Hölder spectrum of typical monotone continuous functions

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Abstract

We compute the Hölder spectrum and the moment sums of the typical (in the sense of Baire category) monotone continuous function, f. We show that these functions are of multifractal nature, for example α equals the Hausdorff dimension of those points where such a function, f is of exact Hölder class $\alpha \in [0,1]$. We also prove that f "grows" only at those points where it has bad Hölder properties, that is, at those points where it is of Hölder class 0. The upper moment sum $\overline{\tau}_f(q) = \infty$ if $q < 0$, $\overline{\tau}_f(q) = 1-q$ if $0 \leq q \leq 1$, and $\overline{\tau}_f(q) = 0$ if $1 < q$. The lower moment sum $\underline{\tau}_f(q) = 1-q$ if $q < 0$, $\underline{\tau}_f(q) = 0$ if $0 \leq q \leq 1$, and $\underline{\tau}_f(q) = 1-q$ if $1 < q$.

1 Introduction

In this paper we investigate the multifractal (Hölder) properties of the typical (in the sense of Baire category) monotone continuous function defined on $[0,1]$. We denote the space of monotone increasing continuous functions equipped with the sup norm by $C_{\nearrow}[0,1]$. It is a separable, complete metric space, being a closed subspace of $C[0,1]$. If a property holds for a residual subset of $C_{\nearrow}[0,1]$ then we say that the typical function f has this property.

In [11, 12] the differentiability properties of typical monotone continuous functions were studied and it was shown that the typical monotone continuous

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function, f is a strictly monotone increasing singular function and its derivative equals 0 wherever it exists. Of course, f' exists almost everywhere in $[0,1]$.

Jaffard [2, 3, 4, 5] has studied the multifractal properties of several specific continuous functions. For these functions many interesting methods, for example, wavelets, Diophantine approximation etc. were used.

It is not difficult to verify that the typical continuous function on $[0,1]$ is quite uninteresting from our point of view, it does not have multifractal Hölder properties, in fact, it is Hölder class 0 everywhere in $[0,1]$. Hence it is much more interesting to look at the class of typical monotone continuous functions which are of multifractal nature. By studying typical monotone continuous functions we study typical continuous measures on $[0,1]$. We point out that typical measures on arbitrary separable metric spaces were considered by [10, 9], for example a typical measure typically has no local dimension.

We denote by E_α^f those points of $[0,1]$ where $f \in C_{\nearrow}[0,1]$ is of exact Hölder class α and by $E_{\leq\alpha}^f$ those points where f is of Hölder class β for any $\beta < \alpha$ and is not of Hölder class β for any $\beta > \alpha$ (the detailed definitions will be given in the preliminaries section).

The main results of this paper are given in Theorem 6. In assertions (i-ii) of Theorem 6 we show that $\dim_H E_\alpha^f = \dim_H E_{\leq\alpha}^f = \alpha$ holds for $\alpha \in [0,1]$, while $E_{\leq\alpha}^f = \emptyset$ for $\alpha > 1$. This implies that for any $\alpha \in [0,1]$ there are "many points" where the typical $f \in C_{\nearrow}[0,1]$ function is of exact Hölder class α. In assertion (vi) we will show that if $\alpha \neq 0$ then f is not "growing" on the sets $\widehat{E}_{\leq\alpha}^f \supset E_\alpha^f$. In fact, all the growth of f takes place on $E_{\leq 0}^f$, by this we mean that $\mu_f(E_{\leq 0}^f) = \mu_f([0,1]) = f(1) - f(0)$, where μ_f is the Borel measure corresponding to f. Thus $\mu_f(\cup_{\alpha\neq 0}\widehat{E}_\alpha^f) = 0$.

Studying multifractal properties quite often the moment sums, $\tau_f(q)$ (for the definition see (1) below), are used [1, 6, 7] and in "lucky situations" the Legendre transform of $\tau_f(q)$ provides the spectrum of f. As one can expect the typical $f \in C_{\nearrow}[0,1]$ is not the "lucky case". In assertions (iii-v) of Theorem 6 we compute these moment sums. It turns out that for the typical $f \in C_{\nearrow}[0,1]$ $\tau_f(0) = 1$, $\tau_f(1) = 0$ and $\tau_f(q)$ does not exist for other vaules of q. If the upper moment sums are considered we have $\overline{\tau}_f(q) = \infty$ if $q < 0$, $\overline{\tau}_f(q) = 1 - q$ if $0 \leq q \leq 1$ and $\overline{\tau}_f(q) = 0$ if $1 < q$. For the lower moment sums, $\underline{\tau}_f(q)$, we show that $\underline{\tau}_f(q) = 1 - q$ if $q < 0$, $\underline{\tau}_f(q) = 0$ if $0 \leq q \leq 1$ and $\underline{\tau}_f(q) = 1 - q$ if $1 < q$. It might be worth to mention that an elementary computation shows that for $f(x) = x$ we have $\tau_f(q) = 1 - q$ for all $q \in \mathbb{R}$.

In Section 4 we present the main construction of this paper. This construction will provide the dense G_δ sets in $C_{\nearrow}[0,1]$ showing that certain properties are typical. The proof of Theorem 6 will be split into parts according to the different assertions made.

2 Preliminaries

The integer part of the real number x, that is, the largest integer not exceeding x, will be denoted by $[x]$.

We use the notation $\lambda(A)$ for the Lebesgue measure of the measurable set $A \subset [0,1]$ while $\mathcal{H}^s(A)$ denotes its s-dimensional Hausdorff measure [6, 7].

The space of continuous and continuously differentiable functions on $[0,1]$ is denoted by $C[0,1]$ and by $C^1[0,1]$, respectively.

Definition 1. Given $\alpha \geq 0$ and a point $x_0 \in [0,1]$ we say that the function f is $C^\alpha(x_0)$ if there exists a polynomial P of degree at most $[\alpha]$ and a constant $C > 0$ such that $|f(x) - P(x - x_0)| \leq C|x - x_0|^\alpha$ for all $x \in [0,1]$. The Hölder exponent of f at x_0 is defined by

$$h_f(x_0) = \sup\{\alpha : f \text{ is } C^\alpha(x_0)\}.$$

Set

$$E^f_{\leq\alpha} = \{x_0 : h_f(x_0) \leq \alpha\} \text{ and } E^f_\alpha = \{x_0 : h_f(x_0) = \alpha, \text{and } f \text{ is } C^\alpha(x_0)\}.$$

The set E^f_α denotes the set of those points where f is of "exact" Hölder class α. It is clear that $E^f_\alpha \subset E^f_{\leq\alpha} \setminus \cup_{\beta<\alpha} E^f_{\leq\beta} = \widehat{E}^f_\alpha$. However this inclusion is proper, for example if we take $g(x) = -\log(x/10) \cdot \sqrt{x/10}$ for $x \in (0,1]$ and $g(0) \stackrel{\text{def}}{=} 0$ then $g \in C_{\nearrow}[0,1]$, $h_g(0) = 1/2$, but g is not $C^{1/2}(0)$.

To compute Hausdorff dimensions we will use Proposition 2.2 of [7] (see also Proposition 4.9 of [6]):

Proposition 1. *Let $E \subset \mathbb{R}^n$ be a Borel set, let μ be a finite Borel measure on $\mathbb{R}^n$ and $0 < c < \infty$.*

(a) If $\limsup_{r\to 0+} \mu(B(x,r))/r^s \leq c$ for all $x \in E$ then $\mathcal{H}^s(E) \geq \mu(E)/c$.

(b) If $\limsup_{r\to 0+} \mu(B(x,r))/r^s \geq c$ for all $x \in E$ then $\mathcal{H}^s(E) \leq 2^s\mu(E)/c$.

3 Elementary lemmas for the moment sums

Given a monotone increasing continuous function $g : [0,1] \to \mathbb{R}$ we denote by μ_g the Borel measure corresponding to g, that is for which $\mu_g((a,b)) = g(b) - g(a)$ for each interval (a,b).

Set

$$\tau_{g,j}(q) = \frac{1}{j\log 2} \log \sum_{k=0}^{2^j-1} \left(\mu_g\left(\left[\frac{k}{2^j}, \frac{k+1}{2^j}\right]\right)\right)^q \text{ and } \tau_g(q) = \lim_{j\to\infty} \tau_{g,j}(q). \quad (1)$$

If the latter limit does not exist we use $\overline{\tau}_g(q) = \limsup_{j\to\infty} \tau_{g,j}(q)$ and $\underline{\tau}_g(q) = \liminf_{j\to\infty} \tau_{g,j}(q)$.

It is not difficult to see that $\tau_{cg}(q) = \tau_g(q)$ holds for any $c > 0$, so there is no problem if the finite measure μ_g is not a probability measure as long as g is a nonconstant increasing function, actually, when we compute moment sums we can assume that μ_g is a probability measure.

