UNIONS OF PRODUCTS OF INDEPENDENT SETS

BY

Zoltán Buczolich*

Mailing address: Eötvös Loránd University, Department of Analysis, Budapest, Múzeum krt. 6-8, Hungary, H-1088.

Email: *buczo@ludens.elte.hu*

Abstract

We show that there exists an open set $H \subseteq [0,1] \times [0,1]$ with $\lambda_2(H) = 1$ such that for any $\epsilon > 0$ there exists a set E satisfying $\lambda_1(E) > \frac{1}{2} - \epsilon$ and H contains the product set $E \times E$ but there is no set S with $\lambda_1(S) = 1/2$ and $S \times S \subseteq H$. Especially this property is verified for sets of the form $H = \cup_{i=1}^{\infty} E_i \times E_i$ where the sets E_i are independent and $\lambda_1(E_i) < 1/2$. The results of this paper answer questions of M. Laczkovich and are related to a paper of D. H. Fremlin.

Introduction

M. Laczkovich asked whether it is possible to find a Lebesgue measurable set $H \subseteq [0,1] \times [0,1]$ which is of full two dimensional measure, that is $\lambda_2(H) = 1$, and which for any $\epsilon > 0$ contains product sets of the form $E \times E$ with $\lambda_1(E) > \frac{1}{2} - \epsilon$, but there is no product set $S \times S \subseteq H$ with $\lambda_1(S) = \frac{1}{2}$. M. Laczkovich also suggested a construction for such a set H. Take stochastically independent sets $E_i \subseteq [0,1]$, $i = 1, 2, ...$ with $\lambda_1(E_i) < 1/2$, and $\lambda_1(E_i) \to 1/2$ and set $H = \cup_{i=1}^{\infty} E_i \times E_i$. In this paper we show that this construction works.

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It follows from §4 of [1] that if (E_i) is a stochastically independent sequence of Lebesgue measurable sets with $\sup_{i\in\mathbf{N}} \lambda_1(E_i) \le 1/2$, and S is a Lebesgue measurable set such that $S\times S \subseteq \cup_{i=1}^{\infty} E_i \times E_i$ then $\lambda_1(S) \le 1/2$. Our result shows that from $\lambda_1(E_i) < 1/2$ it follows that $\lambda_1(S) < 1/2$.

As it is shown in §3 of [1] the number $1/2$ has a special role. For any $\epsilon \in (\frac{1}{2}, 1)$ and $\eta \in (0,1)$ one can choose a sufficiently large n, such that for any finite sequence of independent sets $E_1, ..., E_n$ with $\lambda_1(E_i) = \epsilon$ $(i = 1, ..., n)$ there exists a measurable set S with $S \times S \subseteq \cup_{i=1}^{n} E_i \times E_i$ and $\lambda_1(S) = \eta$.

The referee of the first version of this paper pointed out that our argument can be generalized to perfect probability measure spaces. We state Theorem 1, which is the main result of this paper, by using these spaces. The result is the following. If F_i are given independent subsets of measure less than 1/2 in a perfect probability space then any measurable set S such that $S \times S \subseteq \cup_{i=1}^{\infty} F_i \times F_i$ is also of measure less than 1/2. The results in [1] imply that for more general spaces Theorem 1 is not valid. Theorem 2, which is an easy Corollary of Theorem 1 is the answer to Laczkovich's question.

Preliminaries

By $\mathbf{N}$ we denote the natural numbers $\{1, 2, ...\}$. The one- and the two-dimensional Lebesgue measure is denoted by λ_1 and λ_2 respectively. The closure, and the interior of a set $A \subseteq [0,1]$ is denoted by $\mathrm{cl}(A)$, and $\mathrm{int}(A)$ respectively.

Let $\mathbf{Q}_2$ denote the dyadic rationals, that is $\mathbf{Q}_2 = \{p/2^i : i \in \mathbf{N},\ p \in \{0, \pm 1, \pm 2, ...\}\}$.

The measure space X with measure μ and with the σ-algebra of measurable sets Σ is denoted by (X, Σ, μ) or simply by (X, μ). We recall from [2] the definition of perfect measures.

Definition. The measure space (X, Σ, μ) is perfect if for any Σ-measurable function f and for any set $E \subseteq \mathbf{R}$ satisfying $f^{-1}(E) \in \Sigma$ there exists a Borel set B such that $B \subseteq E$ and $\mu(f^{-1}(E)) = \mu(f^{-1}(B))$.

It is obvious that $([0,1], \lambda_1)$ is a perfect space. It follows from Theorem 6 of [2] that if (Z, ν) is a perfect space then $Z \times [0,1]^{\mathbf{N}}$ with the corresponding product measure is also perfect.

