When tensor products of AC_* charges and Radon measures are AC_* charges

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Abstract

In this paper we solve the tensor problem of AC_* charges and Radon measures. We show that if F is a charge in $\mathbb{R}^m$ which is AC_* in a locally BV set E and μ is an absolutely continuous Radon measure on $\mathbb{R}^n$ with locally bounded Radon-Nikodym derivative then the charge $F \otimes \mu$ is AC_* in $E \times \mathbb{R}^n$.

1 Introduction

In [3] the tensor problem was stated for the tensor product of AC_* charges and the Lebesgue measure. Later W.F. Pfeffer suggested to look at possible generalizations of this problem.

The generalized version of the tensor problem is the following (for the details of the definitions see the Preliminaries section of this paper, or [3]):

Let μ be an absolutely continuous Radon measure in $\mathbb{R}^n$, and let F be a charge in $\mathbb{R}^m$ where m and n are positive integers. (In this paper absolute continuity is always considered with respect to the Lebesgue measure.) Given a bounded BV set $B \subseteq \mathbb{R}^{m+n}$, let

$$B^y = \{x \in \mathbb{R}^m : (x, y) \in B\}$$

and

$$(F \otimes \mu)(B) = \int_{\mathbb{R}^n} F(B^y) d\mu(y).$$

It is not difficult to see that $F \otimes \mu$ is a charge and the question is whether $F \otimes \mu$ is AC_* in $E \times \mathbb{R}^n$ whenever F is AC_* in a locally BV set $E \subseteq \mathbb{R}^m$.

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It is easy to see that if μ has an atomic part then the tensor conjecture is false. Even for absolutely continuous measures one can obtain counterexamples, see [2]. In that construction we gave a μ such that its density function (its Radon-Nikodym derivative) φ is unbounded. The conjecture on tensor products is true if we assume that φ is locally essentially bounded in $\mathbb{R}^n$, that is, $\varphi \in L^\infty_{\text{loc}}(\mathbb{R}^n)$, this will be the subject of this paper.

This question is of interest because for nonabsolute integrals usually there is no Fubini's theorem and hence results concerning products are useful, for details of the background in R-integration see [3]. A function f defined on $\mathbb{R}^m$ is locally R-integrable whenever there is an AC_* charge F such that $F'(x) = f(x)$ for almost all $x \in \mathbb{R}^m$. Assume f is locally R-integrable on $\mathbb{R}^m$ and φ is the locally bounded density function of an absolutely continuous Radon measure μ on $\mathbb{R}^n$. Our result implies that $f \otimes \varphi$ is locally R-integrable on $\mathbb{R}^{m+n}$ and its indefinite R-integral is the charge $F \otimes \mu$.

This paper is organized the following way. After the Preliminaries Section in Section 3 we show that tensor products of almost derivable charges and AC Radon measures are almost derivable. In Section 4 we state the main lemmas and Theorem 7, which is the main result of this paper. Based on Lemma 5 we prove first Lemma 6, then Theorem 7. The main difficulty, the proof of Lemma 5 is postponed to Sections 5 and 6. It is interesting that the $m > 1$ dimensional case of Lemma 5 needs a different argument, which is given in Section 5. The $m = 1$ dimensional case is treated in Section 6. The reason is that the one-dimensional surface measure is the zero-dimensional Hausdorff-measure, which is atomic, this means that if we cut a small one-dimensional figure into two pieces the cutting point increases its surface by two, while for $m > 1$ one can cut off some pieces of small surface. However, one-dimensional BV sets are of a very simple structure and this can help in that case. Since the one-dimensional case is more technical this is postponed to the later Section 6.

2 Preliminaries

We denote by $\text{int}(A)$, $\text{cl}(A)$, $\partial(A)$ $d(A)$, and $|A|_m$ the interior, the closure, the boundary, the diameter, and the Lebesgue measure of the set $A \subseteq \mathbb{R}^m$, respectively. However, in certain situations we will also use the alternate notation $\mathcal{L}^m$ for the m-dimensional Lebesgue measure. The m-dimensional Hausdorff measure of A is denoted by $\mathcal{H}^m(A)$. Recall that there is a constant $c_{\mathcal{H},m}$ for which $c_{\mathcal{H},m}\mathcal{H}^m(A) = \mathcal{L}^m(A)$.

In $\mathbb{R}^m$ the open ball centered at x and of radius r will be denoted by $B_m(x,r)$. We denote by $c_{b,m}$ the constant, for which $|B_m(x,r)|_m = c_{b,m}r^m$. The closed cube centered at $x \in \mathbb{R}^m$ and of sidelength $2d$ is denoted by $Q_m[x,d]$, while for the open cube we use the notation $Q_m(x,d)$. The sidelength of a cube Q is denoted by $\ell(Q)$, that is, $\ell(Q_m[x,d]) = 2d$.

Closed intervals in $\mathbb{R}^m$ are m-fold Cartesian products of closed one-dimensional intervals. Figures in $\mathbb{R}^m$ are finite unions of closed non-degenerate m-dimensional intervals. We will also allow a figure to be the empty set.

Clearly, differences and intersections of figures are not always figures. If A and B are figures we set $A \odot B \stackrel{\text{def}}{=} \text{cl}(\text{int}(A) \cap \text{int}(B))$ and $A \ominus B = \text{cl}(A \setminus B)$.

We also recall the $(m+1)$-dimensional version of Lemma 3.3 of [1].

Lemma 1. *Assume $B \subseteq \mathbb{R}^{m+1}$ is a figure, $Q \subseteq \mathbb{R}^{m+1}$ is a cube of sidelength d, and $|B \cap Q|_{m+1} \leq |Q|_{m+1}/2 = d^{m+1}/2$. Then there exists a constant $c'_{F,m+1}$, depending only on the dimension, such that*

$$\mathcal{H}^m(B \cap \partial Q) \leq c'_{F,m+1} \mathcal{H}^m(\partial B \cap \text{int}(Q)). \tag{1}$$

The Lebesgue density points (see for example [5] Corollary 6.2.6) of a measurable set $A \subseteq \mathbb{R}^m$ form its essential interior, $\text{int}^* A$. The Lebesgue dispersion points (zero-density points) of A form its essential exterior $\text{ext}^* A$. The essential boundary of A equals $\partial^* A \stackrel{\text{def}}{=} \mathbb{R}^m \setminus (\text{int}^* A \cup \text{ext}^* A)$.

The measurable set $A \subseteq \mathbb{R}^m$ is a BV set if its perimeter $||A||_m \stackrel{\text{def}}{=} \mathcal{H}^{m-1}(\partial^* A)$ is finite. We will denote by BV_m the class of bounded m-dimensional BV-sets.

Recall first the isoperimetric inequality see for example [8] 3.2.43, or [10] 2.7.

Theorem 2. *There exists a constant $c_{iso,m+1}$ such that for all BV sets $H \subseteq \mathbb{R}^{m+1}$ we have*

$$|H|_{m+1}^{\frac{m}{m+1}} \leq c_{iso,m+1} ||H||_{m+1}.$$

We also need the relative isoperimetric inequality for cubes in $\mathbb{R}^{m+1}$. It is usually stated for balls ([10] Theorem 5.4.3, p.230), but it is also true with a similar proof for cubes as well.

Theorem 3. *There exists a constant $c_{riso,m+1}$ such that for all BV sets $H \subseteq \mathbb{R}^{m+1}$ and all $(m+1)$-dimensional cubes Q we have*

$$\min\{|Q \cap H|_{m+1}, |Q \setminus H|_{m+1}\}^{\frac{m}{m+1}} \leq c_{riso,m+1} \mathcal{H}^m(\text{int}(Q) \cap \partial H). \tag{2}$$

A function F defined on m-dimensional BV-sets is an m-dimensional charge if it is

1. additive, that is, $F(A \cup B) = F(A) + F(B)$ holds for any two nonoverlapping BV sets A and B (recall that A and B are nonoverlapping if $\text{int}^* A \cap \text{int}^* B = \emptyset$);
2. continuous, that is, for every $\epsilon > 0$ there exists $\rho > 0$ such that $|F(A)| < \epsilon$ if $|A| < \rho$, $||A|| < 1/\epsilon$ and $A \subseteq B(0, 1/\epsilon)$.

Assume $E \subseteq \mathbb{R}^m$, $\delta : E \to [0, \infty)$ is a gage on E if $\{x : \delta(x) = 0\}$ is thin, that is, it is of σ-finite $m-1$-dimensional Hausdorff measure.

A system $\mathcal{P} = \{(A_i, x_i)\}_{i=1}^p$ is a partition if the A_i's are nonoverlapping BV sets and $x_i \in A_i$ for $i = 1, ..., p$. The partition $\mathcal{P}$ is anchored in E if $x_i \in E$ for all $i = 1, ..., p$. If a gage δ is given on E then $\mathcal{P}$ is δ-fine if $A_i \subseteq B_m(x_i, \delta(x_i))$ holds for all $i = 1, ..., p$. Given a partition $\mathcal{P} = \{(A_i, x_i)\}_{i=1}^p$ and a function defined on BV sets (not necessarily a charge) F it will be convenient to use the notation $F(\mathcal{P}) = \sum_{i=1}^{p} |F(A_i)|$. A partition $\mathcal{P}$ is an $\mathcal{F}$-partition if all A_i's are figures.

The regularity, $r(A)$, of $A \in BV_m$ is the number $|A|_m/d(A)||A||_m$ if $d(A)||A||_m \neq 0$, otherwise $r(A) \stackrel{\text{def}}{=} 0$. If $\eta > 0$ and $r(A) > \eta$ we say that A is η-regular. A partition $\mathcal{P}$ is η-regular if $r(A_i) > \eta$ for all $i = 1, ..., p$.

A charge F is *almost derivable* at x if

$$\overline{D}_\eta |F|(x) = \inf_{\delta > 0} \sup_{A} \frac{|F(A)|}{|A|_m} < \infty$$

holds for all $\eta > 0$ where sup is taken for all BV sets A such that $d(A \cup \{x\}) < \delta$ and $r(A \cup \{x\}) > \eta$. By Lemma 3.1 of [4] for charges we obtain the same result if we consider only figures A, instead of BV sets.

Assume F is a function defined on BV sets of $\mathbb{R}^m$, $\eta > 0$, $E \subseteq \mathbb{R}^m$ and $\delta : E \to [0, \infty)$ is a gage. Set

$$V_{\eta,\delta} F(E) = \sup_{\mathcal{P}} F(\mathcal{P}) = \sup_{\mathcal{P}} \sum_{i=1}^{p} |F(A_i)|,$$

where the supremum is taken for all η-regular, δ-fine partitions $\mathcal{P} = \{(A_i, x_i)\}_{i=1}^{p}$ anchored in E. The F-variational outer measure of E is defined as

$$V_* F(E) = \sup_{0 < \eta} \inf_{\delta} V_{\eta,\delta} F(E),$$

where the infimum is taken for all δ gage functions defined on E. It is not difficult to show that $V_* F$ is a metric outer measure. Recently variational measures outside non-absolute integration are also gaining more and more interest see [6], pp. 10-13.