Lemma 2. *If $g \in C^1[0,1]$ and $g' > 0$ then for every $\epsilon > 0$ there exists $K > 0$ such that $\tau_{g,j}(q) \geq 1 - q - \epsilon$ for all $j \geq K$, $q \in [0,1]$.*

Proof. Using the continuity of $g' > 0$ we can choose ϱ_1 such that $g' > \varrho_1 > 0$. Using the mean value theorem for g from the above estimates we infer

$$\mu_g\left(\left[\frac{k}{2^j}, \frac{k+1}{2^j}\right]\right) \geq \varrho_1 \frac{1}{2^j}.$$

Then

$$\tau_{g,j}(q) \geq \frac{1}{j \log 2} \log\left(2^j \left(\frac{\varrho_1}{2^j}\right)^q\right) = 1 - q + \frac{q \log \varrho_1}{j \log 2}.$$

This proves the lemma. $\square$

Lemma 3. *For any $q \in [0,1]$ and $g \in C_{\nearrow}[0,1]$ we have $\overline{\tau}_g(q) \leq 1 - q$ and $\underline{\tau}_g(q) \geq 0$.*

Proof. Set $\mu_{k,j} = \mu_g([\frac{k}{2^j}, \frac{k+1}{2^j}])$. If $q = 0$ then $\sum_{k=0}^{2^j-1} \mu_{k,j}^0 = 2^j$ and hence $\overline{\tau}_g(0) = \tau_g(0) = 1$. In the sequel, we assume $q \in (0,1]$. Obviously, $\sum_{k=0}^{2^j-1} \mu_{k,j} = g(1) - g(0) \stackrel{\text{def}}{=} G_0 > 0$. Since $\mu_{k,j} \geq 0$ we can use some well-known inequalities between means of nonegative numbers and obtain

$$\left(\frac{\sum_{k=0}^{2^j-1} \mu_{k,j}^q}{2^j}\right)^{\frac{1}{q}} \leq \frac{\sum_{k=0}^{2^j-1} \mu_{k,j}}{2^j} = \frac{G_0}{2^j}. \tag{2}$$

This implies $\sum_{k=0}^{2^j-1} \mu_{k,j}^q \leq G_0^q 2^{j(1-q)}$ and an easy calculation implies $\overline{\tau}_g(q) \leq 1 - q$.

Using that $\tau_{cg} = \tau_g$ for any $c > 0$ we can assume $\mu_g([0,1]) = G_0 = 1$. Then $\mu_{k,j}^q \geq \mu_{k,j}$ and hence $\sum_{k=0}^{2^j-1} \mu_{k,j}^q \geq \sum_{k=0}^{2^j-1} \mu_{k,j} = G_0 = 1$. Thus $\tau_{g,j}(q) \geq 0$ and hence $\underline{\tau}_g(q) \geq 0$ holds. $\square$

The next two lemmas are the $q > 1$ versions of the previous two ones. Their proofs are analogous so we omit some details.

Lemma 4. *If $g \in C^1[0,1]$ and $g' \geq 0$ then for every $\epsilon > 0$ there exists $K > 0$ such that $\tau_{g,j}(q) \leq 1 - q + \epsilon$ for all $j \geq K$, $q \in [1, 1/\epsilon]$.*

Proof. Using the continuity of $g' > 0$ we can choose ϱ_2 such that $\varrho_2 > g'$. By the mean value theorem we infer

$$\mu_g\left(\left[\frac{k}{2^j}, \frac{k+1}{2^j}\right]\right) \leq \varrho_2 \frac{1}{2^j}.$$

Then

$$\tau_{g,j}(q) \leq \frac{1}{j\log 2}\log\left(2^j\left(\frac{\varrho_2}{2^j}\right)^q\right) = 1-q+\frac{q\log\varrho_2}{j\log 2}.$$

$\square$

Lemma 5. *For any $q>1$ and $g \in C_{\nearrow}[0,1]$ we have $\underline{\tau}_g(q) \geq 1-q$ and $\overline{\tau}_g(q) \leq 0$.*

Proof. Define $\mu_{k,j}$ and G_0 as in the proof of Lemma 2. From $q>1$ it follows that we have the reversed inequality in (2), that is,

$$\left(\frac{\sum_{k=0}^{2^j-1}\mu_{k,j}^q}{2^j}\right)^{\frac{1}{q}} \geq \frac{G_0}{2^j}.$$

This implies $\sum_{k=0}^{2^j-1}\mu_{k,j}^q \geq G_0^q 2^{j(1-q)}$ and hence $\underline{\tau}_g(q) \geq 1-q$.

Like in Lemma 2 we can assume $\mu_g([0,1])=1$ and using $\mu_{k,j}^q \leq \mu_{k,j}$ we can obtain $\sum_{k=0}^{2^j-1}\mu_{k,j}^q \leq 1$ which implies $\tau_{g,j}(q) \leq 0$ and $\overline{\tau}_g(q) \leq 0$. $\square$

4 The main construction

Assume $m \in \mathbb{N}$ is given. We choose a countable dense subset, $\{f_n\}$, in $C_{\nearrow}[0,1]$, such that each f_n is continuously differentiable and $f_n' > 0$ on $[0,1]$. Using Lemmas 2 and 4 with $\epsilon = 1/n$ choose $K_n > n$ such that for all $q \in [0,1]$ and $j \geq K_n$ we have

$$\tau_{f_n,j}(q) \geq 1-q-\frac{1}{n}; \tag{3}$$

and for all $q \in [1,n]$ and $j \geq K_n$ we have

$$\tau_{f_n,j}(q) \leq 1-q+\frac{1}{n} \tag{4}$$

Using the uniform continuity of f_n choose and fix $N_n \geq K_n$ such that

$$f_n\left(\frac{k+1}{2^{N_n}}\right) - f_n\left(\frac{k}{2^{N_n}}\right) = \mu_{f_n}\left(\left[\frac{k}{2^{N_n}},\frac{k+1}{2^{N_n}}\right]\right) < \frac{1}{2n} \tag{5}$$

holds for $k=0,...,2^{N_n}-1$.

We will choose a large integer $\nu_{n,m} > \max\{N_n^2, m\}$ to be determined later. Set

$$h_{n,m}^* = \frac{1}{4^n 2^{m\nu_{n,m}}} < \frac{1}{4^n}, \qquad h_{n,m} = \frac{h_{n,m}^*}{2^{\nu_{n,m}}}, \tag{6}$$

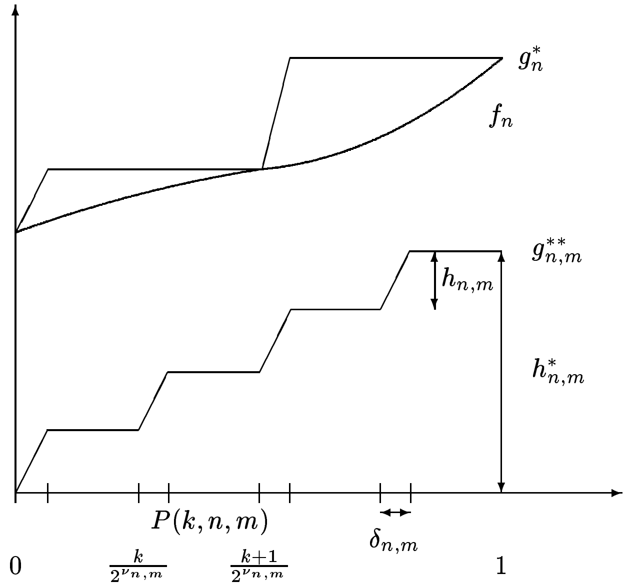
$$\delta_{n,m} = (h_{n,m})^{2m}\frac{1}{4} < \frac{1}{2}\frac{1}{2^{\nu_{n,m}}}, \qquad P(k,n,m) = \frac{k}{2^{\nu_{n,m}}} + \delta_{n,m}, \tag{7}$$

where $k \in \{0,...,2^{\nu_{n,m}}-1\}$. In some applications of our construction we will use different values for $h_{n,m}^*$, however for all these values we will have $0 <$

$h^*_{n,m} \leq 1/4^n \leq 1/4n$. Observe that $h^*_{n,m}$, $h_{n,m}$, $\delta_{n,m}$ and $P(k,n,m)$ are all dyadic rationals.

Set

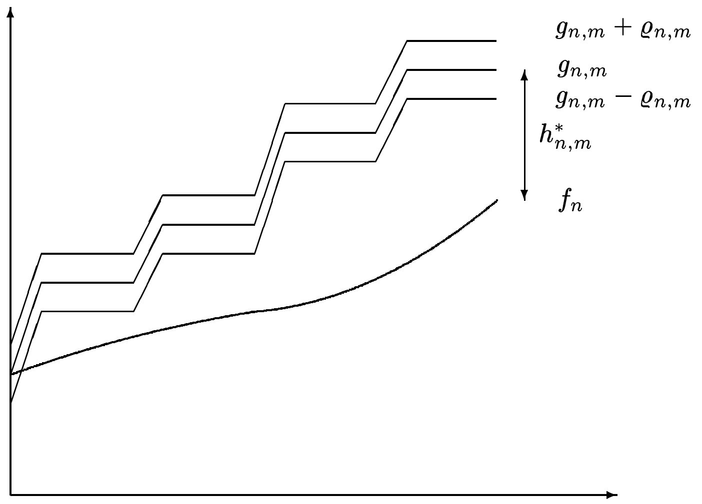
$$W_{n,m} = \bigcup_{k=0}^{2^{\nu_{n,m}}-1} \left(\frac{k}{2^{\nu_{n,m}}}, P(k,n,m) \right).$$



Next we define the auxiliary function $g^*_n(x) = f_n((k'+1)/2^{N_n})$ if $x \in (k'/2^{N_n}, (k'+1)/2^{N_n}] \setminus W_{n,m}$, $(k' = 0, ..., 2^{N_n} - 1)$ and $g^*_n(0) = f_n(0)$. On $W_{n,m}$ we define g^*_n so that it is continuous on $[0,1]$ and is linear on each interval $(k/2^{\nu_{n,m}}, P(k,n,m))$.

Set $g^{**}_{n,m}(x) = (k+1)h_{n,m}$ if $x \in (k/2^{\nu_{n,m}}, (k+1)/2^{\nu_{n,m}}] \setminus W_{n,m}$, $(k = 0, ..., 2^{\nu_{n,m}} - 1)$ and $g^{**}_{n,m}(0) = 0$. On $W_{n,m}$ we define $g^{**}_{n,m}$ so that it is continuous on $[0,1]$ and is linear on each interval $(k/2^{\nu_{n,m}}, P(k,n,m))$.

