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Second Peano Derivatives are not Extendable

Introduction. A well-known theorem of V.Jarník [J] (see also [PL] Th. 5.5.2.) states that if a real function is defined and differentiable on a perfect subset of the real line, then this function can be extended to be a function defined and differentiable everywhere.

C. E. Weil in a personal communication asked what can be said about the extensions of Peano differentiable functions. Since the first Peano derivatives coincide with the ordinary derivatives, this question is interesting for the second and higher Peano derivatives. In this paper we show that in general there is no extension theorem for the second Peano derivative.

Definition. Suppose that $F \subset \mathbf{R}$ is perfect. We say that $f : F \to \mathbf{R}$ is k times Peano differentiable at $x \in F$ if there exist finite numbers $f_{(0)}(x) = f(x)$, $f_{(1)}(x)$,...,$f_{(k)}(x)$ such that

$$\Big|f(x+h) - \sum_{i=0}^{k} \frac{f_{(i)}(x)}{i!} h^i\Big| = o(h^k),$$

when $x + h \in F$ and $h \to 0$. The set of k times Peano differentiable functions on F will be denoted by $\mathbf{D}^k(F)$.

Theorem. *There exists a non-empty perfect set F and a function $f : F \to \mathbf{R}$ such that $f \in \mathbf{D}^2(F)$ and there does not exist $g : \mathbf{R} \to \mathbf{R}$ such that $g\big|_F = f$ and $g \in \mathbf{D}^2(\mathbf{R})$.*

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Proof. Put $d_n := (n+1)^{-(n+1)}$. Denote by $\mathcal{S}$ the set of infinite zero-one sequences. For every $n \in \mathbf{N}$ and $s \in \mathcal{S}$ denote by $s|n$ the first n terms of s. Put $I_\emptyset := [0,1]$. Suppose that for an $n \in \mathbf{N}$ and $s \in \mathcal{S}$ we have defined $I_{s|n-1} = [a_{s|n-1}, b_{s|n-1}]$. If the n'th term of s equals zero (or one), then we put $I_{s|n} := [a_{s|n-1}, a_{s|n-1}+d_n]$ (or $I_{s|n} := [b_{s|n-1}-d_n, b_{s|n-1}]$ respectively). Put $X(s) := \cap_{n=1}^{\infty} I_{s|n}$ and $F := \cup\{X(s) : s \in \mathcal{S}\}$. It is obvious that the mapping $s \mapsto X(s)$ is one-to-one and the set F is non-empty and perfect. Thus for $x \in F$ denote by $S(x) \in \mathcal{S}$ the inverse of x; that is, $X(S(x)) = x$. Denote by $N(s) \in \{0\} \cup \mathbf{N} \cup \{+\infty\}$ the number of the ones in the sequence $s \in \mathcal{S}$. Suppose that $x \in F$. If $i \leq N(S(x))$ is a natural number then choose $x_i \in F$ such that $S(x_i)|n = S(x)|n$ if n denotes the place in the sequence $S(x)$ where the i'th term one appeares, and the remaining terms of $S(x_i)$ equal zero. For $i > N(S(x))$ put $x_i = x$. We also put $x_0 = 0$. It is clear that $x_i \geq x_{i-1}$ for every $x \in F$ and $i = 1, 2, \ldots$. The function f claimed in our theorem will be defined on F by

$$f(x) := \sum_{i=1}^{\infty} \frac{1}{2^i}(x_i - x_{i-1})(x - x_i).$$

First we show that $f \in \mathbf{D}^2(F)$. Namely, we shall show that for every $x \in F$ we have $f_{(1)}(x) = \sum_{i=1}^{\infty} 2^{-i}(x_i - x_{i-1})$ and $f_{(2)}(x) = 0$. Suppose that $x, y \in F$, $S(x)|m = S(y)|m$, $S(x)|(m+1) \neq S(y)|(m+1)$. Denote by j the number of the ones in $S(x)|m$. By the choice of m it is obvious that the $(m+1)$'th terms of $S(x)$ and $S(y)$ are different. Therefore by the definition of F we have that $d_m/2 < |x-y| \leq d_m$. We also have that

$x_i = y_i$ for $i = 1, \ldots, j$,

$|x - x_i| < d_m$, $|y - y_i| < d_m$ for $i \geq j$,

$|x_i - x_{i-1}| < d_m$, $|y_i - y_{i-1}| < d_m$ for $i \geq j+1$.