Using $\mathcal{F}$-partitions $\mathcal{P} = \{(A_i, x_i)\}_{i=1}^{p}$ we obtain the variation with respect to figures, denoted by $V_*^{\mathcal{F}}(E)$.

If F is a charge then by [9] the two variations $V_*(E)$ and $V_*^{\mathcal{F}}(E)$ take the same value.

We say that F is an AC_* (m-dimensional) charge if $V_* F$ is absolutely continuous with respect to the Lebesgue measure (in $\mathbb{R}^m$). It is not difficult to see that if F is almost derivable at all $x \in E$ then it is AC_* on E.

We will also use the constant $c_{f,m} > 1$ for which $||Q||_m = c_{f,m} \ell(Q)^{m-1}$ holds for all cubes $Q \subseteq \mathbb{R}^m$.

Working in $\mathbb{R}^{m+1}$ we will denote the $(m+1)$'th coordinate by t, that is, we work with $(x, t) \in \mathbb{R}^{m+1}$, $x \in \mathbb{R}^m$, $t \in \mathbb{R}$.

If $A \subseteq \mathbb{R}^{m+1}$ then its projection on the t-axis will be denoted by

$$T(A) = \{t : \exists x \in \mathbb{R}^m, \ (x, t) \in A\}.$$

We also put

$$\pi_m(A) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^m : \exists t \in \mathbb{R}, \ (x, t) \in A\}.$$

The projection into $\mathbb{R}^m$ of the horizontal sections of A at height t will be denoted by A_t, that is, $A_t = \{x \in \mathbb{R}^m : (x,t) \in A\}$.

For a figure $A \subseteq \mathbb{R}^{m+1}$ we use the notation

$$||A||_v = \int_{\mathbb{R}} ||A_t||_m dt,$$

where $||A||_v$ is the "vertical" surface of A. Clearly, $c_{\mathcal{H},m-1}||A||_v \le c_{\mathcal{H},m}||A||_{m+1}$. We also put $\partial_v(A) = \cup_{t\in\mathbb{R}}\{(\partial(A_t),t)) \in \mathbb{R}^{m+1}\}$, the "vertical boundary" of A. It is clear that $c_{\mathcal{H},m-1}||A||_v = c_{\mathcal{H},m}\mathcal{H}^m(\partial_v A)$.

3 Tensor products of almost derivable charges and AC Radon measures are almost derivable

Theorem 4. *Assume $E \subseteq \mathbb{R}^m$ and the charge $F : BV_m \to \mathbb{R}$ is almost derivable at every point of E. Assume, furthermore that μ is an AC Borel measure on $\mathbb{R}^n$ and $\varphi \stackrel{\text{def}}{=} d\mu/d\mathcal{L}^n \in L^\infty_{loc}(\mathbb{R}^n)$. Then $F \otimes \mu$ is almost derivable at every point of $E \times \mathbb{R}^n$.*

Proof. We assume that $\varphi < M \in \mathbb{R}$ almost everywhere.

We choose later an $\eta_m > 0$ depending only on the dimension. Since F is almost derivable on E we have

$$\overline{D}_{\eta_m}|F|(x) = \inf_{\delta>0} \sup_A \frac{|F(A)|}{|A|_m} < \infty$$

for all $x \in E$ where sup is taken for all BV sets A such that $d(A \cup \{x\}) < \delta$ and $r(A \cup \{x\}) > \eta_m$.

Assume $\eta > 0$ is arbitrary. We will show that $\overline{D}_\eta|F \otimes \mu|(x,y) < \infty$ for all $(x,y) \in E \times \mathbb{R}^n$.

Now we suppose that (x_0, y_0) is a fixed element of $E \times \mathbb{R}^n$ and B is an $(m+n)$-dimensional figure near (x_0,y_0) such that $B \cup \{(x_0,y_0)\}$ is η-regular and of diameter less than one.

Using our local boundedness assumption we can choose $M \in \mathbb{R}$ for which $\varphi(y) < M$ holds for almost every $y \in B_n(y_0, 1)$.

Put $d_0 = d(B \cup (x_0,y_0))$. Assume $e_{1,m} = (1,0,...,0) \in \mathbb{R}^m$ is the first unit vector and set $A^* = Q_m[x_0 + 2d_0e_{1,m}, d_0]$.

Clearly, $B^y \cup A^* \subseteq B_m(x_0, (\sqrt{m}+2)d_0)$, and hence $d((B^y \cup A^*) \cup \{x_0\}) < 2(\sqrt{m}+2)d_0$.

Furthermore, $|A^*|_m = 2^m d_0^m$, $||A^*||_m = c_{f,m}2^{m-1}d_0^{m-1}$, B^y and A^* are non-overlapping.

Assume $K = \overline{D}_{\eta_m}|F|(x_0)+1$. Choose $\delta_0 \in (0,1)$ such that $|F(A)| = |F|(A) < K|A|_m$ holds for all η_m-regular BV sets such that $d(A \cup \{x_0\}) < \delta_0$.

Set $\kappa(y) \stackrel{\text{def}}{=} [||B^y||_m/d_0^{m-1}] + 1$. For a $t \in \mathbb{R}$ denote the hyperplane perpendicular to $e_{1,m}$ and containing the point $te_{1,m}$ by H_t, that is,

$$H_t = \{(t, x_2, ..., x_m) : x_2, ..., x_m \in \mathbb{R}\}.$$

We also denote by $H_{\leq t}$ and $H_{\geq t}$ the closed halfspaces

$$\{(t', x_2, ..., x_m) : t' \leq t,\ x_2, ..., x_m \in \mathbb{R}\} \text{ and}$$

$$\{(t', x_2, ..., x_m) : t' \geq t,\ x_2, ..., x_m \in \mathbb{R}\},$$

respectively.

Next we choose $t_1^y, ..., t_{\kappa(y)-1}^y \in \mathbb{R}$ and divide B^y by the hyperplanes $H_{t_1^y}, ..., H_{t_{\kappa(y)-1}^y}$ into non-overlapping BV sets $B_1^y, ..., B_{\kappa(y)}^y$, by this we mean that we take $B_1^y = B^y \cap H_{\leq t_1^y}$, $B_j^y = B^y \cap H_{\geq t_{j-1}^y} \cap H_{\leq t_j^y}$ for $j = 2, ..., \kappa(y) - 1$ and $B_{\kappa(y)}^y = B^y \cap H_{\geq t_{\kappa(y)-1}^y}$. The definition of $\kappa(y)$ easily implies that we can choose $t_1^y, ..., t_{\kappa(y)-1}^y$ such that

$$||B_j^y||_m \leq d_0^{m-1}(1 + 2c_{b,m-1}).$$

Now,

$$|B_j^y \cup A^*|_m \geq |A^*|_m = 2^m d_0^m,$$

$$||A^*||_m \leq ||B_j^y \cup A^*||_m \leq d_0^{m-1}(1 + 2c_{b,m-1} + c_{f,m}2^{m-1}),$$

and

$$d(A^* \cup \{x_0\}) \leq d((B_j^y \cup A^*) \cup \{x_0\}) < 2(\sqrt{m} + 2)d_0 < \delta_0,$$

where we can assume that the last inequality also holds. Thus, both $A^* \cup \{x_0\}$ and $B_j^y \cup A^* \cup \{x_0\}$ are

$$\eta_m \stackrel{\text{def}}{=} \frac{2^m}{(1 + 2c_{b,m-1} + c_{f,m}2^{m-1})2(\sqrt{m} + 2)}\text{-regular.}$$

Therefore,

$$|F|(B_j^y \cup A^*) < K|B_j^y \cup A^*|_m \leq K|B_m(x_0, (\sqrt{m}+2)d_0)|_m = Kc_{b,m}(\sqrt{m}+2)^m d_0^m,$$

and similarly

$$|F|(A^*) < K|A^*|_m \leq K|B_m(x_0, (\sqrt{m} + 2)d_0)|_m = Kc_{b,m}(\sqrt{m} + 2)^m d_0^m,$$

which implies that there exists a constant K' depending only on m and K for which

$$|F|(B_j^y) < K'd_0^m.$$

Adding these inequalities for all j we infer

$$|F|(B^y) \leq \kappa(y)K'd_0^m.$$

Now we can do our final estimate showing the almost differentiability of $F \otimes \mu$ at (x_0, y_0), indeed,

$$|F \otimes \mu|(B) \leq \int_{\mathbb{R}^n} |F|(B^y)d\mu(y) \leq M \int_{B_n(y_0,d_0)} K'\kappa(y)d_0^m dy \leq$$

$$MK'\int_{B_n(y_0,d_0)} d_0^m dy + MK'\int_{B_n(y_0,d_0)} \frac{||B^y||_m}{d_0^{m-1}} d_0^m dy \leq$$

$$MK'\left(c_{b,n}d_0^{m+n} + d_0\int_{\mathbb{R}^n} ||B^y||_m dy\right) \leq$$

$$MK'(c_{b,n}d_0^{m+n} + d_0(c_{\mathcal{H},m+n-1}/c_{\mathcal{H},m-1})||B||_{m+n}) \leq$$

$$MK'(c_{b,n}d_0^{m+n} + (c_{\mathcal{H},m+n-1}/c_{\mathcal{H},m-1})|B|_{m+n}/\eta) \leq K''|B|_{m+n}$$

where the constant K'' depends only on m, n, M, η and K. □

4 Statement of the Main Result and Lemmas

Lemma 5. *Assume F is a function defined on m-dimensional figures and $V_*^{\mathcal{F}}F(E) = 0$ holds for an $E \subseteq \mathbb{R}^m$ then $V_*^{\mathcal{F}}(|F| \otimes \mathcal{L})(E \times \mathbb{R}) = 0$.*

We postpone the proof of Lemma 5, which will be the most difficult proof in this paper and will need several steps.

Lemma 6. *Assume F is a function defined on m-dimensional figures $E \subseteq \mathbb{R}^m$ and $V_*^{\mathcal{F}}F(E) = 0$ then $V_*^{\mathcal{F}}(|F| \otimes \mathcal{L}^n)(E \times \mathbb{R}^n) = 0$.*

Based on Lemma 5 the proof of this lemma is not difficult.

Proof. For a fixed m, apply induction on n.