Finally we put $g_{n,m}(x) = g^*_n(x) + g^{**}_{n,m}(x)$.



By (5)

$$|f_n(x) - g_n^*(x)| < \frac{1}{2n} \text{ and } 0 \leq g^{**}_{n,m}(x) \leq g^{**}_{n,m}(1) = h^*_{n,m}. \tag{8}$$

We also have

$$|f_n(x) - g_{n,m}(x)| \leq |f_n(x) - g_n^*(x)| + h^*_{n,m} \leq \frac{3}{4n} \tag{9}$$

Using

$$\varrho_{n,m} = \frac{(h_{n,m})^m}{8} < \frac{h_{n,m}}{8} \tag{10}$$

we set $G_{n,m} = B(g_{n,m}, \varrho_{n,m})$ and $H_m = \cup_{n=m}^{\infty} G_{n,m}$. It is clear that H_m is open in $C_{\nearrow}[0,1]$ and using the density of the functions $\{f_n\}$ and (9) it is easy to see that H_m is dense. To prove the main result of the paper we will use the residual set $\mathcal{F}^* = \cap_{m=1}^{\infty} H_m$.

5 Main results

The purpose of this paper is to prove the following theorem:

Theorem 6. *There is a residual subset $\mathcal{F} \subset C_{\nearrow}[0,1]$ such that for any $f \in \mathcal{F}$ we have*

(i) $\widehat{E}^f_\alpha = \emptyset$ *for* $\alpha > 1$;

(ii) $dim_H E^f_\alpha = dim_H \widehat{E}^f_\alpha = dim_H E^f_{\leq\alpha} = \alpha$ *for all* $\alpha \in [0,1]$;

(iii) $\overline{\tau}_f(q) = 1-q$ *and* $\underline{\tau}_f(q) = 0$ *for* $0 < q \leq 1$, *and* $\overline{\tau}_f(0) = \underline{\tau}_f(0) = \tau_f(0) = 1$;

(iv) $\overline{\tau}_f(q) = 0$ *and* $\underline{\tau}_f(q) = 1-q$ *for* $1 < q$;

(v) $\overline{\tau}_f(q) = \infty$ *and* $\underline{\tau}_f(q) = 1-q$ *for* $q < 0$;

(vi) $\mu_f(E^f_{\leq 0}) = f(1) - f(0)$, *that is* $\mu_f([0,1] \setminus E^f_{\leq 0}) = 0$.

We will split the proof of the theorem according to the assertions (i-vi). It is also clear that if for each assertion we can find a suitable dense G_δ set in $C_{\nearrow}[0,1]$ then the intersection of these sets will provide $\mathcal{F}$.

5.1 Assertion (i)

Proof. Recall that a monotone function is differentiable almost everywhere and by [11, 12] there is a residual, dense G_δ subset, $\mathcal{F}_0$ of $C_{\nearrow}[0,1]$ such that $f'(x) = 0$ at all points where f' exists. We use a slightly modified version of the main construction, in equation (6) we set $h^*_{n,m} = \frac{1}{4^n}$. With these values of $h^*_{n,m}$ the sets H_m will still remain dense and open in $C_{\nearrow}[0,1]$. By choosing $\nu_{n,m} \geq m^2$ sufficiently large we can assume that

$$h_{n,m} = \frac{h^*_{n,m}}{2^{\nu_{n,m}}} = \frac{1}{4^n 2^{\nu_{n,m}}} > 2\left(\frac{1}{2^{\nu_{n,m}}}\right)^{1+\frac{1}{m}}. \tag{11}$$

Denote by $\mathcal{F}_1 = \cap_{m=1}^\infty H_m$ the set obtained by the main construction and suppose that $f \in \mathcal{F}_{(i)} \overset{\text{def}}{=} \mathcal{F}_1 \cap \mathcal{F}_0$.

Proceeding towards a contradiction assume $1 < \alpha < 2$, $x_0 \in [0,1]$ and f is $C^\alpha(x_0)$. Then

$$|f(x) - (a_0 + a_1(x - x_0))| \leq C|x - x_0|^\alpha$$

and, clearly, $a_0 = f(x_0)$, $f'(x_0)$ exists and $a_1 = f'(x_0) = 0$ where we also used that $f \in \mathcal{F}_0$. Hence

$$|f(x) - f(x_0)| \leq C|x - x_0|^\alpha. \tag{12}$$

Fix $m > 2$ such that $\alpha > 1 + \frac{2}{m}$ and $C < m$. Choose $\delta > 0$ such that $m|r|^{1/m} < 1$ holds for $0 < |r| < \delta$; hence

$$C|r|^\alpha < m|r|^\alpha < m|r|^{1+\frac{2}{m}} < |r|^{1+\frac{1}{m}}$$

is also true. From $f \in \mathcal{F}_1$ it follows that there exists $n \geq m$ such that $f \in G_{n,m}$. We can also suppose that $\nu_{n,m}$ is so large that $2^{-\nu_{n,m}} < \delta$. Set $r = 1/2^{\nu_{n,m}}$ if $x_0 + r \in [0,1]$, otherwise set $r = -1/2^{\nu_{n,m}}$. Clearly,

$$|g^{**}_{n,m}(x_0 + r) - g^{**}_{n,m}(x_0)| = h_{n,m} \text{ and } g^*_n,\ g^{**}_{n,m} \text{ are increasing;}$$

hence

$$|g_{n,m}(x_0 + r) - g_{n,m}(x_0)| \geq h_{n,m}.$$

Using $f \in B(g_{n,m}, \varrho_{n,m})$ and $\varrho_{n,m} < h_{n,m}/8$ we obtain

$$|f(x_0 + r) - f(x_0)| > |g_{n,m}(x_0 + r) - g_{n,m}(x_0)| - 2\varrho_{n,m} > \frac{3}{4} h_{n,m}.$$

Therefore by recalling (11) and (12)

$$\frac{3}{4} h_{n,m} < |f(x_0 + r) - f(x_0)| \leq C|r|^\alpha < |r|^{1+\frac{1}{m}} = \left(\frac{1}{2^{\nu_{n,m}}}\right)^{1+\frac{1}{m}} < \frac{h_{n,m}}{2},$$

a contradiction.

We showed that for $1 < \alpha < 2$ there is no $x_0 \in [0,1]$ for which f is $C^\alpha(x_0)$. If f is $C^\beta(x_0)$ and $0 < \alpha < \beta$ then f is $C^\alpha(x_0)$ as well. Thus we showed that $h_f(x_0) \leq 1$ holds for any $f \in \mathcal{F}_{(i)}$ and $x_0 \in [0,1]$. $\square$

5.2 Assertion (ii)

Before turning to this assertion we state and prove a lemma.

Lemma 7. *Assume $g \in C_{\nearrow}[a,b]$ and*

$$U_g = \{x \in [a,b] : \exists y \in [a,b] \text{ such that } |g(y) - g(x)| \geq |y - x|\}$$

then $\lambda(U_g) \leq 2(g(b) - g(a))$.

Proof. Set

$$U_1 = \{x \in [a,b] : \exists y \in (x,b] \text{ such that } g(y) > g(x) + (y - x)\}$$

and

$$U_2 = \{x \in [a,b] : \exists y \in [a,x) \text{ such that } g(y) < g(x) + (y - x)\}.$$

Clearly, $U_g = U_1 \cup U_2$. We will show that $\lambda(U_1) \leq g(b) - g(a)$, the proof for U_2 is similar.

Letting $g_0(x) = g(x) - x$ we have

$$U_1 = \{x \in [a,b] : \exists y \in (x,b] \text{ such that } g_0(y) > g_0(x)\}.$$

Set $g_1(x) = \max\{g_0(t) : t \in [x,b]\}$. Then $g_1(b) = g_0(b)$, $g_1(a) \geq g_0(a)$ and $g_1'(b) \geq -1$ where this derivative exists.

It is easy to see that $g_1(x)$ is monotone decreasing and hence (by using Theorem 2.9.19 of [8])

$$\int_{[a,b]} g_1' \leq g_1(b) - g_1(a) \leq g_0(b) - g_0(a) = g(b) - g(a) + (a-b).$$

On the other hand $g_1' = 0$ almost everywhere on U_1 and hence

$$\int_{[a,b]} g_1' \geq \int_{[a,b]\setminus U_1} (-1) = (a-b) + \lambda(U_1).$$

From the above two displayed inequalities it follows that $\lambda(U_1) \leq g(b) - g(a)$. $\square$

Now we verify assertion (ii).

Proof. Any $f \in \mathcal{F}_1$ is almost everywhere differentiable, and hence it is $C^1(x_0)$ at almost every $x_0 \in [0,1]$. This and assertion (i) imply $\dim_H(E_1^f) = 1$.

Assume $0 < \alpha < 1$ is fixed and $1 > \beta > \alpha$. Next we show that $\dim_H E^f_{\leq\alpha} \leq \alpha$. Actually, this is valid for any $f \in C_{\nearrow}[0,1]$ and follows easily from Proposition 1. If $x_0 \in E^f_{\leq\alpha}$ then f is not $C^\beta(x_0)$ and hence there exists $x_n \to x_0$ such that

$$|f(x_n) - f(x_0)| \geq |x_n - x_0|^\beta.$$

Letting $r_n = |x_n - x_0|$ we have

$$\frac{\mu_f(B(x_0, r_n))}{r_n^\beta} \geq 1.$$

This implies $\limsup_{r\to 0+} \mu_f(B(x_0,r))/r^\beta \geq 1$. From Proposition 1 it follows that $\mathcal{H}^\beta(E^f_{\leq\alpha}) \leq 2^\beta \mu_f(E^f_{\leq\alpha}) < \infty$ and hence $\dim_H E^f_{\leq\alpha} \leq \beta$. Since this holds for any $\beta > \alpha$ we obtain $\dim_H E^f_{\leq\alpha} \leq \alpha$.

