).

[S] S. Saks, Theory of the integral, Hafner New-York, (1937).

[St] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton (1970).

[T] B. S. Thomson, On the level set structure of a continuous function, (to appear).