If $n = 1$ use Lemma 5. Assume Lemma 6 is true for $n-1$, that is,

$$V_*^{\mathcal{F}}(|F| \otimes \mathcal{L}^{n-1})(E \times \mathbb{R}^{n-1}) = 0.$$

Then apply Lemma 5 for $|F| \otimes \mathcal{L}^{n-1}$ using $E \times \mathbb{R}^{n-1}$ instead of E to obtain that $V_*^{\mathcal{F}}((|F| \otimes \mathcal{L}^{n-1}) \otimes \mathcal{L})((E \times \mathbb{R}^{n-1}) \times \mathbb{R}) = 0$. Since $(|F| \otimes \mathcal{L}^{n-1}) \otimes \mathcal{L} = |F| \otimes \mathcal{L}^n$ we obtain the required result. □

Theorem 7. *Assume F is a charge on $\mathbb{R}^m$ and μ is an absolutely continuous Radon measure in $\mathbb{R}^n$ with $\varphi \stackrel{\text{def}}{=} d\mu/d\mathcal{L}^n \in L^\infty_{loc}(\mathbb{R}^n)$. If F is AC_* in the locally BV set $E \subseteq \mathbb{R}^m$ then $F \otimes \mu$ is AC_* in $E \times \mathbb{R}^n$.*

Proof. First we assume $\varphi < M$ almost everywhere in $\mathbb{R}^n$. By [9]

$$V_*(F \otimes \mu)(E \times \mathbb{R}^n) = V_*^{\mathcal{F}}(F \otimes \mu)(E \times \mathbb{R}^n) \leq V_*^{\mathcal{F}}(|F| \otimes \mu)(E \times \mathbb{R}^n).$$

We assumed that F is an AC_* charge in E, that is, $V_*F = V_*^{\mathcal{F}}F$ is absolutely continuous on E. By the results in [4] F is almost everywhere differentiable on E. Denote by E_d the set of those points where DF exists and put $E_{nd} = E \setminus E_d$. Then $|E_{nd}|_m = 0$ and hence $V_*^{\mathcal{F}}F(E_{nd}) = 0$. By Lemma 6, $V_*^{\mathcal{F}}(|F| \otimes \mathcal{L}^n)(E_{nd} \times \mathbb{R}^n) = 0$. For any partition $\mathcal{P} = \{(A_j, x_j)\}_{j=1}^p$, we have

$$(F \otimes \mu)(\mathcal{P}) = \sum_{j=1}^p \int_{\mathbb{R}^n} |F|(A_j^y) d\mu(y) \leq$$

$$M\sum_{j=1}^{p}\int_{\mathbb{R}^n}|F|(A_j^y)dy = M(|F|\otimes\mathcal{L}^n)(\mathcal{P}).$$

This implies $V_*^{\mathcal{F}}(F\otimes\mu)(E_{nd}\times\mathbb{R}^n) \le MV_*^{\mathcal{F}}(|F|\otimes\mathcal{L}^n)(E_{nd}\times\mathbb{R}^n) = 0$. Since $V_*^{\mathcal{F}}(F\otimes\mu) = V_*(F\otimes\mu)$ we obtain $V_*(F\otimes\mu)(E_{nd}\times\mathbb{R}^n) = 0$.

Since F is differentiable, hence almost differentiable, on E_d Theorem 4 implies that $F\otimes\mu$ is almost differentiable at every point of $E_d\times\mathbb{R}^n$. Therefore, $V_*(F\otimes\mu)$ is AC on $E_d\times\mathbb{R}^n$ and hence $V_*(F\otimes\mu)$ is AC on $(E_d\times\mathbb{R}^n)\cup(E_{nd}\times\mathbb{R}^n) = E\times\mathbb{R}^n$.

Finally we remark that if φ is only locally bounded then we can show by using the above argument that $F\otimes\mu$ is AC_* in $E\times B_n(0,r)$ for any $r>0$, which implies that it is AC_* on $E\times\mathbb{R}^n$. □

5 Details of the proof of Lemma 5 when $m > 1$

5.1 First steps

Proceeding towards a contradiction assume that $V_*^{\mathcal{F}}(|F|\otimes\mathcal{L})(E\times\mathbb{R}) \ne 0$, that is, there exists $\gamma > 0$ such that $V_*^{\mathcal{F}}(|F|\otimes\mathcal{L})(E\times\mathbb{R}) > \gamma$. We choose $\eta_0 > 0$ such that $\inf_\delta V_{\eta_0,\delta}^{\mathcal{F}}(|F|\otimes\mathcal{L})(E\times\mathbb{R}) > \gamma$. We will show that this leads to a contradiction. Using $V_*F(E) = 0$ we choose a suitably small $\epsilon^* > 0$. We know that for any $\eta_0^* > 0$ there exists a δ^* gage function on E such that

$$V_{\eta_0^*,\delta^*}(F)(E) = V_{\eta_0^*,\delta^*}(|F|)(E) = \sup_{\mathcal{P}^*}\sum_j |F|(A_j^*) < \epsilon^*, \tag{3}$$

where the supremum is taken for all $\mathcal{P}^* = \{(A_j^*, x_j^*)\}_{j=1}^p$ η_0^*-regular, δ^*-fine m-dimensional BV-partitions. Of course, since $\mathcal{F}$-partitions are special BV-partitions we have (3) for $\mathcal{F}$-partitions as well. Actually, we will use the above estimate for the special case when our BV sets will consist of a horizontal section $A_{j,t}$ of a figure together with a point x_j^*. In this case $A_{j,t}\cup\{x_j^*\}$ will be an η_0^*-regular BV-set.

In the sequel we will show that this yields a contradiction. First we choose a small η_0^* depending only on η_0 and on the dimension m. Then we choose δ^* for this η_0^* and for $x = (x^*, t) \in E\times\mathbb{R}$ we set $\delta((x^*,t)) = \eta_1\delta^*(x^*)$ where η_1 is a constant depending only on η_0 and on the dimension. By Chapter 7 of [7] the direct product of a thin set in $\mathbb{R}^m$ and of $\mathbb{R}$ will be a thin set in $\mathbb{R}^{m+1}$ hence the zero set of δ will be thin and δ will be an $(m+1)$-dimensional gage.

We show that for this gage function δ

$$(|F|\otimes\mathcal{L})(\mathcal{P}_0) = \sum_{j=1}^{p}(|F|\otimes\mathcal{L})(A_j) = \sum_j\int_{\mathbb{R}}|F|(A_{j,t})dt \le \gamma/2 \tag{4}$$

holds for each η_0-regular, δ-fine partition, $\mathcal{P}_0 = \{(A_j, x_j)\}_{j=1}^p$, anchored in $E\times\mathbb{R}$. Which implies

$$\gamma/2 \ge V_{\eta_0,\delta}^{\mathcal{F}}(|F|\otimes\mathcal{L})(E\times\mathbb{R}) > \gamma,$$

a contradiction.

Without limiting generality we assume $\delta < 1/8$.

Assume now that (4) does not hold, that is, $(|F| \otimes \mathcal{L})(\mathcal{P}_0) > \gamma/2$ for an η_0-regular, δ-fine partition $\mathcal{P}_0$ anchored in $E \times \mathbb{R}$. In the sequel we show that this is impossible.

5.2 Introducing net partitions

Assume $\eta_0 > 0$ is given and the partition $\mathcal{P}_0 = \{(A_j, x_j)\}_{j=1}^p$ is a given $(m+1)$-dimensional η_0-regular δ-fine $\mathcal{F}$-partition. Then $d_j = d(A_j) < \delta(x_j) < 1/8$ for all j's.

Assume $\tau \in \mathbb{R}^{m+1}$ and we consider, $\mathcal{Q}_{m+1}^\tau$, the dyadic net-cube system translated by τ. This means that we consider the cube systems

$$\mathcal{Q}_{m+1}^\tau(k) = \left\{ \tau + \prod_{i=1}^{m+1} \left[\frac{k_i - 1}{2^k}, \frac{k_i}{2^k} \right] : k_i \in \mathbb{Z},\ i = 1, ..., m+1 \right\}$$

and set $\mathcal{Q}_{m+1}^\tau = \cup_{k=0}^\infty \mathcal{Q}_{m+1}^\tau(k)$.

For a fixed j we choose $k_j \geq 0$ such that $2^{-k_j - 1} \leq 8d_j < 2^{-k_j}$. If there exists $Q_j^\tau \in \mathcal{Q}_{m+1}^\tau(k_j)$ such that $A_j \subseteq \text{int}(Q_j^\tau)$ then we say that (A_j, x_j) is τ-well located. Set

$$T_j = \{\tau \in [0,1]^{m+1} : (A_j, x_j) \text{ is } \tau\text{-well located}\}$$

it is easy to see that $|T_j|_{m+1} > 1/2$. For a fixed $\tau \in [0,1]^{m+1}$ we set

$$\mathcal{J}^\tau = \{j : (A_j, x_j) \text{ is } \tau\text{-well located}\}.$$

For a given $\tau \in [0,1]^{m+1}$ consider $\mathcal{P}_0^\tau = \{(A_j, Q_j^\tau)\}_{j \in \mathcal{J}^\tau}$.
It is not difficult to see that from $|T_j|_{m+1} > 1/2$ it follows that we can choose a $\tau \in [0,1]^{m+1}$ such that

$$(|F| \otimes \mathcal{L})(\mathcal{P}_0^\tau) \geq (|F| \otimes \mathcal{L})(\mathcal{P}_0)/2 > \gamma/4.$$

It is also clear that

$$c_{b,m+1} d_j^{m+1} \geq |A_j|_{m+1} \geq \eta_0 d_j ||A_j||_{m+1}$$

and by the isoperimetric inequality

$$|A_j|_{m+1}^{\frac{m}{m+1}} \leq c_{iso,m+1} ||A_j||_{m+1}.$$

Thus $(c_{b,m+1}/\eta_0) d_j^m \geq ||A_j||_{m+1}$ and from

$$|A_j|_{m+1} \geq \eta_0 d_j ||A_j||_{m+1} \geq \eta_0 d_j |A_j|_{m+1}^{\frac{m}{m+1}} / c_{iso,m+1}$$

it follows that $|A_j|_{m+1} \geq (\eta_0 / c_{iso,m+1})^{m+1} d_j^{m+1}$.

By the choice of k_j and Q_j^τ for $j \in \mathcal{J}^\tau$ we have $|Q_j^\tau|_{m+1} \leq 16^{m+1} d_j^{m+1}$ and $||Q_j^\tau||_{m+1} > 8^m c_{f,m+1} d_j^m$.

Therefore,

$$|A_j|_{m+1} \geq (\eta_0/c_{iso,m+1})^{m+1} d_j^{m+1} \geq (\eta_0/16c_{iso,m+1})^{m+1} |Q_j^\tau|_{m+1},$$

and

$$||A_j||_{m+1} \leq (c_{b,m+1}/\eta_0) d_j^m < (c_{b,m+1}/8^m c_{f,m+1}\eta_0) ||Q_j^\tau||_{m+1}.$$

This means that there exists $\eta_1 > 0$ depending only on the dimension and on η_0 such that

$$|A_j|_{m+1} \geq \eta_1 |Q_j^\tau|_{m+1}, \tag{5}$$

and

$$||A_j||_{m+1} \leq \frac{1}{\eta_1} ||Q_j^\tau||_{m+1}, \tag{6}$$

without limiting generality we also assume that $\eta_1 < 1/16\sqrt{m+1}$. If inequalities (5) and (6) hold and $A_j \subseteq \mathbb{Q}_j^\tau$ then we say that (A_j, Q_j^τ) is an *η_1-regular net-partition element.* If, furthermore, $A_j \subseteq \mathrm{int}(Q_j^\tau)$ then we say that (A_j, Q_j^τ) is a *strongly η_1-regular net-partition element.*

Furthermore, $d_j < \delta(x_j)$ implies $Q_j^\tau \subseteq B_{m+1}(x_j, \delta(x_j)/\eta_1)$ also holds. That is, the net-partition element is δ/η_1-fine. In general, if a gage function $\widetilde{\delta}$ is given on a set E we say that *the net-partition element (A_j, Q_j) is $\widetilde{\delta}$-fine* if there exists $x \in Q_j \cap E$ such that $Q_j \subseteq B_{m+1}(x, \widetilde{\delta}(x))$.