We turn to the much more difficult task to verify that $\dim_H E^f_\alpha \geq \alpha$, where $0 < \alpha < 1$ is fixed. Since $E^f_\alpha \subset E^f_{\leq\alpha}$ this will imply $\dim_H E^f_\alpha = \dim_H E^f_{\leq\alpha} = \alpha$. We will use the main construction again with $h^*_{n,m} = 1/4^n$ and assume that $f \in \mathcal{F}_1 = \cap_{m=1}^\infty H_m$. This means that for each m there exists $n(m) \geq m$ such that $f \in \cap_{m=1}^\infty G_{n(m),m}$. We will choose a suitable subsequence $m_\ell \geq \ell$. For ease of notation we let $G_\ell = G_{n(m_\ell),m_\ell}$, $N_\ell = N_{n(m_\ell)}$, $\nu_\ell = \nu_{n(m_\ell),m_\ell}$, $h_\ell = h_{n(m_\ell),m_\ell}$, $\delta_\ell = \delta_{n(m_\ell),m_\ell}$, $\varrho_\ell = \varrho_{n(m_\ell),m_\ell}$, and $P_\ell(k) = \frac{k}{2^{N_\ell}} + \delta_\ell$ for $k = 0, ..., 2^{N_\ell} - 1$. Observe that each of the above numbers is a dyadic rational.

We choose h'_ℓ such that it is a dyadic rational and

$$2^{\ell-1} h_\ell^{1/\alpha} \leq h'_\ell \leq 2^\ell h_\ell^{1/\alpha}. \tag{13}$$

Set $h''_\ell = h'_\ell / 2^\ell$. Thus

$$\frac{h_\ell^{1/\alpha}}{2} \leq h''_\ell \leq h_\ell^{1/\alpha}. \tag{14}$$

We assume that the subsequence m_ℓ is sufficiently rapidly increasing, namely, $2^{m_{\ell+1}}$ exceeds the denominator of the above numbers at "level ℓ". This assumption will be convenient, for example, this way the intervals used in the definitions of $g^*_{\ell+1} = g^*_{n(m_{\ell+1})}$ and $g^{**}_{\ell+1} = g^*_{n(m_{\ell+1}),m_{\ell+1}}$, will belong to a subdivision of the intervals occuring at "level ℓ".

Next we define closed sets F'_ℓ and F_ℓ and using their intersection we define some Cantor type sets. Using Proposition 1 we will estimate the Hausdorff dimension of these sets, furthermore we will also obtain information about the Hölder properties of f on these sets. Put

$$F'_\ell = \bigcup_{k=0}^{2^{\nu_\ell}} [P_\ell(k) + h''_\ell, P_\ell(k) + h'_\ell] \text{ and } F_\ell = \bigcap_{\ell' \leq \ell} F'_{\ell}.$$

We also let $F_\infty = \cap_{\ell=1}^\infty F_\ell$.

It is clear that if we choose the sequence m_ℓ to be sufficiently rapidly increasing then each F_ℓ consists of $\kappa(\ell) > 2^\ell$ many intervals of the form $J_{\ell,k} = [P_\ell(k) + h''_\ell, P_\ell(k) + h'_\ell]$. By repeated subdivision ([7] pp. 9-10) we define a Borel probability measure, μ_α with support on F_∞ by setting $\mu_\alpha(J_{\ell,k}) = 1/\kappa(\ell)$, that is, we assign equal weight to each interval $J_{\ell,k}$.

Assume $\ell \geq 2$. Choose a component of $F_{\ell-1}$. This component is a closed interval of length $h'''_{\ell-1} \stackrel{\text{def}}{=} h'_{\ell-1} - h''_{\ell-1}$ and F'_ℓ has one subinterval in each interval $[k/2^{\nu_\ell}, (k+1)/2^{\nu_\ell}]$ (we assumed that $m(\ell)$ is increasing sufficiently fast; hence from $h'''_{\ell-1}$ being at "level $\ell - 1$" it follows that it is an integer multiple of $1/2^{\nu_\ell}$). Then $\kappa(\ell) \geq h'''_{\ell-1} 2^{\nu_\ell}$ and hence

$$\mu_\alpha(J_{\ell,k}) \leq \frac{1}{h'''_{\ell-1} 2^{\nu_\ell}}. \tag{15}$$

Our next purpose is to show that for all $x \in F_\infty$ we have

$$\limsup_{r \to 0+} \frac{\mu_\alpha(B(x,r))}{r^{\alpha-\epsilon}} < 1 \tag{16}$$

for all $\epsilon \in (0, \alpha)$. We consider $r \in (1/2^{\nu_{\ell+1}}, 1/2^{\nu_\ell}]$ for each $\ell = 2, 3, \dots$. We split each such case into two subcases

I. $h'_\ell < r \leq 1/2^{\nu_\ell}$;

II. $1/2^{\nu_{\ell+1}} < r \leq h'_\ell$.

Assuming $\epsilon \in (0, \alpha)$ is fixed first we consider subcase I. In this case $B(x, r)$ can intersect at most three different $J_{\ell,k}$ and using (13) and (6)

$$\frac{\mu_\alpha(B(x,r))}{r^{\alpha-\epsilon}} \leq \frac{\frac{3}{h'''_{\ell-1}2^{\nu_\ell}}}{(h'_\ell)^{\alpha-\epsilon}} \leq \frac{\frac{3}{h'''_{\ell-1}2^{\nu_\ell}}}{2^{(\ell-1)(\alpha-\epsilon)} h_\ell^{(\alpha-\epsilon)/\alpha}} =$$

$$= \frac{3 \cdot 2^{(\epsilon-\alpha)(\ell-1)}}{h'''_{\ell-1}(h^*_\ell)^{(\alpha-\epsilon)/\alpha}} 2^{\nu_\ell(\frac{\alpha-\epsilon}{\alpha}-1)} \leq \frac{3 \cdot 4^{m_\ell}}{h'''_{\ell-1}} 2^{\nu_\ell(\frac{\alpha-\epsilon}{\alpha}-1)} < \frac{3 \cdot 4^{m_\ell}}{h'''_{\ell-1}} 2^{m_\ell^2(\frac{\alpha-\epsilon}{\alpha}-1)} < \frac{1}{4}$$

where the last inequality holds when m_ℓ is chosen to be sufficiently large.

Subcase II. In this case $B(x,r)$ can intersect less than $(2r/2^{-\nu_{\ell+1}})+2 \le 4r2^{\nu_{\ell+1}}$ many components of $F_{\ell+1}$, each of μ_α measure $\mu_\alpha(J_{\ell,k})2^{-\nu_{\ell+1}}/h_\ell'''$. By (15)

$$\mu_\alpha(B(x,r)) \le \frac{4r}{h_\ell'''}\mu_\alpha(J_{\ell,k}) \le 4\frac{r}{h_\ell'}\frac{1}{h_{\ell-1}'''2^{\nu_\ell}}.$$

Thus

$$\frac{\mu_\alpha(B(x,r))}{r^{\alpha-\epsilon}} \le \frac{4\frac{r}{h_\ell'}\frac{1}{h_{\ell-1}'''2^{\nu_\ell}}}{\left(\frac{r}{h_\ell'}h_\ell'\right)^{\alpha-\epsilon}} =$$

$$4\left(\frac{r}{h_\ell'}\right)^{1-(\alpha-\epsilon)}\frac{\frac{1}{h_{\ell-1}'''2^{\nu_\ell}}}{(h_\ell')^{\alpha-\epsilon}} \le 4\frac{\frac{1}{h_{\ell-1}'''2^{\nu_\ell}}}{(h_\ell')^{\alpha-\epsilon}} < 1$$

where in the last line we used an estimate similar to the one used in subcase I. Hence for all $x \in F_\infty$ we have (16).

Next we investigate the Hölder properties of f at points of F_∞.

Since $f \in B(g_\ell, \varrho_\ell)$ we have

$$|f(y)-f(x)| \ge |g_\ell(y)-g_\ell(x)| - 2\varrho_\ell \ge |g_\ell(y)-g_\ell(x)| - h_\ell/4. \tag{17}$$

Recall that $g_\ell = g_\ell^* + g_\ell^{**}$ and g_ℓ^{**} increases by h_ℓ on each interval $[k/2^{\nu_\ell}, P_\ell(k)]$.