Main Results

Lemma 1. *Assume that $1 > e_i > 0$, $i = 1, 2, \dots$*

$$\sum_{i=1}^{\infty} e_i < +\infty, \tag{1}$$

and

$$0 < (1 - e_j) \prod_{i=j+1}^{\infty} (1 + e_i) < (1 + e_j) \prod_{i=j+1}^{\infty} (1 - e_i) \tag{2}$$

holds for $j = 1, 2, \dots$. For $x \in [0, 1)$ denote by $x_1, x_2, \dots$ its (terminating) binary expansion, set

$$\omega(x) = \prod_{i=1}^{\infty} (1 + (-1)^{x_i} e_i),$$

and $\omega(1) = \prod_{i=1}^{\infty}(1 - e_i)$. Then ω is strictly positive and strictly decreasing on $[0, 1]$ and $\int_{[0,1]} \omega = 1$. Writing $\mu(A) = \int_A \omega(x) d\lambda_1(x)$, the sets $E_i = \operatorname{int}\{x : x_i = 1\}$ are μ independent and $\mu(E_i) = (1 - e_i)/2$ for each i.

Proof of Lemma 1. It is easy to see that (1) implies that ω is strictly positive on $[0, 1]$. Assume that $x, y \in [0, 1)$, $x < y$ then there exists a $j \in \mathbf{N}$ such that $x_i = y_i$ for $i = 1, \dots, j - 1$ and $x_j < y_j$. Thus $x_j = 0$, $y_j = 1$, and

$$\frac{\omega(y)}{\omega(x)} = \frac{\prod_{i=1}^{\infty}(1 + (-1)^{y_i} e_i)}{\prod_{i=1}^{\infty}(1 + (-1)^{x_i} e_i)} =$$

$$\frac{(1 - e_j) \prod_{i=j+1}^{\infty}(1 + (-1)^{y_i} e_i)}{(1 + e_j) \prod_{i=j+1}^{\infty}(1 + (-1)^{x_i} e_i)} < \frac{(1 - e_j) \prod_{i=j+1}^{\infty}(1 + e_i)}{(1 + e_j) \prod_{i=j+1}^{\infty}(1 - e_i)} < 1$$

where in the last estimation we used assumption (2) of Lemma 1. This implies $\omega(x) > \omega(y)$. It is also clear that $\omega(1) < \omega(x)$ holds for $x \in [0, 1)$. Let $\omega_n(x) = \prod_{i=1}^{n}(1 + (-1)^{x_i} e_i)$, and $\mu_n(A) = \int_A \omega_n$. Assumption (1) and Lebesgue's theorem on dominated convergence implies that $\lim_{n\to\infty} \mu_n(A) = \mu(A)$. It is easy to compute that $\mu_n([0, 1]) = 1$, and $\mu_n(E_i) = (1 - e_i)/2$ holds for each i and $n \geq i$. This implies $\int_{[0,1]} \omega = 1$, and $\mu(E_i) = (1 - e_i)/2$ for each i.

Lemma 2. *Using the notation introduced in Lemma 1, suppose that (Z, Σ, ν) is a perfect probability space and F_i are independent subsets of Z such that $\nu(F_i) \leq (1 - e_i)/2$*

for each i. Assume that S is a measurable subset of Z such that $S \times S \subseteq \cup_{i=1}^{\infty} F_i \times F_i$. Then there is a Borel- measurable set $T \subseteq [0,1]$ such that $T \times T \subseteq \cup_{i=1}^{\infty} E_i \times E_i$ and $\mu(T) = \nu(S)$.

Proof of Lemma 2. Replacing Z by $Z \times [0,1]^{\mathbf{N}}$, S by $S \times [0,1]^{\mathbf{N}}$, F_i by $(F_i \times [0,1]^{\mathbf{N}}) \cup (Z \times \{y : y_i \leq \alpha_i\})$ for suitable α_i, it is enough to deal with the case when $\nu(F_i) = (1-e_i)/2$. Denote by χ_i the function which equals 1 on F_i and 0 elsewhere on Z. Put $f = \sum_{i=1}^{\infty} \chi_i/2^i$. Then $\nu(f^{-1}(E_i)) = (1-e_i)/2 = \mu(E_i)$ and hence $\nu(f^{-1}(H)) = \mu(H)$ for every Borel $H \subseteq [0,1]$. Since (Z,ν) is perfect there is a Borel set $T \subseteq f(S) \setminus \mathbf{Q}_2$ such that $\mu(T) = \nu(S)$. If $(x,y) \in T \times T$ then there exists $(z,w) \in S \times S$ such that $f(z) = x$, and $f(w) = y$. Since $(z,w) \in S \times S \subseteq \cup_{i=1}^{\infty} F_i \times F_i$ there exists j such that $(z,w) \in F_j \times F_j$, thus the definition of f implies $x_j = y_j = 1$. Using $x, y \notin \mathbf{Q}_2$ we obtain $(x,y) \in E_j \times E_j \subseteq \cup_{i=1}^{\infty} E_i \times E_i$. This completes the proof of Lemma 2.