From now on (by shifting the coordinate system and changing notation) without limiting generality we assume that $\tau = 0 \in \mathbb{R}^{m+1}$ and we drop the superscript τ. So we assume that we work with a strongly η_1-regular, δ/η_1-fine net-partition $\mathcal{P}_{0,0} = \{(A_j, Q_j)\}_{j=1}^{p_{0,0}}$ such that $(|F| \otimes \mathcal{L})(\mathcal{P}_{0,0}) > \gamma/4$. Without the superscript τ we denote by $\mathcal{Q}_{m+1}$ the net system and by $\mathcal{Q}_{m+1}(k)$ the netcubes of sidelength 2^{-k}.

It may happen that for $j \neq j'$ we have $Q_j = Q_{j'}$. However, by η_1-regularity $|A_j|_{m+1} \geq \eta_1 |Q_j|_{m+1}$ and since the A_j's are non-overlapping this implies that we can have at most $[1/\eta_1]$ many different j''s for which $Q_{j'} = Q_j$ for a fixed j. This means that we can select a subpartition of $\mathcal{P}_{0,0}$, denoted by $\mathcal{P}_{0,1} = \{(A_j, Q_j)\}_{j=1}^{p_{0,1}}$ such that $(|F| \otimes \mathcal{L})(\mathcal{P}_{0,1}) > \gamma\eta_1/4$ and for $j \neq j'$ we have $Q_j \neq Q_{j'}$.

From now on if we say that a system $\mathcal{P}_{0,1} = \{(A_j, Q_j)\}_{j=1}^{p_{0,1}}$ is a net-partition we will always mean that if $Q_j \neq Q_{j'}$ for a $j \neq j'$ then at least one of the sets A_j and $A_{j'}$ is empty, and hence can be dropped from the partition.

5.3 A variant of Lemma 1

Before we proceed further we state and prove a variant of Lemma 1.

Lemma 8. *Assume $\eta_1 > 0$, and $B \subseteq \mathbb{R}^{m+1}$ is a figure, $Q \subseteq \mathbb{R}^{m+1}$ is a cube of sidelength d, and $|B \cap Q|_{m+1} \leq (1-\eta_1)|Q|_{m+1}$. Then there exists a constant $c_{F,m+1}$, depending only on the dimension and η_1, such that*

$$\mathcal{H}^m(B \cap \partial Q) \leq c_{F,m+1}\mathcal{H}^m(\partial B \cap \operatorname{int}(Q)).$$

(Without limiting generality we also assume that $c_{F,m+1} > 1$.)

Proof. If $|B \cap Q|_{m+1} \leq |Q|_{m+1}/2$ then by Lemma 1 we have (1). If $|B \cap Q|_{m+1} > |Q|_{m+1}/2$ then by our assumption and by the relative isoperimetric inequality (2)

$$\frac{\eta_1^{\frac{m}{m+1}}||Q||_{m+1}}{c_{f,m+1}} = \eta_1^{\frac{m}{m+1}}|Q|_{m+1}^{\frac{m}{m+1}} \leq |Q \setminus B|_{m+1}^{\frac{m}{m+1}} \leq c_{riso,m+1}\mathcal{H}^m(\partial B \cap \operatorname{int}(Q)).$$

On the other hand, $\mathcal{H}^m(B \cap \partial Q) \leq ||Q||_{m+1}$ and hence

$$\mathcal{H}^m(B \cap \partial Q) \leq \frac{c_{f,m+1}c_{riso,m+1}}{\eta_1^{\frac{m}{m+1}}}\mathcal{H}^m(\partial B \cap \operatorname{int}(Q)).$$

Therefore $c_{F,m+1} = \max\{c'_{F,m+1}, c_{f,m+1}c_{riso}/\eta_1^{\frac{m}{m+1}}\}$ is a suitable constant. □

5.4 Choosing the net constant ν

Recall that the choice of η_1 depends only on η_0 and on the dimension, m. Next we choose $\epsilon > 0$ such that

- $\epsilon + \frac{\epsilon}{\eta_1} \leq \eta_1$, (this clearly implies $\epsilon < \eta_1 < 1$);
- $(1 + c_{F,m+1})\epsilon c_{\mathcal{H},m}/c_{\mathcal{H},m-1} \leq 0.01/c_{f,m+1}$.

Observe that ϵ depends only on η_1 and m.

Theorem 9. *Given $\eta_1, \epsilon \in (0,1)$ we can choose $\nu \in \mathbb{N}$ depending on η_1, ϵ and m such that if (A, Q) is an η_1-regular dyadic net-partition element with $Q \in \mathcal{Q}_{m+1}(k_A)$ then there exists $\widetilde{Q} \subseteq Q$, $\widetilde{Q} \in \mathcal{Q}_{m+1}(k_A + \nu)$ such that*

$$|\widetilde{Q} \cap A|_{m+1} > (1-\epsilon)|\widetilde{Q}|_{m+1}$$

$$\mathcal{H}^m(\operatorname{int}(\widetilde{Q}) \cap \partial A) < \epsilon||\widetilde{Q}||_{m+1}, \text{ and}$$

$$8 \cdot 4^{\nu m}\sum_{j=1}^{\infty} 64^{-j\nu(m-1)} = \frac{1}{\eta^*}\sum_{j=1}^{\infty} 64^{-j\nu(m-1)} < \frac{1}{4}, \tag{7}$$

where $\eta^* \stackrel{\text{def}}{=} \frac{1}{8 \cdot 4^{\nu m}}$.

For later use we remark that apart from (7) the rest of the above theorem holds for $m = 1$ as well.

Proof. Clearly, we can choose ν_0 such that for all $\nu > \nu_0$ (7) is satisfied.

For a given $\nu > \nu_0$ denote by $Q_1, ..., Q_{2^\nu}$ the netcubes $Q_j \subseteq Q$, $Q_j \in \mathcal{Q}_{m+1}(k_A + \nu)$.

We say that $j \in \mathcal{J}_\nu$ if $|Q_j \cap A|_{m+1} \geq (\eta_1/2)|Q_j|_{m+1}$. Then

$$| \cup_{j \notin \mathcal{J}_\nu} Q_j \cap A|_{m+1} \leq \frac{\eta_1}{2} | \cup_{j \notin \mathcal{J}_\nu} Q_j|_{m+1} < \frac{\eta_1}{2}|Q|_{m+1}$$

and by η_1-regularity of (A, Q) we also have

$$\eta_1|Q|_{m+1} < |A|_{m+1} = | \cup_{j=1}^{2^\nu} Q_j \cap A|_{m+1} =$$

$$| \cup_{j \in \mathcal{J}_\nu} Q_j \cap A|_{m+1} + | \cup_{j \notin \mathcal{J}_\nu} Q_j \cap A|_{m+1}.$$

Therefore,

$$| \cup_{j \in \mathcal{J}_\nu} Q_j \cap A|_{m+1} > \frac{\eta_1}{2}|Q|_{m+1}.$$

From $\# \mathcal{J}_\nu < \eta_1 2^{(m+1)\nu - 1}$ it would follow

$$(\eta_1 2^{(m+1)\nu-1}) 2^{-(m+1)\nu} |Q|_{m+1} > | \cup_{j \in \mathcal{J}_\nu} Q_j \cap A|_{m+1} > \frac{\eta_1}{2}|Q|_{m+1},$$

which is a contradiction. Hence $\# \mathcal{J}_\nu \geq \eta_1 2^{\nu(m+1)-1}$.

Set

$$\epsilon' = \min \left\{ \epsilon, \frac{\epsilon}{c_{f,m+1} c_{riso,m+1}}, \frac{((\eta_1/2)|\widetilde{Q}|_{m+1})^{\frac{m}{m+1}}}{c_{riso,m+1}||\widetilde{Q}||_{m+1}} \right\}.$$

Observe that ϵ' depends only on η_1, ϵ, and m and does not depend on the choice of the cube $\widetilde{Q}$.

Denote by $\mathcal{J}'_\nu$ the set of those $j \in \mathcal{J}_\nu$ for which $\mathcal{H}^m(\mathrm{int}(Q_j) \cap \partial A) \geq \epsilon'||Q_j||_{m+1} = 2^{-m\nu}\epsilon'||Q||_{m+1}$. Then

$$||Q||_{m+1}/\eta_1 \geq ||A|| \geq$$

$$\sum_{j \in \mathcal{J}'_\nu} \mathcal{H}^m(\mathrm{int}(Q_j) \cap \partial A) \geq (\#\mathcal{J}'_\nu) 2^{-m\nu} \epsilon' ||Q||_{m+1},$$

that is, $2^{m\nu}/\eta_1 \epsilon' \geq \#\mathcal{J}'_\nu$.

Hence, if ν is chosen so large that $2^{m\nu}/\eta_1\epsilon' < \eta_1 2^{\nu(m+1)-1}$ then we can select a $j \in \mathcal{J}_\nu \setminus \mathcal{J}'_\nu$. For $\widetilde{Q} = Q_j$ we have

$$|\widetilde{Q} \cap A|_{m+1} \geq (\eta_1/2)|\widetilde{Q}|_{m+1}, \text{ and}$$

$$\mathcal{H}^m(\mathrm{int}(\widetilde{Q}) \cap \partial A) < \epsilon'||\widetilde{Q}||_{m+1} < \epsilon||\widetilde{Q}||_{m+1}.$$

Using the relative isoperimetric inequality (2)

$$\min\{|\widetilde{Q} \cap A|_{m+1}, |\widetilde{Q} \setminus A|_{m+1}\}^{\frac{m}{m+1}} \leq c_{riso,m+1}\mathcal{H}^m(\mathrm{int}(\widetilde{Q}) \cap \partial A) < c_{riso,m+1}\epsilon'||\widetilde{Q}||_{m+1}. \quad (8)$$

On the other hand, using the definition of ϵ'

$$c_{riso,m+1}\epsilon'||\widetilde{Q}||_{m+1} = c_{riso,m+1}\frac{||\widetilde{Q}||_{m+1}}{|\widetilde{Q}|_{m+1}^{\frac{m}{m+1}}}\epsilon'|\widetilde{Q}|_{m+1}^{\frac{m}{m+1}} \leq$$

$$(\eta_1/2)^{\frac{m}{m+1}}|\widetilde{Q}|_{m+1}^{\frac{m}{m+1}} \leq |\widetilde{Q}\cap A|_{m+1}^{\frac{m}{m+1}},$$

and hence (8) implies

$$|\widetilde{Q}\setminus A|_{m+1}^{\frac{m}{m+1}} < c_{riso,m+1}\epsilon'||\widetilde{Q}||_{m+1}.$$

Therefore,

$$|\widetilde{Q}\setminus A|_{m+1} < (c_{riso,m+1}\epsilon')^{\frac{m+1}{m}}||\widetilde{Q}||_{m+1}^{\frac{m+1}{m}} =$$

$$(c_{riso,m+1}\epsilon')^{\frac{m+1}{m}}c_{f,m+1}^{\frac{m+1}{m}}|\widetilde{Q}|_{m+1} \leq \epsilon^{\frac{m+1}{m}}|\widetilde{Q}|_{m+1} < \epsilon|\widetilde{Q}|_{m+1}.$$

This implies $|\widetilde{Q}\cap A|_{m+1} > (1-\epsilon)|\widetilde{Q}|_{m+1}$. □

Now we assume that ν is chosen and fixed according to Theorem 9. One can easily see that ν is depending on constants which, after all, depend only on η_0 and m.