Assume $x \in F_\infty \subset F_\ell$ and $\alpha + \frac{1}{\ell} < \beta < 1$. Then using $\delta_\ell < h_\ell'$ there exists $y \in [x-2h_\ell', x]$ such that $|g_\ell^{**}(y) - g_\ell^{**}(x)| = h_\ell$ and hence using (17) and (13) we obtain

$$|f(y)-f(x)| \ge \frac{3}{4}h_\ell \ge \frac{3}{4}\left(\frac{h_\ell'}{2^\ell}\right)^\alpha \ge \frac{3}{8}\frac{1}{(h_\ell')^{1/\ell}2^{\ell\alpha}}(2h_\ell')^{\alpha+\frac{1}{\ell}} \ge$$

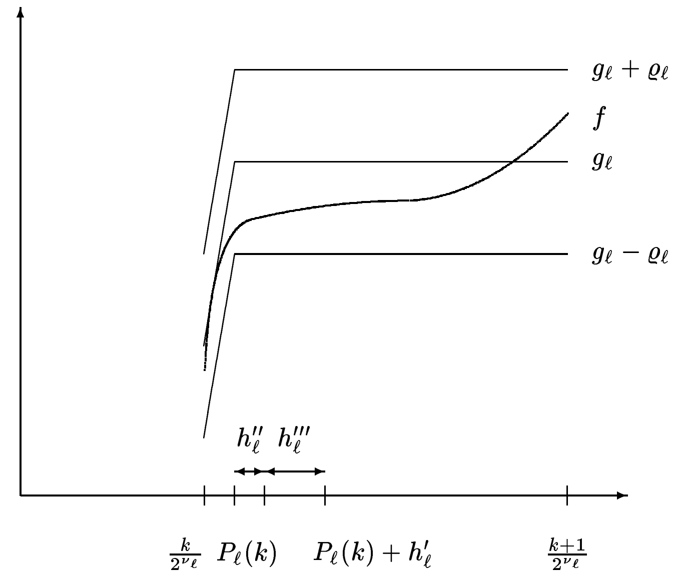
$$\frac{3}{16}\frac{1}{h_\ell^{1/\ell\alpha}2^{\ell\alpha}}(2h_\ell')^\beta \ge \frac{3}{16}\frac{1}{h_\ell^{1/\ell\alpha}2^{\ell\alpha}}|x-y|^\beta > |x-y|^\beta$$

where the last inequality holds if we choose a sufficiently large ℓ and ν_ℓ, namely, choosing $\nu_\ell > m_\ell^3 \ge \ell^3$ we have

$$\frac{3}{16}\frac{1}{h_\ell^{1/\ell\alpha}2^{\ell\alpha}} = \frac{3}{16}\frac{1}{(h_\ell^*)^{1/\ell\alpha}}2^{\frac{\nu_\ell}{\ell\alpha}-\ell\alpha} > \frac{3}{16}2^{\frac{\ell^2}{\alpha}-\ell\alpha} > 1.$$

Since the above type estimations are valid for all $\beta \in (\alpha, 1)$ when ℓ is sufficiently large we obtain that $F_\infty \subset E^f_{\le\alpha}$. From (16) and Proposition 1 it follows that $\dim_H F_\infty \ge \alpha$ which implies $\dim_H(E^f_{\le\alpha}) \ge \alpha$.

Next we refine our argument to obtain $\dim_H E^f_\alpha \ge \alpha$. To obtain this result we find a subset $F^0_\infty \subset F_\infty$ for which $\mu_\alpha(F^0_\infty) > 0$ and $F^0_\infty \subset E^f_\alpha$. By Proposition 1 this will imply the result about the Hausdorff dimension. To obtain F^0_∞ we delete some components of F_ℓ to obtain the closed sets F^0_ℓ. We will have a lower estimate on the μ_α measures of these sets yielding $\mu_\alpha(F^0_\infty) = \mu_\alpha(\cap_{\ell=1}^\infty F^0_\ell) > 0$.



Assume $x \in F'_\ell$. Then $x \in [P_\ell(k) + h''_\ell, P_\ell(k) + h'_\ell] \subset [k/2^{\nu_\ell}, (k+1)/2^{\nu_\ell}]$ for a suitable k. If $y \in [P_\ell(k), (k+1)/2^{\nu_\ell}]$ then $g_\ell^{**}(x) - g_\ell^{**}(y) = 0$. If $y \notin [P_\ell(k), (k+1)/2^{\nu_\ell}]$ then $h''_\ell \leq |y-x|$ and one can easily see that

$$|g_\ell^{**}(y) - g_\ell^{**}(x)| \leq \left(\left[\frac{|y-x|}{2^{-\nu_\ell}} \right] + 1 \right) h_\ell \leq h_\ell + |y-x| 2^{\nu_\ell} h_\ell = \tag{18}$$

$$h_\ell + |y-x| h_\ell^* < (2h''_\ell)^\alpha + |y-x| \leq 2^\alpha |y-x|^\alpha + |y-x| \leq 3|y-x|^\alpha, \tag{19}$$

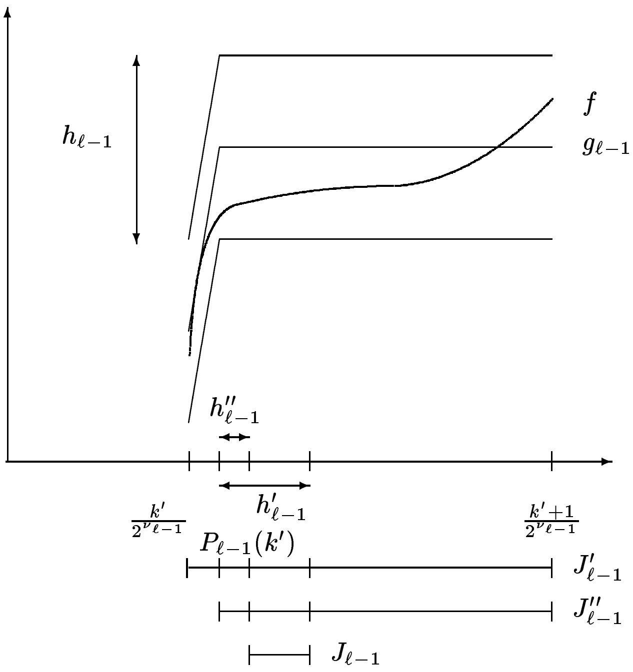
where we also used (6) and (14).

In the next argument we assume that ℓ_0 is sufficiently large and $\ell > \ell_0$. We define the sets F_ℓ^0 by induction on ℓ.

We assume that ℓ_0 is so large that for all $\ell \geq \ell_0$ we have

$$2h_\ell^\ell < h_\ell^{\frac{1}{\alpha}+2}. \tag{20}$$

To start our induction we choose a component $J_{\ell_0} \stackrel{\text{def}}{=} [P_{\ell_0}(k') + h''_{\ell_0}, P_{\ell_0}(k') + h'_{\ell_0}]$ of F_{ℓ_0} in a way that $g_{\ell_0}^*$ is constant on $J'_{\ell_0} \stackrel{\text{def}}{=} [k'/2^{\nu_{\ell_0}}, (k'+1)/2^{\nu_{\ell_0}}]$ and set $F_{\ell_0}^0 = J_{\ell_0}$.



Assume $\ell > \ell_0$ and $J_{\ell-1} \overset{\text{def}}{=} [P_{\ell-1}(k') + h''_{\ell-1}, P_{\ell-1}(k') + h'_{\ell-1}]$ is a component of $F^0_{\ell-1}$. Then $J_{\ell-1} \subset J'_{\ell-1} \overset{\text{def}}{=} [k'/2^{\nu_{\ell-1}}, (k'+1)/2^{\nu_{\ell-1}}]$. We also suppose that $g^*_{\ell-1}$ is constant on $J'_{\ell-1}$. Set $J''_{\ell-1} = [P_{\ell-1}(k'), (k'+1)/2^{\nu_{\ell-1}}]$. Observe that $g^{**}_{\ell-1}$ is constant on $J''_{\ell-1}$; hence $g_{\ell-1}$ is also constant there. From $f \in B(g_{\ell-1}, \varrho_{\ell-1})$ we infer

$$f\left(\frac{k'+1}{2^{\nu_{\ell-1}}}\right) - f(P_{\ell-1}(k')) \leq 2\varrho_{\ell-1} = \frac{h^{m_{\ell-1}}_{\ell-1}}{4} < h^{\ell-1}_{\ell-1} < h^{\frac{1}{\alpha}+2}_{\ell-1} \tag{21}$$

where we used (10), $m_\ell \geq \ell$ and (20).

From $f \in B(g_\ell, \varrho_\ell)$ and $2\varrho_\ell < h_\ell^{\frac{1}{\alpha}+2}$ it follows that

$$|g_\ell(y) - g_\ell(x)| - 2h_\ell^{\frac{1}{\alpha}+2} \leq |f(y) - f(x)| \leq |g_\ell(y) - g_\ell(x)| + 2h_\ell^{\frac{1}{\alpha}+2} \tag{22}$$

for all x, y.

Assume $x \in J_{\ell-1} \subset F_{\ell-1}$ and $y \in [k'/2^{\nu_{\ell-1}}, P_{\ell-1}(k')]$. Then (18), (14), $|y - x| \geq h''_{\ell-1}$ and $|f - g_{\ell-1}| \leq \varrho_{\ell-1} < h^{\frac{1}{\alpha}+2}_{\ell-1}$ yield

$$\begin{aligned} &|f(y) - f(x)| \leq 2h^{\frac{1}{\alpha}+2}_{\ell-1} + |g_{\ell-1}(y) - g_{\ell-1}(x)| = 2h^{\frac{1}{\alpha}+2}_{\ell-1} + \\ &|g^{**}_{\ell-1}(y) - g^{**}_{\ell-1}(x)| \leq 2h^{\frac{1}{\alpha}}_{\ell-1} h^2_{\ell-1} + 3|y - x|^\alpha \leq \\ &4h''_{\ell-1} h^2_{\ell-1} + 3|y - x|^\alpha \leq |y - x| + 3|y - x|^\alpha \leq 4|y - x|^\alpha. \end{aligned} \tag{23}$$

Using (21) and (22) we obtain

$$g_\ell\left(\frac{k'+1}{2^{\nu_{\ell-1}}}\right) - g_\ell(P_{\ell-1}(k')) \le h_{\ell-1}^{\frac{1}{\alpha}+2} + 2h_\ell^{\frac{1}{\alpha}+2} < 2h_{\ell-1}^{\frac{1}{\alpha}+2}.$$

Since $g_\ell = g_\ell^* + g_\ell^{**}$ and the functions $g_\ell{}^*$, g_ℓ^{**} are monotone we also have

$$g_\ell^*\left(\frac{k'+1}{2^{\nu_{\ell-1}}}\right) - g_\ell^*(P_{\ell-1}(k')) < 2h_{\ell-1}^{\frac{1}{\alpha}+2} \le 4h''_{\ell-1}h_{\ell-1}^2 < \frac{1}{2}h'''_{\ell-1}h_{\ell-1} \tag{24}$$

where we assumed that $\ell_0 \le \ell$ is sufficiently large and hence $4h_{\ell-1} < 1/2$.