Using these estimations we obtain

$$\Big|\sum_{i=1}^{\infty} 2^{-i}(x_i - x_{i-1})(x - x_i) + \sum_{i=1}^{\infty} 2^{-i}(x_i - x_{i-1})(y - x) -$$

$$\sum_{i=1}^{\infty} 2^{-i}(y_i - y_{i-1})(y - y_i)\Big| \leq$$

$$\Big|\sum_{i=1}^{j} 2^{-i}(x_i - x_{i-1})(x - x_i) + \sum_{i=1}^{j} 2^{-i}(x_i - x_{i-1})(y - x) -$$

$$\sum_{i=1}^{j} 2^{-i}(x_i - x_{i-1})(y - x_i)\Big| +$$

$$\Big|\sum_{i=j+1}^{\infty} 2^{-i}(x_i - x_{i-1})(x - x_i) + \sum_{i=j+1}^{\infty} 2^{-i}(x_i - x_{i-1})(y - x) -$$

$$\sum_{i=j+1}^{\infty} 2^{-i}(y_i - y_{i-1})(y - y_i)\Big| \leq$$

$$0 + \Big|\sum_{i=j+1}^{\infty} \frac{d_m}{2^i} d_m\Big| + \Big|\sum_{i=j+1}^{\infty} \frac{d_m}{2^i}\Big| \cdot |y - x| + \Big|\sum_{i=j+1}^{\infty} \frac{d_m}{2^i} d_m\Big| \leq d_m^2 \cdot 3 \cdot 2^{-j}.$$

Therefore

$$\frac{|f(y) - f(x) - f_{(1)}(x)(y - x) - \frac{f_{(2)}(x)}{2!}(y - x)^2|}{|y - x|^2} \leq \frac{d_m^2 \cdot 3 \cdot 2^{-j}}{2^{-2} \cdot d_m^2} \leq 2^{-j+4}$$

If $N(S(x)) = \infty$, then $y \to x$ implies $j \to \infty$ in the above computations and hence f is twice Peano differentiable at x.

For $N(S(x)) < \infty$ we have to modify the above computations a little. We can obviously suppose that y is so close to x that $m > n(N(S(x)), S(x))$. Then $j = N(S(x))$ and we also have $x_i - x_{i-1} = 0$ for $i \geq j + 1$. This implies that

$$|\sum_{i=j+1}^{\infty} 2^{-i}(x_i - x_{i-1})(x - x_i) + \sum_{i=j+1}^{\infty} 2^{-i}(x_i - x_{i-1})(y - x) -$$

$$\sum_{i=j+1}^{\infty} 2^{-i}(y_i - y_{i-1})(y - y_i)\big| = |\sum_{i=j+1}^{\infty} 2^{-i}(y_i - y_{i-1})(y - y_i)\big|.$$

From $j = N(S(x)) < \infty$ it also follows that $S(x)$ has a zero at the $(m+1)$'th place and $S(y)$ has a one there. Thus $|y - y_i| \le d_{m+1}$ for $i \ge j+1$.

Hence

$$|\sum_{i=j+1}^{\infty} 2^{-i}(y_i - y_{i-1})(y - y_i)\big| \le \sum_{i=j+1}^{\infty} \frac{d_m}{2^i} d_{m+1} \le d_m \cdot d_{m+1}.$$

Therefore

$$\frac{|f(y) - f(x) - f_{(1)}(x)(y-x) - \frac{f_{(2)}(x)}{2!}(y-x)^2|}{|y-x|^2} \le \frac{d_m \cdot d_{m+1}}{2^{-2} \cdot d_m^2}.$$

Using that $\frac{d_{m+1}}{d_m} \to 0$ as $m \to \infty$, and $y \to x$ implies $m \to \infty$ we obtain that in this case f is also twice Peano differentiable at x.

Suppose that there exists a $g \in \mathbf{D}^2(\mathbf{R})$ such that $g|_F = f$. From this it follows that for every $x \in \mathbf{R}$ we can find a $\delta(x) > 0$ such that $|y - x| < \delta(x)$ implies

$$|g(y) - g(x) - g_{(1)}(x)(y-x) - \frac{g_{(2)}(x)}{2!}(y-x)^2| \le (y-x)^2.$$

Since F is a non-empty perfect set, by the Baire Category Theorem we can find a portion $I \cap F$ of F such that denoting its length by h the set $\{x : \delta(x) > 2 \cdot h\}$ is dense in $I \cap F$. Choose an $x \in I \cap F$ such that $S(x)$ contains finitely many, say j, ones and the last one in the sequence $S(x)$ is the m_0'th term.