We also assume that a ν' is also given. Set $\mathcal{Q}_{m+1,\nu,\nu'}(k)\stackrel{\text{def}}{=}\mathcal{Q}_{m+1}(k\nu+\nu')$ and $\mathcal{Q}_{m+1,\nu,\nu'}\stackrel{\text{def}}{=}\cup_{k=0}^{\infty}Q_{m+1,\nu,\nu'}(k)$. We call $\mathcal{Q}_{m+1,\nu,\nu'}$ the 2^ν-net starting at ν'. In the sequel we will also use the $64^\nu = 2^{6\nu}$-net based at ν', that is, we use $Q_{m+1,6\nu,\nu'}$.

If $Q\in Q_{m+1,\nu,\nu'}(k)$ then the collection of those cubes $Q'\subsetneq Q$ which are in $Q_{m+1,\nu,\nu'}(k+\alpha)$ are called the $2^{\alpha\nu}$-net children of Q. We denote this collection by $2^{\alpha\nu}CHDRN(Q)$.

Consider the net-partitions

$$\mathcal{P}_{0,1,\nu'} = \{(A_j,Q_j): j\in\{1,...,p_{0,1}\};\ Q_j\in\mathcal{Q}_{m+1,6\nu,\nu'}\}$$

for $\nu'=0,...,6\nu-1$.

Then $\mathcal{P}_{0,1}=\cup_{\nu'=0}^{6\nu-1}\mathcal{P}_{0,1,\nu'}$ with this union disjoint.

Therefore, we can choose a ν' for which

$$(|F|\otimes\mathcal{L})(\mathcal{P}_{0,1,\nu'}) > \frac{\gamma\eta_1}{4\cdot 6\nu} = \frac{\gamma\eta_1}{24\nu}.$$

We fix this ν' and to simplify our notation we denote $\mathcal{P}_{0,1,\nu'}$ by $\mathcal{P}$. We also omit some other indices. Thus, from now on we deal with $\mathcal{P}=\{(A_j,Q_j)\}_{j=1}^{p}$ which is a fixed strongly η_1-regular, δ/η_1-fine 64^ν-net partition, satisfying $(|F|\otimes\mathcal{L})(\mathcal{P}) > \frac{\gamma\eta_1}{24\nu}$. Without limiting generality we also assume that $Q_j\subsetneq Q^0=[0,1]^{m+1}$ holds for all j.

5.5 Choosing the heart system

Assume $\mathcal{P} = \{(A_j, Q_j)\}_{j=1}^p$ is a fixed strongly η_1-regular 64^ν-net partition with the properties given in the end of Section 5.4.

Recall that we assumed that $\epsilon < \eta_1 < 1$.

For each j choose $\widetilde{Q}_j \in 2^\nu CHDRN(Q_j)$ according to Theorem 9, that is, we have

$$|\widetilde{Q}_j \cap A_j|_{m+1} > (1-\epsilon)|\widetilde{Q}_j|_{m+1}, \text{ and}$$

$$\mathcal{H}^m(\operatorname{int}(\widetilde{Q}_j) \cap \partial A_j) < \epsilon ||\widetilde{Q}_j||_{m+1}.$$

Denote by $\mathcal{J}_j'$ the set of those $j' \neq j$ for which $Q_{j'} \subseteq \widetilde{Q}_j$.

Since $Q_{j'} \in 64^\nu CHDRN(Q_j)$ we have $Q_{j'} \subsetneq \widetilde{Q}_j$. Denote by $\mathcal{J}_j$ the set of those $j' \in \mathcal{J}_j'$ for which $Q_{j'}$ is maximal, that is, for all $j'' \in \mathcal{J}_j'$, $j'' \neq j'$ we have $Q_{j'} \not\subseteq Q_{j''}$.

Set $A_j' = (A_j \cap \widetilde{Q}_j) \ominus \cup_{j' \in \mathcal{J}_j} Q_{j'}$.

From

$$|A_{j'} \cap Q_{j'}|_{m+1} = |A_{j'}|_{m+1} \geq \eta_1 |Q_{j'}|_{m+1} \text{ and } |A_{j'} \cap A_j|_{m+1} = 0$$

it follows that $|A_j \cap Q_{j'}|_{m+1} \leq (1-\eta_1)|Q_{j'}|_{m+1}$ for $j' \in \mathcal{J}_j'$ and by Lemma 8

$$\mathcal{H}^m(A_j \cap \partial Q_{j'}) \leq c_{F,m+1} \mathcal{H}^m(\partial A_j \cap \operatorname{int}(Q_{j'})).$$

Hence

$$\mathcal{H}^m(\operatorname{int}(\widetilde{Q}_j) \cap \partial A_j') \leq \mathcal{H}^m(\operatorname{int}(\widetilde{Q}_j) \cap \partial A_j) + \sum_{j' \in \mathcal{J}_j} \mathcal{H}^m(A_j \cap \partial Q_{j'}) \leq \tag{9}$$

$$\mathcal{H}^m(\operatorname{int}(\widetilde{Q}_j) \cap \partial A_j) + \sum_{j' \in \mathcal{J}_j} c_{F,m+1} \mathcal{H}^m(\operatorname{int}(Q_{j'}) \cap \partial A_j) \leq$$

$$(1 + c_{F,m+1}) \mathcal{H}^m(\operatorname{int}(\widetilde{Q}_j) \cap \partial A_j) < (1 + c_{F,m+1}) \epsilon ||\widetilde{Q}_j||_{m+1}. \tag{10}$$

Since $|A_j \cap Q_{j'}|_{m+1} < (1-\eta_1)|Q_{j'}|_{m+1}$, for $j' \in \mathcal{J}_j$, we also have

$$|A_j \cap \cup_{j' \in \mathcal{J}_j} Q_{j'}|_{m+1} < (1-\eta_1)| \cup_{j' \in \mathcal{J}_j} Q_{j'}|_{m+1}.$$

Thus

$$\epsilon |\widetilde{Q}_j|_{m+1} > |\widetilde{Q}_j \setminus A_j|_{m+1} \geq | \cup_{j' \in \mathcal{J}_j} Q_{j'} \setminus A_j|_{m+1} > \eta_1 | \cup_{j' \in \mathcal{J}_j} Q_{j'}|_{m+1}$$

and hence

$$(\epsilon/\eta_1)|\widetilde{Q}_j|_{m+1} > |A_j \setminus A_j'|_{m+1}.$$

This implies

$$|A_j'|_{m+1} \geq |\widetilde{Q}_j \cap A_j|_{m+1} - |A_j \setminus A_j'|_{m+1} > (1 - (\epsilon + \frac{\epsilon}{\eta_1}))|\widetilde{Q}_j|_{m+1}.$$

We assumed that $\epsilon + \frac{\epsilon}{\eta_1} \le 0.01\eta_1$ and hence $|A'_j|_{m+1} > (1-0.01\eta_1)|\widetilde{Q}_j|_{m+1}$. This implies that denoting by $T^1_{j,H}$ the set of those t's in $T(\widetilde{Q}_j)$ for which

$$|A'_{j,t}|_m > (1-\eta_1)|\widetilde{Q}_{j,t}|_m$$

we have $|T^1_{j,H}|_1 > 0.9|T(\widetilde{Q}_j)|_1$.

We also assumed that $(1+c_{F,m+1})\epsilon c_{\mathcal{H},m}/c_{\mathcal{H},m-1} \le 0.01/c_{f,m+1}$. Then from (9)-(10) it follows that

$$c_{\mathcal{H},m}\mathcal{H}^m(\mathrm{int}(\widetilde{Q}_j)\cap \partial A'_j)/c_{\mathcal{H},m-1} < 0.01||\widetilde{Q}_j||_{m+1}/c_{f,m+1} =$$

$$0.01\ell(\widetilde{Q}_j)^m = 0.01|T(\widetilde{Q}_j)|_1^m.$$

Hence

$$0.01|T(\widetilde{Q}_j)|_1^m > c_{\mathcal{H},m}\mathcal{H}^m(\mathrm{int}(\widetilde{Q}_j)\cap \partial A'_j)/c_{\mathcal{H},m-1} \ge$$

$$\int_{T(\widetilde{Q}_j)} \mathcal{H}^{m-1}(\mathrm{int}(\widetilde{Q}_{j,t})\cap \partial A'_{j,t})dt.$$

This implies that denoting by $T^2_{j,H}$ the set of those t's in $T(\widetilde{Q}_j)$ for which

$$\mathcal{H}^{m-1}(\mathrm{int}(\widetilde{Q}_{j,t})\cap \partial A'_{j,t}) < (1/2)|T(\widetilde{Q}_j)|_1^{m-1} \tag{11}$$

we have $|T^2_{j,H}|_1 > 0.9|T(\widetilde{Q}_j)|_1$. Using that A'_j is a figure, choose $T_{j,H} \subseteq T^1_{j,H} \cap T^2_{j,H}$ such that $T_{j,H}$ is a one-dimensional figure (finite union of closed intervals) and

$$|T_{j,H}|_1 > \frac{1}{2}|T(\widetilde{Q}_j)|_1 = \frac{1}{2^{\nu+1}}\ell(Q_j). \tag{12}$$

Set

$$A^H_j \stackrel{\mathrm{def}}{=} (\mathbb{R}^m \times T_{j,H}) \cap A'_j.$$

We call A^H_j the heart of A_j. It is clearly a figure. If $t \in T(A^H_j) = T_{j,H}$ then we have the following inequalities:

$$|A^H_{j,t}|_m > (1-\eta_1)|\widetilde{Q}_j|_m > \frac{1}{2}2^{-\nu m}|Q_{j,t}|_m, \tag{13}$$

and

$$\begin{aligned}||A^H_{j,t}||_m &< ||\widetilde{Q}_{j,t}||_m + \mathcal{H}^{m-1}(\mathrm{int}(\widetilde{Q}_{j,t})\cap \partial A'_{j,t}) < \\ ||\widetilde{Q}_{j,t}||_m &+ \frac{1}{2}|T(\widetilde{Q}_j)|_1^{m-1} < ||\widetilde{Q}_{j,t}||_m + \frac{1}{2}||\widetilde{Q}_{j,t}||_m \le ||Q_{j,t}||_m.\end{aligned} \tag{14}$$

Furthermore, the definition of $A'_j \supset A^H_j$ implies that if $j \neq j'$ and the size of $Q_{j'}$ is no greater than that of Q_j then

$$Q_{j'} \odot A^H_j = \emptyset. \tag{15}$$

5.6 Preparation to Spreading and Swapping

Recall that we assumed that $\mathcal{P} = \{(A_j, Q_j)\}_{j=1}^p$ is a fixed $(m+1)$-dimensional strongly η_1-regular 64^ν-net partition. Without limiting generality we also assumed that $Q_j \subseteq Q^{down} = [0,1]^{m+1}$.