Now Lemma 7 implies that there is an exceptional set Ψ^* such that

$$\lambda(\Psi^*) < 2\left(g_\ell^*\left(\frac{k'+1}{2^{\nu_{\ell-1}}}\right) - g_\ell^*(P_{\ell-1}(k'))\right) < h'''_{\ell-1}h_{\ell-1}$$

and from $x \in J''_{\ell-1} \setminus \Psi^*$ it follows that $|g_\ell^*(y) - g_\ell^*(x)| \le |y-x|$ holds for all $y \in J''_{\ell-1}$.

We can assume that ν_ℓ is chosen so large that $2/2^{\nu_\ell}$ is less than all the jumps $g_\ell^*((k'+1)/2^{N_n}) - g_\ell^*(k'/2^{N_n}) > 0$, $(k' \in \{0,...,2^{N_n}-1\})$. Hence g_ℓ^* is either constant on an interval $[k/2^{\nu_\ell},(k+1)/2^{\nu_\ell}]$ or increases by at least $2/2^{\nu_\ell}$.

This implies that $(b-a) > 1/2^{\nu_\ell}$ for each component (a,b) of Ψ^*. Denote by Ψ_0^* the union of those intervals of the form $[k/2^{\nu_\ell},(k+1)/2^{\nu_\ell}]$ which intersect Ψ^*. By adding to each component (a,b) of Ψ^* those intervals $[k/2^{\nu_\ell},(k+1)/2^{\nu_\ell}]$ which contain a and b we obtain Ψ_0^*. Hence $\lambda(\Psi_0^*) \le 3\lambda(\Psi^*) < 3h'''_{\ell-1}h_{\ell-1}$. Denote by F_ℓ^* the union of those components of $F_\ell \cap J''_{\ell-1}$ which do not intersect Ψ_0^*. We define F_ℓ^0 so that $F_\ell^0 \cap J''_{\ell-1}$ equals F_ℓ^*. Observe that $F_\ell \cap J_{\ell-1} = F_\ell \cap J'_{\ell-1}$ consists of $h'''_{\ell-1}2^{\nu_\ell}$ many components and we need to delete $\lambda(\Psi_0^*)2^{\nu_\ell} < 3h_{\ell-1}h'''_{\ell-1}2^{\nu_\ell}$ many of them to obtain $F_\ell^0 \cap J'_{\ell-1} = F_\ell^0 \cap J_{\ell-1}$.

Since the μ_α measure of each component of F_ℓ equals $\mu_\alpha(F_\ell \cap J_{\ell-1})2^{-\nu_\ell}/h'''_{\ell-1}$ we obtain

$$\mu_\alpha(F_\ell^0 \cap J_{\ell-1}) \ge (1-3h_{\ell-1})\mu_\alpha(F_\ell \cap J_{\ell-1}) = (1-3h_{\ell-1})\mu_\alpha(F_{\ell-1}^0 \cap J_{\ell-1}).$$

Adding the above inequalities for all $J_{\ell-1}$ type components of $F_{\ell-1}^0$ we have

$$\mu_\alpha(F_\ell^0) \ge (1-3h_{\ell-1})\mu_\alpha(F_{\ell-1}^0). \tag{25}$$

Using $\nu_\ell \ge n_\ell \ge m_\ell \ge \ell$ and (6) we infer

$$\mu_\alpha(F_\infty^0) \ge \prod_{\ell=\ell_0+1}^{\infty}(1-3h_{\ell-1})\mu_\alpha(F_{\ell_0}^0) > 0. \tag{26}$$

For each $x \in F_\ell^0$ and $y \in J''_{\ell-1}$ we have $|g_\ell^*(y) - g_\ell^*(x)| \le |y-x|$. We also know that $x \in F_\ell^0 \subset F'_\ell$ and (18) implies

$$|g_\ell^{**}(y) - g_\ell^{**}(x)| \le 3|y-x|^\alpha.$$

Hence $|g_\ell(y)-g_\ell(x)| \le 4|y-x|^\alpha$ holds for all $x \in F_\ell^0$ and $y \in J''_{\ell-1}$. By (22) we obtain $|f(y)-f(x)| \le 4|y-x|^\alpha + 2h_\ell^{\frac{1}{\alpha}+2}$. We also note that if $J_\ell \overset{\text{def}}{=} [P_\ell(k)+h''_\ell, P_\ell(k)+h'_\ell]$ is a component of F_ℓ^0 then $J_\ell \subset J'_\ell \overset{\text{def}}{=} [k/2^{\nu_\ell},(k+1)/2^{\nu_\ell}]$ and g_ℓ^* is constant on J'_ℓ, since otherwise $|g_\ell^*(y)-g_\ell^*(x)| \le |y-x|$ cannot hold for all $y \in J'_\ell$ (recall that we assumed that the jumps of g_ℓ^* are at least $2 \cdot 2^{\nu_\ell}$).

Assume $x \in F_\ell^0$ and $y \in J''_{\ell-1} \setminus J'_\ell$. Then $|y-x| \ge h''_\ell \ge h_\ell^{1/\alpha}/2$; hence $2^\alpha |y-x|^\alpha \ge h_\ell$. Therefore

$$|f(y)-f(x)| \le 4|y-x|^\alpha + 2h_\ell^{\frac{1}{\alpha}+2} \le 4|y-x|^\alpha + 4h_\ell^2|y-x| \le 5|y-x|^\alpha.$$

If $x \in F_\ell^0 \subset F_{\ell-1}$ and $y \in [k'/2^{\nu_{\ell-1}}, P_{\ell-1}(k')) = J'_{\ell-1} \setminus J''_{\ell-1}$ we have already seen in (23) that $|f(y)-f(x)| \le 4|y-x|^\alpha$. Hence $|f(y)-f(x)| \le 5|y-x|^\alpha$ holds for all $x \in F_\ell^0$ and $y \in J'_{\ell-1} \setminus J'_\ell$.

Finally, if $x \in \cap_{\ell=\ell_0}^\infty F_\ell^0 = F_\infty^0$ and $x \ne y \in J'_{\ell_0}$ then there exists $\ell \ge \ell_0+1$ such that $y \in J'_{\ell-1} \setminus J'_\ell$ and hence $|f(y)-f(x)| \le 5|y-x|^\alpha$ holds. Therefore $F_\infty^0 \subset E_\alpha^f$ and by Proposition 1, (26) and (16) we infer $\dim_H(F_\infty^0) \ge \alpha$.

This shows that for functions in $\mathcal{F}_{(ii)} \overset{\text{def}}{=} \mathcal{F}_1$ assertion (ii) holds.

□

5.3 Assertion (iii)

Proof. Assume $\mathcal{F}_{(iii)}$ is the dense open set $\mathcal{F}^*$ of the main construction (used with the value of $h^*_{n,m}$ given in (6)) and $f \in \mathcal{F}_{(iii)}$. Given $j_0 \in \mathbb{N}$ set $m = j_0$ and choose $n \ge m$ such that $f \in G_{n,m}$. Then $N_n \ge K_n > n \ge m = j_0$; using (5) and the definition of g_n^* we obtain

$$\mu_{f_n,k'} \overset{\text{def}}{=} \mu_{f_n}\left(\left[\frac{k'}{2^{N_n}}, \frac{k'+1}{2^{N_n}}\right]\right) = g_n^*\left(\frac{k'+1}{2^{N_n}}\right) - g_n^*\left(\frac{k'}{2^{N_n}}\right).$$

Since $f'_n > 0$ on $[0,1]$ we have $0 < \mu_{f_n,\min} \overset{\text{def}}{=} \min\{\mu_{f_n,k'} : k' = 0,\ldots,2^{N_n}-1\}$ and we can assume that $\nu_{n,m}$ was chosen so large that $2^{-\nu_{n,m}} < \mu_{f_n,\min}/8$; hence $h^*_{n,m} < 2^{-\nu_{n,m}} < \mu_{f_n,\min}/8$ and $\varrho_{n,m} < h_{n,m}/8 < h^*_{n,m}/8 < \mu_{f_n,\min}/8$.

Recalling (8) we obtain

$$\begin{aligned} &\mu_f\left(\left[\frac{k'}{2^{N_n}}, \frac{k'+1}{2^{N_n}}\right]\right) = \left|f\left(\frac{k'+1}{2^{N_n}}\right) - f\left(\frac{k'}{2^{N_n}}\right)\right| \ge \\ &\left|g_{n,m}\left(\frac{k'+1}{2^{N_n}}\right) - g_{n,m}\left(\frac{k'}{2^{N_n}}\right)\right| - 2\varrho_{n,m} \ge \\ &\left|g_n^*\left(\frac{k'+1}{2^{N_n}}\right) - g_n^*\left(\frac{k'}{2^{N_n}}\right)\right| - 2\varrho_{n,m} - 2h^*_{n,m} > \frac{\mu_{f_n,k'}}{2}. \end{aligned} \tag{27}$$

Thus, using (3) we have

$$\tau_{f,N_n}(q) \ge \frac{1}{N_n \log 2} \log \sum_{k'=0}^{2^{N_n}-1} (\mu_{f_n,k'}/2)^q = \tau_{f_n,N_n}(q) - \frac{q}{N_n} \ge 1 - q - \frac{2}{j_0}.$$

This implies $\overline{\tau}_f(q) = \limsup_{j\to\infty} \tau_{f,j}(q) \geq 1-q$. By Lemma 3 we obtain $\overline{\tau}_f(q) = 1-q$.

Since f is strictly monotone increasing one can easily show that $\tau_f(0) = 1$.

Assume $0 < q \leq 1$ is given and fixed. We verify that $\underline{\tau}_f(q) = 0$.