Lemma 3. *Suppose that $T \subseteq [0,1]$ is a measurable set such that $T \times T \subseteq \cup_{i=1}^{\infty} E_i \times E_i$ where the sets E_i are defined in Lemma 1. Then $f(x) = \lambda_1(T \cap [0,x]) \leq x/2$ for every $x \in [0,1]$.*

Proof of Lemma 3. First we verify that if $z \in [0, 2^{-(n-1)}] \setminus \mathbf{Q}_2$ and $z^* = (2^{-(n-1)} - z)$ then $z \in T$ implies $z^* \notin T$. Denote the binary expansions of z and z^* by $z_1, z_2, ...$ and $z_1^*, z_2^*, ...$ respectively. From $z \in [0, 2^{-(n-1)}] \setminus \mathbf{Q}_2$ it follows that $z_i = z_i^* = 0$ for $i = 1, ..., n-1$. The definition of z^* implies that $z_i^* = 1 - z_i$ holds for $i = n, n+1, ...$ If $z, z^* \in T$ then $(z, z^*) \in \cup_{i=1}^{\infty} E_i \times E_i$ and hence there would exist a $j \in \mathbf{N}$ such that $(z, z^*) \in E_j \times E_j$, that is, $z_j = z_j^* = 1$ which is impossible.

Obviously it is enough to deal with the case when $\lambda_1(T) > 0$. Since f is continuous it is sufficient to show that if $x \in [0,1] \setminus \mathbf{Q}_2$ then $x/2 > f(x)$.

For any $x \in [0,1] \setminus \mathbf{Q}_2$ choose $n \in \mathbf{N}$ such that $x \in (2^{-n}, 2^{-(n-1)})$ and put $t(x) = 2^{-(n-1)} - x$. It is clear that $t(x) \in [0,1] \setminus \mathbf{Q}_2$. Put $a_1 = x$. If a_j is defined set $a_{j+1} = t(a_j)$. It is easy to see that the sequence (a_j) is strictly monotone decreasing and converges to 0. Assume that for a $j \in \mathbf{N}$ we have

$$\frac{a_j}{2} \leq f(a_j). \tag{3}$$

Choose $n_j \in \mathbf{N}$ such that $a_j \in (2^{-n_j}, 2^{-(n_j-1)})$. Since $a_{j+1} = 2^{-(n_j-1)} - a_j$ we have $(a_j + a_{j+1})/2 = 2^{-n_j}$ and by the first paragraph of this proof from $z, z^* \in [0, 2^{-(n_j-1)}] \setminus \mathbf{Q}_2$,

$(z+z^*)/2 = 2^{-n_j}$ it follows that at most one of the points z, z^* can belong to T. Therefore $(a_j - a_{j+1})/2 \geq \lambda_1([a_{j+1}, a_j] \cap T)$. Multiplying by -1 the last inequality and adding to (3) we obtain $a_{j+1}/2 \leq f(a_j) - \lambda_1([a_{j+1}, a_j] \cap T) = f(a_j) - (f(a_j) - f(a_{j+1})) = f(a_{j+1})$.

Proceeding towards a contradiction assume that $a_1/2 \leq f(a_1)$. Then this implies $a_j/2 \leq f(a_j)$ for $j \in \mathbf{N}$. We verify that this contradicts $\lambda_1(T) > 0$ and Lebesgue's density theorem. If $z \in ([0, a_j] \setminus \mathbf{Q}_2) \cap T$ with binary expansion $z_1, z_2, \ldots$ then from $0 < a_j < 2^{-(n_j-1)}$ it follows that $z_1 = z_2 = \ldots = z_{n_j-1} = 0$. If $z' \in [0,1]$ with binary expansion $z'_1, z'_2, \ldots$ and $z'_i = 1 - z_i$ for $i = n_j, n_j+1, \ldots$, then $z_i \cdot z'_i = 0$ holds for $i = 1, 2, \ldots$ and hence $(z, z') \notin \cup_{i=1}^{\infty} E_i \times E_i$. Therefore $z' \notin T$. Assume that $I = [p/2^{n_j-1}, (p+1)/2^{n_j-1}]$ with a $p \in \{1, 2, \ldots, 2^{n_j-1} - 1\}$. Next we show that from

$$\frac{1}{4} \cdot 2^{-(n_j-1)} \leq \frac{a_j}{2} \leq f(a_j) = \lambda_1([0, a_j] \cap T) \tag{4}$$

it follows that

$$\lambda_1(I \setminus T) \geq \frac{1}{4} \cdot 2^{-(n_j-1)} = \frac{1}{4}\lambda_1(I). \tag{5}$$