Since computing $F \otimes \mathcal{L}$ we are interested only in the horizontal sections of the partition elements we introduce the concept of h-partitions. We denote $Q_j^* = \pi_m(Q_j)$, the projection of Q_j onto $\mathbb{R}^m$ the hyperplane of the first m coordinates. We say that $\mathcal{P} = \{(A_j, Q_j^*)\}_{j=1}^p$ is an h-partition if for almost all $t \in \mathbb{R}$ the horizontal section $\mathcal{P}_t = \{(A_{j,t}, Q_j^*)\}_{j=1}^p$ is an m-dimensional 64^ν-net partition. Observe that $Q_{j'}^* = Q_j^*$ is possible when $j' \neq j$. However, the original $(m+1)$-net-partition properties of $\mathcal{P}$ imply that for almost all t from $j' \neq j$ and $Q_{j'}^* = Q_j^*$ it follows that at least one of the horizontal sections $A_{j',t}$, or $A_{j,t}$ is empty, so it could be dropped from the partition at this level. We say that $\mathcal{P}$ is η'-regular if for almost all sections $\mathcal{P}_t$ is an η'-regular m-dimensional partition. (If $A_{j,t} = \emptyset$ then the partition element $(A_{j,t}, Q_j^*)$ is regarded to be η'-regular by definition for all $\eta' > 0$.)

Clearly,

$$\frac{\gamma\eta_1}{24\nu} < (|F| \otimes \mathcal{L})(\mathcal{P}) \leq \int_{\mathbb{R}} |F|(\mathcal{P}_t)dt.$$

We say that two h-partitions $\mathcal{P}$ and $\mathcal{P}'$ are $\otimes$-equivalent if $(|F| \otimes \mathcal{L})(\mathcal{P}) = (|F| \otimes \mathcal{L})(\mathcal{P}')$ holds for all charges, F defined on $\mathbb{R}^m$.

In the sequel we give an algorithm which will "essentially" replace our original $(m+1)$-dimensional, η_1-regular 64^ν-net partition $\mathcal{P}$ with another $\otimes$-equivalent h-partition $\mathcal{P}'$ such that for almost all t the horizontal sections $\mathcal{P}'_t$ are m-dimensional η^*-regular 64^ν-net partitions where η^* was chosen in Theorem 9. This will make it possible for us to estimate $|F(\mathcal{P}'_t)|$ from above for almost all t's.

To do our swapping algorithm we need the heart system defined previously. We use these "hearts" to keep our modified sections "alive", that is, to make them η^*-regular. This means that we want to increase the volume of our horizontal sections without increasing too much their surface (for this we use (13)-(14)). We also use an integer constant $c_{aff} = 2^{\nu+1}c_{sw}$, where the integer constant c_{sw} will be chosen in (39) and depends only on η_0 and m (in fact, it depends on η_1 and m, but η_1 depends on η_0 and m).

Denote by Ψ_{aff} the affine transformation which maps $Q^{down} = [0,1]^{m+1}$ onto $Q^{up} \stackrel{\text{def}}{=} [0,1]^m \times [1, 1+c_{aff}]$ and is the identity on the first m coordinates. Choose the heart system $\{(A_j^H, Q_j)\}_{j=1}^p$ as it was given in Section 5.5 and set

$$A_j^0 \stackrel{\text{def}}{=} A_j \cup \Psi_{aff}(A_j^H), \text{ and } Q_j^0 \stackrel{\text{def}}{=} Q_j \cup \Psi_{aff}(Q_j)$$

for each j. Consider the h-partition $\mathcal{P}^0 = \{(A_j^0, Q_j^*)\}_{j=1}^p$. Clearly,

$$\frac{\gamma\eta_1}{24\nu} < (|F| \otimes \mathcal{L})(\mathcal{P}^0).$$

Without limiting generality we assume that our partition is indexed in a nondecreasing order of the size of the Q_j's, that is, for $j < j'$ we have $\ell(Q_j) \leq \ell(Q_{j'})$.

In the sequel we will define the h-partitions $\mathcal{P}^i = \{(A^i_j, Q^*_j)\}_{j=1}^p$ by induction so that the A^i_j's are $(m+1)$-dimensional figures, $\mathcal{P}^i$ is $\otimes$-equivalent to $\mathcal{P}^0$ and if we let $\mathcal{P}^i_{\le} = \{(A^i_j, Q^*_j)\}_{j=1}^i$ then it is an η^*-regular 64^ν-net h-partition.

Assume that we have defined $\mathcal{P}^p = \{(A^p_j, Q^*_j)\}_{j=1}^p$ such that it is an η^*-regular, 64^ν-net h-partition.

Since $\mathcal{P} = \{(A_j, Q_j)\}_{j=1}^p$ was a δ/η_1-fine $(m+1)$-dimensional partition each Q_j contains a point x_j such that $Q_j \subseteq B_{m+1}(x_j, \delta(x_j)/\eta_1)$. Then each Q^*_j contains the point $x^*_j \overset{\text{def}}{=} \pi_m(x_j)$ such that $Q_j \subseteq B_m(x^*_j, \delta^*(x^*_j))$ and hence $A_{j,t} \subseteq B_m(x^*_j, \delta^*(x^*_j))$ for all t. It is clear that for η^* we can choose $\eta^*_0 > 0$ such that $(A_{j,t} \cup \{x^*_j\}, x^*_j)_{j=1}^p$ will be an η^*_0-regular m-dimensional BV-partition for almost every t and by (3) $|F|(\mathcal{P}^p_t) < \epsilon^*$ holds. This implies

$$\frac{\gamma\eta_1}{24\nu} < (|F| \otimes \mathcal{L})(\mathcal{P}^p) = (|F| \otimes \mathcal{L})(\mathcal{P}^0) \le \int_0^{1+c_{aff}} |F|(\mathcal{P}^p_t)dt < \epsilon^*(1 + c_{aff})$$

where the constant c_{aff} depends only on m and η_0. Thus choosing ϵ^* so that $\epsilon^*(1 + c_{aff}) < \gamma\eta_1/24\nu$ we obtain a contradiction which concludes the proof of Lemma 5.

Next we show how to obtain the partitions $\mathcal{P}^i$. Before turning to the details we say a few words about the construction of these partitions.

Partition $\mathcal{P}^0$ was given above. Assume $i \ge 1$ and $\mathcal{P}^{i-1} = \{(A^{i-1}_j, Q^*_j)\}_{j=1}^p$ is defined. We define our partitions by induction and one can easily see that all our assumptions are true when $i = 1$, that is, $i - 1 = 0$. We assume that

$$A^{i-1}_j \odot Q^{up} = \Psi_{aff}(A^H_j) \text{ for } j = i, ..., p. \tag{16}$$

Observe that by (15)

$$A^{i-1}_j \odot (Q^{up} \cap Q^0_i) = \Psi_{aff}(A^H_j) \cap Q^0_i = \emptyset \text{ holds for all } j > i. \tag{17}$$

To define $\mathcal{P}^i$ we modify the sets A^{i-1}_j only in Q^0_i, that is,

$$A^i_j \ominus Q^0_i = A^{i-1}_j \ominus Q^0_i \text{ holds for all } j = 1, ..., p, \tag{18}$$

and using (17) we have

$$A^i_j \odot Q^{up} = A^{i-1}_j \odot Q^{up} \text{ for all } j = i+1, ..., p. \tag{19}$$

From (19) it follows that after step i we will have

$$A^i_j \odot Q^{up} = \Psi_{aff}(A^H_j) \text{ for } j = i+1, ..., p. \tag{20}$$

So, according to (18) we modify the figures A^{i-1}_j only in Q^0_i. This modification will consist of two steps. We call the first step spreading and the second step swapping. In fact, spreading is just a preparation to the swapping algorithm.

What happens at the spreading step? We denote by $\{A^{i,r}_j\}_{j=1}^p$ the system of figures which we obtain after this step. For $j < i$ we do not change anything, that is, $A^{i,r}_j \overset{\text{def}}{=} A^{i-1}_j$ for $j = 1, ..., i-1$.

If $j \geq i$ we define $A_j^{i,r}$ so that $A_j^{i,r} \ominus Q_i = A_j^{i-1} \ominus Q_i$ and within $Q_i \subseteq Q_i^0$ these sets will be spreaded in the sense that for almost every $t \in T(Q_i) \subseteq [0,1]$ *there is only at most one $j \geq i$ for which the horizontal section $A_{j,t}^{i,r} \odot Q_i^* \neq \emptyset$*, that is, the sections spread across Q_i^*. Of course, we will also make sure that the system $\mathcal{P}^{i,r} = \{(A_j^{i,r}, Q_j^*)\}_{j=1}^p$ will be $\otimes$-equivalent to $\mathcal{P}^i$.

This will be assured by the property that for almost all $t \in T(Q_i)$ we will have $\cup_{j \geq i} A_{j,t}^{i-1} = \cup_{j \geq i} A_{j,t}^{i,r}$.

At the swapping step we do not change the sets $A_j^{i,r}$ for $j > i$, that is,

$$A_j^i = A_j^{i,r} \text{ for } j = i+1, ..., p. \tag{21}$$

Based on (13) and (14) the good regularity properties of the affine copy $\Psi_{aff}(A_i^H) \subseteq \Psi_{aff}(Q_i) \subseteq Q_i^0$ of the heart of A_i will be used in the swapping algorithm to obtain η^*-regular horizontal sections of A_i^i. Here we also need to watch out not to ruin the similar regularity properties of sections of $A_j^{i,r} = A_j^{i-1}$ for $j < i$. During the swapping step we obtain A_i^i such that the partition element $(A_{i,t}^i, Q_i^*)$ is η^*-regular for almost all t. We also need to make sure that after the swapping steps $\mathcal{P}^i$ will be $\otimes$-equivalent to $\mathcal{P}^{i,r}$. In the swapping algorithm sections $A_{j,t}^{i,r}$, $(j < i)$ are either translated vertically, or are deleted and, after a vertical translation being added to $A_i^{i,r}$. This will ensure the η^*-regularity of the partition elements (A_j^i, Q_j^*) for $j < i$.

The main technical difficulty, not mentioned so far, is that we need to keep control of the surfaces of the sets A_j^i. In fact what we need to control is not the total $(m+1)$-dimensional surface, $||A_j^i||_{m+1}$ but the "vertical surface" $||A_j^i||_v$. The details will be provided during the discussion of the spreading and swapping steps.

5.7 Spreading

First we recall that we started in Section 5.6 with an $(m+1)$-dimensional partition $\mathcal{P} = \{(A_j, Q_j)\}_{j=1}^p$. We say that Q_i is a *defect cube* if

$$\sum_{j>i} \mathcal{H}^m(\partial A_j \cap \text{int}(Q_i)) \leq 2c_{F,m+1}||Q_i||_{m+1}, \tag{22}$$

that is the total surface of pieces of larger partition elements within Q_i cannot be much larger than the surface of Q_i.