Given $f \in \mathcal{F}_{(iii)}$ and $j_0 > 1/q$ we choose n, m as above. Observe that out of the $2^{\nu_{n,m}}$ intervals of the form $[k/2^{\nu_{n,m}}, (k+1)/2^{\nu_{n,m}}]$, $(k = 0, ..., 2^{\nu_{n,m}} - 1)$ there are only 2^{N_n} many intervals on which g_n^* is nonconstant. We call these intervals irregular and the remaining intervals are called regular. We denote by Γ the set of those $k \in \{0, ..., 2^{\nu_{n,m}} - 1\}$ for which $[k/2^{\nu_{n,m}}, (k+1)/2^{\nu_{n,m}}]$ is regular. For irregular k's we use the trivial estimate

$$\left(\mu_f\left(\left[\frac{k}{2^{\nu_{n,m}}}, \frac{k+1}{2^{\nu_{n,m}}}\right]\right)\right)^q \leq \mu_f([0,1]) = 1,$$

where we used that replacing f by cf we can assume that μ_f is a probability measure. For $k \in \Gamma$ using $g_n^*((k+1)/2^{\nu_{n,m}}) - g_n^*(k/2^{\nu n,m}) = 0$ we obtain

$$\begin{aligned}
&\mu_f\left(\left[\frac{k}{2^{\nu_{n,m}}}, \frac{k+1}{2^{\nu_{n,m}}}\right]\right) = f\left(\frac{k+1}{2^{\nu_{n,m}}}\right) - f\left(\frac{k}{2^{\nu_{n,m}}}\right) \leq \\
&g_{n,m}\left(\frac{k+1}{2^{\nu_{n,m}}}\right) - g_{n,m}\left(\frac{k}{2^{\nu_{n,m}}}\right) + 2\varrho_{n,m} = \\
&g_{n,m}^{**}\left(\frac{k+1}{2^{\nu_{n,m}}}\right) - g_{n,m}^{**}\left(\frac{k}{2^{\nu_{n,m}}}\right) + 2\varrho_{n,m} = \\
&h_{n,m} + 2\varrho_{n,m} < 2h_{n,m}.
\end{aligned}$$

If j_0 and hence $n \geq m = j_0 > 1/q$ is sufficiently large then we can assume that $\nu_{n,m} > N_n^2$ satisfies $2^{\nu_{n,m}} \cdot 2^q h_{n,m}^q = 2^{\nu_{n,m}} \cdot 2^q (h_{n,m}^*/2^{\nu_{n,m}})^q = 2^{\nu_{n,m}} \cdot 2^q (1/4^n 2^{(m+1)\nu_{n,m}})^q \leq 1 < 2^{N_n}$. Now using $\#(\{0, ..., 2^{\nu_{n,m}} - 1\} \setminus \Gamma) = 2^{N_n}$ $\#\Gamma < 2^{\nu_{n,m}}$, and the above estimates we obtain

$$\begin{aligned}
&\tau_{f,\nu_{n,m}}(q) \leq \frac{1}{\nu_{n,m}\log 2}\log(2^{N_n} + 2^{\nu_{n,m}} \cdot 2^q h_{n,m}^q) \leq \\
&\frac{1}{\nu_{n,m}\log 2}\log(2\cdot 2^{N_n}) \leq \frac{1}{\nu_{n,m}\log 2}\log(2^{\sqrt{\nu_{n,m}}}) \leq \frac{1}{\sqrt{\nu_{n,m}}} \leq \frac{1}{j_0},
\end{aligned} \tag{28}$$

Hence $\underline{\tau}_f(q) \leq 0$ and Lemma 3 completes the proof showing that $\underline{\tau}_f(q) = 0$. □

5.4 Assertion (iv)

The proof of this assertion is similar at many places to that of Assertion (iii).

Proof. Assume $q > 1$ is fixed. Let $\mathcal{F}_{(iv)} = \mathcal{F}^*$ of the main construction and for given $j_0 \in \mathbb{N}$ choose m, n and define $\mu_{f_n,k'}$, $\mu_{f_n,\min}$ as in the proof of Assertion (iii). We can again assume that $h_{n,m}^* < \mu_{f_n,\min}/8$ and $\varrho_{n,m} < \mu_{f_n\,\min}/8$.

Then similarly to (27) we obtain

$$\mu_f\left(\left[\frac{k'}{2^{N_n}},\frac{k'+1}{2^{N_n}}\right]\right)\leq\left|g_n^*\left(\frac{k'+1}{2^{N_n}}\right)-g_n^*\left(\frac{k'}{2^{N_n}}\right)\right|+$$
$$2\varrho_{n,m}+2h^*_{n,m}<2\mu_{f_n,k'}.$$

We can use (4) when $n\geq j_0\geq q$ and we have

$$\tau_{f,N_n}(q)\leq\frac{1}{N_n\log 2}\log\sum_{k'=0}^{2^{N_n}-1}(2\mu_{f_n,k'})^q\leq 1-q+\frac{q+1}{j_0}.$$

This implies $\underline{\tau}_f(q)\leq 1-q$. By Lemma 5 we obtain $\underline{\tau}_f(q)=1-q$.

Next we show $\overline{\tau}_f(q)\geq 0$.

Observe that from $\nu_{n,m}>N_n^2$ and (7) it follows that if we choose j_n such that $2^{-j_n}=\delta_{n,m}$ then $\sqrt{j_n}>N_n$. We can also assume that $j_0\leq m\leq n$ is so large that

$$g_n^*(1)-g_n^*(0)\geq f(1)-f(0)-2\varrho_{n,m}-h^*_{n,m}\geq\frac{f(1)-f(0)}{2}.$$

Observe that if $k''\in\{0,...,2^{j_n}-1\}$ is chosen so that $k''/2^{j_n}=k'/2^{N_n}$ for a suitable $k'\in\{0,...,2^{N_n}-1\}$ then

$$g_n^*\left(\frac{k''+1}{2^{j_n}}\right)-g_n^*\left(\frac{k''}{2^{j_n}}\right)=\mu_{f_n,k'},$$

we denote the set of these k'' by Γ''.

For $k''\in\Gamma''$ similarly to (27) we obtain

$$\mu_f\left(\left[\frac{k''}{2^{j_n}},\frac{k''+1}{2^{j_n}}\right]\right)\geq g_n^*\left(\frac{k'+1}{2^{N_n}}\right)-g_n^*\left(\frac{k'}{2^{N_n}}\right)-2\varrho_{n,m}-2h^*_{n,m}\geq\frac{\mu_{f_n,k'}}{2}.$$

By using an inequality between means of non negative numbers we have

$$\left(\frac{\sum_{k'=0}^{2^{N_n}-1}\mu^q_{f_n,k'}}{2^{N_n}}\right)^{\frac{1}{q}}\geq\frac{\sum_{k'=0}^{2^{N_n}-1}\mu_{f_n,k'}}{2^{N_n}}=\frac{g_n^*(1)-g_n^*(0)}{2^{N_n}}.$$

Hence

$$\sum_{k'=0}^{2^{N_n}-1}\mu^q_{f_n,k'}\geq(g_n^*(1)-g_n^*(0))^q 2^{N_n(1-q)}\geq\left(\frac{f(1)-f(0)}{2}\right)^q 2^{N_n(1-q)}.$$

Thus

$$\tau_{f,j_n}(q) = \frac{1}{j_n \log 2} \log \sum_{k''=0}^{2^{j_n}-1} \left(\mu_f\left(\left[\frac{k''}{2^{j_n}}, \frac{k''+1}{2^{j_n}}\right]\right)\right)^q \geq$$

$$\frac{1}{j_n \log 2} \log \sum_{k'' \in \Gamma''} \left(\mu_f\left(\left[\frac{k''}{2^{j_n}}, \frac{k''+1}{2^{j_n}}\right]\right)\right)^q \geq$$

$$\frac{1}{j_n \log 2} \log \left(\left(\frac{f(1)-f(0)}{2}\right)^q 2^{N_n(1-q)} \frac{1}{2^q}\right) \geq$$

$$\frac{1}{j_n \log 2} \log \left(\left(\frac{f(1)-f(0)}{4}\right)^q 2^{\sqrt{j_n}(1-q)}\right) \geq \frac{1-q}{\sqrt{j_n}} + \frac{\log\left(\left(\frac{f(1)-f(0)}{4}\right)^q\right)}{j_n \log 2}.$$

This implies $\overline{\tau}_f(q) \geq 0$. By Lemma 5 we obtain $\overline{\tau}_f(q) = 0$. $\square$

5.5 Assertion (v)

Proof. First we show $\underline{\tau}_f(q) \geq 1-q$. Assume $f \in \mathcal{F}_{(v)} \stackrel{\text{def}}{=} \mathcal{F}_{(i)} = \mathcal{F}_1 \cap \mathcal{F}_0$ then $f' = 0$ almost everywhere and hence there exists $\delta > 0$ and a measurable set $H \subset [0,1]$ such that $\lambda(H) > 1/2$ and for any $x \in H$, $y \in (x-\delta, x+\delta)$ we have

$$|f(y) - f(x)| < |y - x|. \tag{29}$$

Assume j is chosen so that $1/2^j < \delta$. Denote by Γ_H the set of those $k \in \{0, ..., 2^j - 1\}$ for which $H \cap [k/2^j, (k+1)/2^j] \neq \emptyset$. From $\lambda(H) > 1/2$ it follows that $\#\Gamma_H > 2^{j-1}$. From (29) it follows that $\mu_f([k/2^j, (k+1)/2^j]) < 2^{-j}$ holds when $k \in \Gamma_H$. Hence using $q < 0$ we have

$$\tau_{f,j}(q) \geq \frac{1}{j \log 2} \log \sum_{k \in \Gamma_H} \left(\mu_f\left(\left[\frac{k}{2^j}, \frac{k+1}{2^j}\right]\right)\right)^q >$$

$$\frac{1}{j \log 2} \log(2^{j-1} 2^{-jq}) = \frac{j(1-q)-1}{j}.$$

This implies $\underline{\tau}_f(q) \geq 1-q$.