Since g is, by assumption, twice Peano differentiable at x, we can choose a $0 < \eta < h$ such that $|y - x| < \eta$ implies that

$$|g(y) - g(x) - g_{(1)}(x)(y-x) - \frac{g_{(2)}(x)}{2!}(y-x)^2| < 2^{-3(j+4)}(y-x)^2.$$

Choose a $y \in I \cap F$ such that $\delta(y) > 2h$, $0 < y - x < \eta/4$, and $S(x)|m_0 = S(y)|m_0$. Denote by $m_1(\geq m_0)$ the number chosen so that $S(x)|m_1 = S(y)|m_1$ and $S(x)|(m_1+1) \neq S(y)|(m_1+1)$. Since $y - x > 0$, it is obvious that the (m_1+1)'th terms of $S(y)$ and $S(x)$ are one and zero respectively. We also have

$d_{m_1}/2 < y - x < d_{m_1} < 2(y-x) < \eta/2$,

$x_j = y_j = x$, and $x_i = y_i$ for $i = 1, ..., j-1$,

$x_i = x < y_i$ for $i \geq j+1$,

$y_{j+1} - x \geq d_{m_1}/2$.

Using also that $z_k > z_{k-1}$ for every $z \in F$, and $k = 1, 2, ...$ we obtain

$$|g_{(1)}(y) - g_{(1)}(x)| = \Big|\sum_{i=1}^{\infty} 2^{-i}(y_i - y_{i-1}) - \sum_{i=1}^{j-1} 2^{-i}(x_i - x_{i-1})\Big| =$$

$$2^{-j-1}(y_{j+1} - y_j) + \sum_{i=j+2}^{\infty} 2^{-i}(y_i - y_{i-1} - 0) \geq 2^{-j-1}(y_{j+1} - y_j) =$$

$$2^{-j-1}(y_{j+1} - x) \geq 2^{-j-2} \cdot d_{m_1}.$$

Since $g_{(2)}(z) = f_{(2)}(z) = 0$ for $z \in F$ and $|y - x| < \eta$ implies that

$$\Big|g(y) - g(x) - g_{(1)}(x)(y-x) - \frac{g_{(2)}(x)}{2!}(y-x)^2\Big| =$$

$$\Big|g(y) - g(x) - g_{(1)}(x)(y-x)\Big| < 2^{-3(j+4)}(y-x)^2$$

we infer that for $c \in \mathbf{R}$ we have

$$\Big|g(x) + g_{(1)}(x)(y + c \cdot d_{m_1} - x) + \frac{g_{(2)}(x)}{2!}(y + c \cdot d_{m_1} - x)^2 -$$

$$g(y) - g_{(1)}(y)(c \cdot d_{m_1}) - \frac{g_{(2)}(y)}{2!}(c \cdot d_{m_1})^2\Big| \geq$$

$$\Big|\,|g_{(1)}(x) - g_{(1)}(y)|c \cdot d_{m_1}\Big| - \Big|g(y) - g(x) - g_{(1)}(x)(y-x)\Big| > |c| \cdot 2^{-j-2} \cdot d_{m_1}^2 - 2^{-3(j+4)} d_{m_1}^2.$$

From the choice of η, $|y-x| < \eta/4$, and $d_{m_1} < \eta/2$. It follows that for $|c| < 1$ we have $|y + c \cdot d_{m_1} - x| < \eta$ and hence

$$\left| g(y + c \cdot d_{m_1}) - g(x) - g_{(1)}(x)(y + c \cdot d_{m_1} - x) \right| < 2^{-3(j+4)}(2d_{m_1})^2.$$

Furthermore from $\delta(y) > 2h > \eta > d_{m_1}$ it follows that

$$\left| g(y + c \cdot d_{m_1}) - g(y) - g_{(1)}(y) \cdot c \cdot d_{m_1} \right| < c^2 \cdot d_{m_1}^2.$$

The last three displayed inequalities imply that

$$(4 \cdot 2^{-3(j+4)} + c^2) d_{m_1}^2 > (|c| \cdot 2^{-j-2} - 2^{-3(j+4)}) d_{m_1}^2.$$

Choosing $c = 2^{-(j+4)}$ we obtain

$$2^{-3(j+4)+2} + 2^{-2(j+4)} > 2^{-(2j+6)} - 2^{-3j-12} =$$

$$2^{-(j+4)}(2^{-(j+2)} - 2^{-2j-8}) > 2^{-(j+4)} \cdot 2^{-(j+3)} = 2^{-2j-7}$$

and hence dividing by 2^{-2j-7} we obtain $2^{-j-3} + 2^{-1} > 1$ a contradiction proving our theorem.

References

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