After spreading we will have different estimations for defect and non-defect cubes. The reason is that for $j > i$ we need to cut the sets A_j^{i-1} by the surface of Q_i and hence we need to control the cutting surfaces which increase the surfaces of the A_j^{i-1}'s, not to allow these surfaces to become too large.

Assume that $\mathcal{P}^{i-1} = \{(A_j^{i-1}, Q_j^*)\}_{j=1}^p$ is an h-partition obtained after step $i-1$ of our spreading and swapping process.

We denote by $\mathcal{I}_{max}^{i-1}$ the set of those i' for which $Q_{i'}$ is a maximal cube among the collection of cubes $\{Q_{i''} : i'' = 1, ..., i-1\}$, that is, $i' \in \mathcal{I}_{max}^{i-1}$ if and only if

$Q_{i'} \not\subseteq Q_{i''}$ for $i'' \in \{1, ..., i-1\} \setminus \{i'\}$. We assume by induction that if $i' \in \mathcal{I}_{max}^{i-1}$ and $Q_{i'}$ is a defect cube then

$$\cup_{j>i-1} A_j^{i-1} \cap \mathrm{int}(Q_{i'}) = \emptyset. \tag{23}$$

If $i' \in \mathcal{I}_{max}^{i-1}$ and $Q_{i'}$ is not a defect cube then we will assume

$$\sum_{j>i-1} \mathcal{H}^m(\partial_v A_j^{i-1} \cap \mathrm{int}(Q_{i'})) \leq \tag{24}$$

$$2c_{F,m+1} \sum_{j>i'} \mathcal{H}^m(\partial A_j \cap \mathrm{int}(Q_{i'})) - ||Q_{i'}||_{m+1} = \tag{25}$$

$$2c_{F,m+1} \sum_{j>i-1} \mathcal{H}^m(\partial A_j \cap \mathrm{int}(Q_{i'})) - ||Q_{i'}||_{m+1} \tag{26}$$

where the last equality holds since i' was a maximal subcube of Q_i and hence for $i > j > i'$, Q_j and $Q_{i'}$ are non-overlapping and $A_j \subseteq \mathrm{int}(Q_j)$. This implies

$$\sum_{j>i-1} \mathcal{H}^m(\partial_v A_j^{i-1} \cap Q_{i'}) \leq 2c_{F,m+1} \sum_{j>i-1} \mathcal{H}^m(\partial A_j \cap \mathrm{int}(Q_{i'})).$$

Keeping in mind that that $c_{F,m+1} > 1$ and for non-defect cubes the negation of (22) holds (25) and (26) are positive.

We denote by $\mathcal{I}_{max}^{i-0.5}$ the set of those $i' \in \mathcal{I}_{max}^{i-1}$ for which $Q_{i'} \subsetneq Q_i$. We use the notation $\mathcal{I}_{max,D}$, or $\mathcal{I}_{max,ND}$ for the set of those $i' \in \mathcal{I}_{max}^{i-0.5}$ for which $Q_{i'}$ is a defect, or a non-defect cube, respectively.

Taking into consideration that A_j and $A_{i'}$ are non-overlapping and $(A_{i'}, Q_{i'})$ is strongly η_1-regular, by Lemma 8 we have

$$\mathcal{H}^m(A_j \cap \partial Q_{i'}) \leq c_{F,m+1} \mathcal{H}^m(\partial A_j \cap \mathrm{int}(Q_{i'}))$$

for each $i' \in \mathcal{I}_{\max,D}$.

By (18) applied for all steps $1, 2, ..., i-1$ we obtain that for $j \geq i$ we have

$$A_j^{i-1} \cap (\mathrm{int}(Q_i) \setminus \bigcup_{i' \in \mathcal{I}_{max}^{i-0.5}} Q_{i'}) = A_j \cap (\mathrm{int}(Q_i) \setminus \bigcup_{i' \in \mathcal{I}_{max}^{i-0.5}} Q_{i'}). \tag{27}$$

Set

$$R_{i,D} = \bigcup_{i' \in \mathcal{I}_{max,D}} Q_{i'} \text{ and } R_{i,ND} = \bigcup_{i' \in \mathcal{I}_{max,ND}} Q_{i'}.$$

Since all A_j^{i-1}'s are (nondegenerate) figures from (23) it follows that $\cup_{j \geq i} A_j^{i-1} \cap \mathrm{int}(R_{i,D}) = \emptyset$. This and (27) imply that if $j \geq i$ then

$$\partial_v A_j^{i-1} \cap (R_{i,D} \setminus R_{i,ND}) \subseteq A_j \cap \partial R_{i,D} \subseteq \bigcup_{i' \in \mathcal{I}_{max,D}} (A_j \cap \partial Q_{i'}).$$

Hence

$$\sum_{j\geq i}\mathcal{H}^m(\partial_v A_j^{i-1}\cap \operatorname{int}(Q_i)) \leq \sum_{j\geq i}\mathcal{H}^m\big((\partial_v A_j\cap \operatorname{int}(Q_i))\setminus \bigcup_{i'\in\mathcal{I}_{max}^{i-0.5}} Q_{i'}\big) + \tag{28}$$

$$\sum_{j\geq i}\mathcal{H}^m(\partial A_j^{i-1}\cap R_{i,ND}) + \sum_{j\geq i}\mathcal{H}^m(\partial_v A_j^{i-1}\cap (R_{i,D}\setminus R_{i,ND})) \leq$$

$$\sum_{j\geq i}\mathcal{H}^m\big((\partial_v A_j\cap \operatorname{int}(Q_i))\setminus \bigcup_{i'\in\mathcal{I}_{max}^{i-0.5}} Q_{i'}\big) + \sum_{i'\in\mathcal{I}_{max,ND}}\sum_{j\geq i}\mathcal{H}^m(\partial_v A_j^{i-1}\cap Q_{i'}) +$$

$$\sum_{i'\in\mathcal{I}_{max,D}}\sum_{j\geq i}\mathcal{H}^m(A_j\cap \partial Q_{i'}) \leq$$

$$\sum_{j\geq i}\mathcal{H}^m\big((\partial A_j\cap \operatorname{int}(Q_i))\setminus \bigcup_{i'\in\mathcal{I}_{max}^{i-0.5}} Q_{i'}\big) +$$

$$\sum_{i'\in\mathcal{I}_{max,ND}} 2c_{F,m+1}\sum_{j\geq i}\mathcal{H}^m(\partial A_j\cap \operatorname{int}(Q_{i'})) +$$

$$\sum_{i'\in\mathcal{I}_{max,D}} c_{F,m+1}\sum_{j\geq i}\mathcal{H}^m(\partial A_j\cap \operatorname{int}(Q_{i'})) \leq$$

$$\sum_{j\geq i} 2c_{F,m+1}\mathcal{H}^m(\partial A_j\cap \operatorname{int}(Q_i)). \tag{29}$$

Assume first that Q_i is a non-defect cube, that is,

$$\sum_{j>i}\mathcal{H}^m(\partial A_j\cap \operatorname{int}(Q_i)) > 2c_{F,m+1}||Q_i||_{m+1}. \tag{30}$$

Then there exists a smallest $j>i$ for which $Q_i \subsetneq Q_j$, denote this j by j_i. A moments's reflection shows that there is such a j_i, indeed, if there is no such j_i then $\partial A_j\cap \operatorname{int}(Q_i)=\emptyset$ for all $j>i$, which contradicts (30).

Assume $T(Q_i)=[t_1,t_2]$.

For $t\in[t_1,t_2]$ denote by $H_{\geq t}$ the closed halfspace consisting of points $(x_1,\dots,x_m,t')$ with $t'\geq t$ and set $H_{\leq t}=\operatorname{cl}(\mathbb{R}^{m+1}\setminus H_{\geq t})$. If

$$\sum_{j\geq i}\mathcal{H}^m(\partial_v A_j^{i-1}\cap \operatorname{int}(Q_i)) \geq 2c_{F,m+1}\mathcal{H}^m(\partial A_i\cap \operatorname{int}(Q_i)) + 2||Q_i||_{m+1} =$$

$$2c_{F,m+1}||A_i||_{m+1} + 2||Q_i||_{m+1}$$

then choose $t'\in[t_1,t_2]$ such that

$$\sum_{j\geq i}\mathcal{H}^m(\partial_v A_j^{i-1}\cap H_{\geq t'}\cap \operatorname{int}(Q_i)) = 2c_{F,m+1}\mathcal{H}^m(\partial A_i\cap \operatorname{int}(Q_i)) + 2||Q_i||_{m+1}$$

one can do this since the lefthandside of this equality is a continuous function of t', this is an advantage of using ∂_v instead of ∂.

Otherwise set $t'=t_1$. In both cases if we set

$$A_i^{i,r}\stackrel{\text{def}}{=}(A_i^{i-1}\ominus Q_i)\cup\cup_{j\geq i}(A_j^{i-1}\odot(Q_i\cap H_{\geq t'})) \tag{31}$$

then

$$\mathcal{H}^m(\partial_v A_i^{i,r} \cap \text{int}(Q_i)) \leq 2c_{F,m+1}\mathcal{H}^m(\partial A_i \cap \text{int}(Q_i)) + 2||Q_i||_{m+1} \tag{32}$$

holds and hence

$$\begin{aligned}\mathcal{H}^m(\partial_v A_i^{i,r} \cap Q_i) =& \mathcal{H}^m(\partial_v A_i^{i,r} \odot Q^{down}) < \\ & 2c_{F,m+1}\mathcal{H}^m(\partial A_i \cap \text{int}(Q_i)) + 3||Q_i||_{m+1}.\end{aligned} \tag{33}$$

On the other hand, set

$$A_{j_i}^{i,r} \stackrel{\text{def}}{=} (A_{j_i}^{i-1} \ominus Q_i) \cup \cup_{j \geq i}(A_j^{i-1} \odot (Q_i \cap H_{\leq t'}))$$

and $A_j^{i,r} = A_j^{i-1} \ominus Q_i$ for $j > i$, $j \neq i_i$. Clearly, for almost every t

$$\cup_{j \geq i}\partial A_{j,t}^{i,r} \cap \text{int}(Q_{i,t}) \subseteq \cup_{j \geq i}\partial A_{j,t}^{i-1} \cap \text{int}(Q_{i,t})$$

and hence

$$\sum_{j \geq i} \mathcal{H}^m(\partial_v A_j^{i,r} \cap \text{int}(Q_i)) \leq \sum_{j \geq i} \mathcal{H}^m(\partial_v A_j^{i-1} \cap \text{int}(Q_i)).$$