Indeed if $z' \in I \setminus \mathbf{Q}_2$ with binary expansion $z'_1, z'_2, \ldots$ and there exists $z \in ([0, a_j] \setminus \mathbf{Q}_2) \cap T$ with binary expansion $z_1, z_2, \ldots$ satisfying $z'_i = 1 - z_i$ for $i = n_j, n_j + 1, \ldots$ then $z' \notin T$. Using (4) we obtain (5). Thus $\lambda_1(I \cap T)/\lambda_1(I) \leq 3/4$ holds for all intervals of the form $I = [p/2^{n_j-1}, (p+1)/2^{n_j-1}]$ with $j = 1, 2, \ldots$ and $p = 1, 2, \ldots, 2^{n_j-1} - 1$. From $a_j \to 0$ it follows that $n_j \to \infty$ and hence T cannot have Lebesgue density points contradicting $\lambda_1(T) > 0$. This shows that $f(x) < x$ holds for all $x \in [0,1] \setminus \mathbf{Q}_2$ and completes the proof of Lemma 3.

Lemma 4. *Suppose that $T \subseteq [0,1]$ is a measurable set such that $\lambda_1(T \cap [0,x]) \leq x/2$ for every $x \in [0,1]$, and that ω is a bounded non-negative strictly decreasing function on $[0,1]$. Then $\int_T \omega < \frac{1}{2} \int \omega$.*

Proof of Lemma 4. Set $F(x) = \lambda_1(T \cap [0,x])$, $G(x) = x - F(x)$ for $x \in [0,1]$. Obviously $F(0) = G(0) = 0$. Then both F and G are continuous monotonic functions and $0 \leq F(x) \leq G(x) \leq x$ for every x. By Lebesgue's density theorem $F'(x) \in \{0,1\}$ for almost all x. Thus there must be some x such that $F(x) < x/2 < G(x)$. Since ω is strictly decreasing, that is $d\omega(x)/dx < 0$ a.e., we have $\int_0^1 G\ d\omega < \int_0^1 F\ d\omega$. Consequently

$$\int_T \omega = \int_0^1 \omega\ dF = \omega(1)F(1) - \int_0^1 F\ d\omega < \omega(1)G(1) - \int_0^1 G\ d\omega = \int_0^1 \omega\ dG = \int_{[0,1]\setminus T} \omega.$$

Theorem 1. *Suppose that (Z, Σ, ν) is a perfect probability space and F_i are independent subsets of Z such that $\nu(F_i) < 1/2$. Then from $S \in \Sigma$, $S \times S \subseteq \cup_{i=1}^{\infty} F_i \times F_i$ it follows that $\nu(S) < 1/2$.*

Proof. It is easy to see that one can choose a sequence (e_i) which satisfies (1), (2), and $\nu(F_i) \leq e_i < 1/2$. By using Lemma 1 we can choose the measure μ and the sets E_i. If $S \in \Sigma$, $S \times S \subseteq \cup_{i=1}^{\infty} F_i \times F_i$ then using Lemma 2 choose a Borel measurable set $T \in [0,1]$ such that $T \times T \subseteq \cup_{i=1}^{\infty} E_i \times E_i$ and $\mu(T) = \nu(S)$. By Lemma 3 $f(x) = \lambda_1(T \cap [0,x]) \leq x/2$ for every $x \in [0,1]$, thus Lemma 4 is applicable. Therefore $\nu(S) = \mu(T) = \int_T \omega < \frac{1}{2} \int \omega = \frac{1}{2}$. Q.e.d.

Theorem 2. *There exists an open set $H \subseteq [0,1] \times [0,1]$ of full λ_2-measure in $[0,1] \times [0,1]$ such that for any $\epsilon > 0$ there exists a measurable set $E \subseteq [0,1]$ satisfying $\lambda_1(E) > \frac{1}{2} - \epsilon$, and $E \times E \subseteq H$ but there is no Lebesgue measurable $S \subseteq [0,1]$ with $\lambda_1(S) = 1/2$ and $S \times S \subseteq H$.*

Proof. Since $([0,1], \lambda_1)$ is a perfect probability space Theorem 1 is applicable. It is an easy exercise to show that one can choose open independent sets $E_i \subseteq [0,1]$ such that $\lambda_1(E_i) < 1/2$ and $\lambda_1(E_i) \to 1/2$ as $i \to \infty$. Set $H = \cup_{i=1}^{\infty} E_i \times E_i$. Then $H \subseteq [0,1] \times [0,1]$ is open and of full λ_2-measure and clearly satisfies our requirements.

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