Next we show $\underline{\tau}_f(q) \leq 1-q$. For any $n \geq m$ we have

$$g_{n,m}\left(\frac{k+1}{2^{\nu_{n,m}}}\right) - g_{n,m}\left(\frac{k}{2^{\nu_{n,m}}}\right) \geq h_{n,m}.$$

Since $\varrho_{n,m} < h_{n,m}/8$ we have

$$\mu_{f,n,m,k} = f\left(\frac{k+1}{2^{\nu_{n,m}}}\right) - f\left(\frac{k}{2^{\nu_{n,m}}}\right) >$$

$$g_{n,m}\left(\frac{k+1}{2^{\nu_{n,m}}}\right) - g_{n,m}\left(\frac{k}{2^{\nu_{n,m}}}\right) - 2\varrho_{n,m} >$$

$$\frac{3}{4} h_{n,m} = \frac{3}{4^{n+1} 2^{\nu_{n,m}}} > \frac{1}{4^{2n} 2^{\nu_{n,m}}}.$$

Keeping in mind that $q<0$ we infer

$$\tau_{f,\nu_{n,m}}(q) = \frac{1}{\nu_{n,m}\log 2}\log \sum_{k=0}^{2^{\nu_{n,m}}-1} \mu^q_{f,n,m,k} <$$
$$\frac{1}{\nu_{n,m}\log 2}\log(2^{\nu_{n,m}}4^{-2nq}2^{-q\nu_{n,m}}) = (1-q) + \frac{-4nq}{\nu_{n,m}}.$$

From $\nu_{n,m} > N_n^2 \geq K_n^2 > n^2 \geq m^2$ easily follows that $\frac{-4nq}{\nu_{n,m}} \to 0$ as m, and hence n, tends to ∞.

Therefore $\underline{\tau}_f(q) \leq 1-q$.

Now we verify that $\overline{\tau}_f(q) = \infty$.

Observe that by (7), $P(k,n,m) < (2k+1)/2^{\nu_{n,m}+1}$ and hence $g_{n,m}$ is constant on $[(2k+1)/2^{\nu_{n,m}+1}, (k+1)/2^{\nu_{n,m}}]$. Therefore,

$$f\left(\frac{k+1}{2^{\nu_{n,m}}}\right) - f\left(\frac{2k+1}{2^{\nu_{n,m}+1}}\right) < 2\varrho_{n,m} = \frac{(h_{n,m})^m}{4}.$$

Thus in line (31) below recalling (11) we obtain

$$\tau_{f,\nu_{n,m}+1}(q) =$$
$$\frac{1}{(\nu_{n,m}+1)\log 2}\log \sum_{k'=0}^{2^{\nu_{n,m}+1}-1}\left(f\left(\frac{k'+1}{2^{\nu_{n,m}+1}}\right) - f\left(\frac{k'}{2^{\nu_{n,m}+1}}\right)\right)^q > \tag{30}$$
$$\frac{1}{(\nu_{n,m}+1)\log 2}\log \sum_{k=0}^{2^{\nu_{n,m}}-1}\left(f\left(\frac{k+1}{2^{\nu_{n,m}}}\right) - f\left(\frac{2k+1}{2^{\nu_{n,m}+1}}\right)\right)^q >$$
$$\frac{1}{(\nu_{n,m}+1)\log 2}\log\left(2^{\nu_{n,m}}\frac{(h_{n,m})^{mq}}{4^q}\right) =$$
$$\frac{1}{(\nu_{n,m}+1)\log 2}\log(2^{\nu_{n,m}}(4^n 2^{\nu_{n,m}})^{-mq}4^{-q}) = \tag{31}$$
$$\frac{\nu_{n,m}(1-mq)}{\nu_{n,m}+1} + \frac{-2mnq\log 2 + \log(4^{-q})}{(\nu_{n,m}+1)\log 2}.$$

Recalling that $q<0$ and $\nu_{n,m} > N_n^2 \geq K_n^2 > n^2 \geq m^2$ we obtain $\overline{\tau}_f(q) = \infty$. □

5.6 Assertion (vi)

Assume $f \in \mathcal{F}_{(vi)} \stackrel{\text{def}}{=} \mathcal{F}_{(ii)}$. We will use the notation and choice of $h^*_{n,m}$ introduced in the proof of assertion (ii) and hence choose a suitable subsequence $m_\ell \geq \ell$. We will denote $W_\ell = W_{n(m_\ell),m_\ell}$. Clearly,

$$\mu_{g_\ell}(W_\ell) = \sum_{k=0}^{2^{\nu_\ell}-1} g_\ell(P_\ell(k)) - g_\ell\left(\frac{k}{2^{\nu_\ell}}\right) = \sum_{k=0}^{2^{\nu_\ell}-1} g_\ell\left(\frac{k+1}{2^{\nu_\ell}}\right) - g_\ell\left(\frac{k}{2^{\nu_\ell}}\right) =$$
$$g_\ell(1) - g_\ell(0) \geq f(1) - f(0) - 2\varrho_\ell.$$

Assuming $m_\ell \geq 2$, we have by (6) and (10) $\varrho_\ell < h_\ell^2/2 < h_\ell/2^{\nu_\ell+1}$. Therefore

$$\mu_f(W_\ell) = \sum_{k=0}^{2^{\nu_\ell}-1} f(P_\ell(k)) - f\left(\frac{k}{2^{\nu_\ell}}\right) \geq$$

$$\sum_{k=0}^{2^{\nu_\ell}-1} \left(g_\ell(P_\ell(k)) - g_\ell\left(\frac{k}{2^{\nu_\ell}}\right) - 2\varrho_\ell\right) \geq f(1) - f(0) - 2^{\nu_\ell+1}\varrho_\ell >$$

$$f(1) - f(0) - h_\ell > f(1) - f(0) - \frac{1}{2^\ell}.$$

Thus $\mu_f([0,1] \setminus W_\ell) < 1/2^\ell$. By the Borel-Cantelli lemma μ_f-almost every $x \in [0,1]$ belongs to finitely many of the sets $[0,1] \setminus W_\ell$, that is, for μ_f- almost every x there exists ℓ_x such that for all $\ell \geq \ell_x$ we have $x \in W_\ell$.

If $x \in W_\ell$, that is, $x \in (k/2^{\nu_\ell}, P_\ell(k))$ for a suitable k then

$$f(P_\ell(k)) - f\left(\frac{k}{2^{\nu_\ell}}\right) \geq g_\ell(P_\ell(k)) - g_\ell\left(\frac{k}{2^{\nu_\ell}}\right) - 2\varrho_\ell \geq$$

$$g_\ell^{**}(P_\ell(k)) - g_\ell^{**}\left(\frac{k}{2^{\nu_\ell}}\right) - 2\varrho_\ell = h_\ell - 2\varrho_\ell >$$

$$\frac{h_\ell}{2} > \frac{\delta_\ell^{\frac{1}{2m_\ell}}}{2} > \frac{\delta_\ell^{\frac{1}{2\ell}}}{2}.$$

Taking $y = k/2^{\nu_\ell}$ or $y = P_\ell(k)$ we have

$$|f(y) - f(x)| \geq \frac{f(P_\ell(k)) - f(k/2^{\nu_\ell})}{2} > \frac{\delta_\ell^{\frac{1}{2\ell}}}{4}.$$

On the other hand $|y - x| < \delta_\ell$ and hence $|f(y) - f(x)| > |y-x|^{1/2\ell}/4$. Since we can choose any $\ell \geq \ell_x$ we have $h_f(x) = 0$. This shows $h_f = 0$, μ_f-almost everywhere in $[0,1]$, that is, $\mu_f(E_{\leq 0}^f) = f(1) - f(0) = \mu_f([0,1])$.

References

[1] G. Brown, G. Michon and J. Peyrière, On the multifractal analysis of measures, *J. Statist. Phys.* **66** no. 3-4 (1992), 775-790.

[2] S. Jaffard, Multifractal formalism for functions Part I: Results valid for all functions, *SIAM J. Math. Anal*, **28** No. 4 (1997), 944-970.

[3] S. Jaffard, Multifractal formalism for functions Part II: Self-similar functions, *SIAM J. Math. Anal*, **28** No. 4 (1997), 971-998.

[4] S . Jaffard, The spectrum of singularities of Riemann's function, *Rev. Mat. Iberoamericana*, **12** (1996) 441-460.

[5] S. Jaffard, Old friends revisited: the multifractal nature of some classical functions, *J. of Fourier Anal. and Appl.* **3** No. 1 (1997), 1-22.

[6] K. Falconer, *Fractal Geometry*, John Wiley & Sons (1990).

[7] K. Falconer, *Techniques in Fractal Geometry*, John Wiley & Sons (1997).

[8] H. Federer, *Geometric Measure Theory*, Springer Verlag, New York, 1969.

[9] J. Genyuk, A typical measure typically has no local dimension, *Real Anal. Exchange*, **23**(2), 1997/8, 525-538.

[10] H. Haase, A survey on the dimension of measures, *Topology, measures, and fractals (Warnemünde, 1991)*, 66-75, *Math. Res., 66*, Akademie-Verlag, Berlin 1992.

[11] T. Zamfirescu, Most monotone functions are singular, *Amer. Math. Monthly*, **88** (1) (1981), 47–49.

[12] T. Zamfirescu, Typical monotone continuous functions, *Arch. Math.*, **42** (1984), 151–156.