If $t' \neq t_1$ by using (28)-(29) it follows that

$$\sum_{j > i} \mathcal{H}^m(\partial_v A_j^{i,r} \cap \text{int}(Q_i)) \leq$$

$$\left(\sum_{j \geq i} \mathcal{H}^m(\partial_v A_j^{i,r} \cap \text{int}(Q_i))\right) - 2c_{F,m+1}\mathcal{H}^m(\partial A_i \cap \text{int}(Q_i)) - 2||Q_i||_{m+1} \leq$$

$$\left(\sum_{j \geq i} \mathcal{H}^m(\partial_v A_j^{i-1} \cap \text{int}(Q_i))\right) - 2c_{F,m+1}\mathcal{H}^m(\partial A_i \cap \text{int}(Q_i)) - 2||Q_i||_{m+1} \leq$$

$$\left(\sum_{j \geq i} 2c_{F,m+1}\mathcal{H}^m(\partial A_j \cap \text{int}(Q_i))\right) - 2c_{F,m+1}\mathcal{H}^m(\partial A_i \cap \text{int}(Q_i)) - 2||Q_i||_{m+1} =$$

$$\sum_{j > i} 2c_{F,m+1}\mathcal{H}^m(\partial A_j \cap \text{int}(Q_i)) - 2||Q_i||_{m+1}.$$

From (21) it follows that

$$\sum_{j > i} \mathcal{H}^m(\partial_v A_j^i \cap \text{int}(Q_i)) = \sum_{j > i} \mathcal{H}^m(\partial_v A_j^{i,r} \cap \text{int}(Q_i)) <$$

$$\sum_{j > i} 2c_{F,m+1}\mathcal{H}^m(\partial A_j \cap \text{int}(Q_i)) - ||Q_i||_{m+1},$$

this is (24)-(26) with $i-1$ and i' both replaced by i. If $i' \in \mathcal{I}^i_{max} \setminus \{i\}$ then $Q_{i'}$ and Q_i are non-overlapping and we can easily see that

$$A^{i-1}_j \cap \mathrm{int}(Q_{i'}) = A^i_j \cap \mathrm{int}(Q_{i'}) \text{ holds for } j = i+1, ..., p \tag{34}$$

hence (23) or (24)-(26) will hold with $i-1$ replaced by i.

If $t' = t_1$ then $A^{i,r}_{j_i}$ also equals $A^{i-1}_{j_i} \ominus Q_i$ and

$$\sum_{j>i} \mathcal{H}^m(\partial_v A^{i,r}_j \cap Q_i) \le ||Q_i||_{m+1} < \sum_{j>i} \mathcal{H}^m(\partial A_j \cap \mathrm{int}(Q_i)) - ||Q_i||_{m+1},$$

where the last inequality holds since Q_i was assumed to be a non-defect cube and hence the negation of (22) holds. Thus again using (21)

$$\sum_{j>i} \mathcal{H}^m(\partial_v A^i_j \cap \mathrm{int}(Q_i)) = \sum_{j>i} \mathcal{H}^m(\partial_v A^{i,r}_j \cap \mathrm{int}(Q_i)) \le$$

$$\sum_{j>i} 2c_{F,m+1}\mathcal{H}^m(\partial A_j \cap \mathrm{int}(Q_i)) - ||Q_i||_{m+1}.$$

In case Q_i is a defect cube we set $A^{i,r}_i = (A^{i-1}_i \ominus Q_i) \cup (Q_i \odot \cup_{j \ge i} A^{i-1}_j)$ and for $j > i$ we put $A^{i,r}_j = A^{i-1}_j \ominus Q_i$. This will imply $\cup_{j>i} A^{i,r}_j \cap \mathrm{int}(Q_i) = \emptyset$. On the other hand, (22) and (28)-(29) imply

$$\mathcal{H}^m(\partial_v(A^{i,r}_i) \cap \mathrm{int}(Q_i)) \le \sum_{j \ge i} \mathcal{H}^m(\partial_v(A^{i-1}_j) \cap \mathrm{int}(Q_i)) \le \tag{35}$$

$$\sum_{j \ge i} 2c_{F,m+1}\mathcal{H}^m(\partial A_j \cap \mathrm{int}(Q_i)) \le$$

$$2c_{F,m+1}\mathcal{H}^m(\partial A_i \cap \mathrm{int}(Q_i)) + 2c_{F,m+1}||Q_i||_{m+1}.$$

Again, one can easily see (as at (34)) that (23) and (24)-(26) will hold with $i-1$ replaced by i.

It is also obvious that $\mathcal{P}^{i,r}$ is $\otimes$-equivalent to $\mathcal{P}^i$.

5.8 Swapping, general plan

We assume that at step i the "spreaded" h-partition $\mathcal{P}^{i,r} = \{(A^{i,r}_j, Q^*_j)\}^p_{j=1}$ is given. We put $Q^{up}_i = \Psi_{aff}(Q_i)$, $A^{i,r,u}_j = A^{i,r}_j \odot Q^{up}_i$, $A^{i,r,d}_j = A^{i,r}_j \odot Q_i$.

We assume that from the earlier steps of our construction it follows that for the h-partition $\mathcal{P}^{i,r}_< = \{(A^{i,r}_j, Q^*_j)\}^{i-1}_{j=1}$ the sections $\mathcal{P}^{i,r}_{<,t}$ are η^*-regular m-dimensional 64^ν-net partitions for almost all t.

From (20) it follows that for $j \ge i$ we have $A^{i,r,u}_j = \Psi_{aff}(A^H_j)$. It also follows from the spreading algorithm that if $t \in T(A^{i,r,d}_i)$ then $A^{i,r}_{j,t} \odot Q^*_i = \emptyset$ holds for all $j > i$ and by (33) or (35)

$$\mathcal{H}^m(\partial_v(A^{i,r}_i) \cap Q_i) \le 2c_{F,m+1}\mathcal{H}^m(\partial A_i \cap \mathrm{int}(Q_i)) + 3c_{F,m+1}||Q_i||_{m+1} = \tag{36}$$

$$2c_{F,m+1}||A_i||_{m+1} + 3c_{F,m+1}||Q_i||_{m+1} \le \left(\frac{2}{\eta_1} + 3\right) c_{F,m+1}||Q_i||_{m+1} \tag{37}$$

where at (37) we used the η_1-regularity of A_i.

Set $T_{down} = T(A_i^{i,r,d})$. Then $|T_{down}|_1 \leq \ell(Q_i)$. Since $A_i^{i,r}$ is a figure T_{down} consists of the union of some closed intervals.

Set $T_{up} = T(A_i^{i,r,u})$, again T_{up} also consists of the union of finitely many intervals.

By $A_i^{i,r,u} = \Psi_{aff}(A_i^H)$ and by the choice of A_i^H, (12) implies

$$|T_{up}|_1 \geq \frac{c_{aff}}{2}\frac{\ell(Q_i)}{2^\nu} = c_{sw}\ell(Q_i) \tag{38}$$

where $c_{sw} \in \mathbb{N}$ is the swapping constant satisfying

$$c_{sw} > \frac{2(\frac{2}{\eta_1}+6)c_{F,m+1}c_{\mathcal{H},m}||[0,1]^{m+1}||_{m+1}}{c_{\mathcal{H},m-1}||[0,1]^m||_m} = \frac{2(\frac{2}{\eta_1}+6)c_{F,m+1}c_{\mathcal{H},m}||Q_i||_{m+1}}{\ell(Q_i)c_{\mathcal{H},m-1}||Q_i^*||_m}. \tag{39}$$

Furthermore, by the choice of the heart system there exists $\widetilde{Q}_i \in 2^\nu CHDRN(Q_i)$ such that $A_i^H \subseteq \widetilde{Q}_i$ and hence if $\widetilde{Q}_i^u = \Psi_{aff}(\widetilde{Q}_i)$ then $A_i^{i,r,u} = \Psi_{aff}(A_i^H) \subseteq \widetilde{Q}_i^u$. For all $t \in T_{up}$ we also have

$$|A_{i,t}^{i,r,u}|_m > (1-\eta_1)|\widetilde{Q}_i^*|_m > 0.9|\widetilde{Q}_i^*|_m > 2^{-\nu m-1}|Q_i^*|_m$$

and

$$||A_{i,t}^{i,r,u}||_m < ||Q_i^*||_m,$$

by (13)-(15) of the choice of the heart system.

Since $A_j^{i,r}$, $j = 1,...,p$ are all figures it is not difficult to see that one can choose finitely many non-overlapping intervals $I_1,....,I_\kappa$, and $J_1,....,J_\kappa$ such that

- $\cup_{k=1}^{\kappa} I_k = T_{down}$,
- for each $k = 1,...,\kappa$ there exist possibly empty m-dimensional figures $B_j^{k,d}$ and $B_j^{k,u}$ for $j = 1,...,p$ such that

  $$A_j^{i,r} \odot (Q_i^* \times I_k) = B_j^{k,d} \times I_k \text{ and}$$

  $$A_j^{i,r} \odot (Q_i^* \times J_k) = B_j^{k,u} \times J_k.$$

  This means that we choose the intervals I_k and J_k so that the horizontal slices of $A_j^{i,r}$ in $Q_i^* \times I_k$ and $Q_i^* \times J_k$ can be obtained as a direct product of the m-dimensional figures $B_j^{k,d}$ and $B_j^{k,u}$ by I_k and J_k, respectively.
- Set $L_k = [||B_i^{k,d}||_m/||Q_i^*||_m] + 3$ where [.] denotes integer part. Then we can choose our system so that $|J_k|_1 = L_k|I_k|_1$. Actually, this property needs some justification and is based on estimate (40) below which says that the length of T_{up} is sufficiently large.

In this paragraph we give an estimate why (40) can be satisfied. By the choice of L_k we have

$$L_k|I_k|_1||Q_i^*||_m \leq |I_k|_1 3||Q_i^*||_m + |I_k|_1||B_i^{k,d}||_m \leq$$

$$|I_k|_1 3||Q_i^*||_m + \frac{c_{\mathcal{H},m}}{c_{\mathcal{H},m-1}}\mathcal{H}^m(\partial_v A_i^{i,r} \cap Q_i \cap (\mathbb{R}^m \times I_k))$$

and hence using (36)-(37) we have

$$\sum_{k=1}^{\kappa} L_k|I_k|_1||Q_i^*||_m \leq \ell(Q_i)3||Q_i^*||_m + \frac{c_{\mathcal{H},m}}{c_{\mathcal{H},m-1}}\mathcal{H}^m(\partial_v A_i^{i,r} \cap Q_i) \leq$$

$$\frac{c_{\mathcal{H},m}}{c_{\mathcal{H},m-1}}\left(\frac{2}{\eta_1}+6\right)c_{F,m+1}||Q_i||_{m+1}$$

and by (39)

$$|T_{up}|_1||Q_i^*||_m \geq c_{sw}\ell(Q_i)||Q_i^*||_m > 2\frac{c_{\mathcal{H},m}}{c_{\mathcal{H},m-1}}\left(\frac{2}{\eta_1}+6\right)c_{F,m+1}||Q_i||_{m+1}.$$

Therefore

$$2\sum_{k=1}^{\kappa}|J_k|_1 = 2\sum_{k=1}^{\kappa}L_k|I_k|_1 < |T_{up}|_1. \tag{40}$$

Our construction implies that for $j > i$ both $B_j^{k,u}$ and $B_j^{k,d}$ are empty. Since $\ell(Q_j)$ is nondecreasing in j from $B_j^{k,u} \neq \emptyset$, or $B_j^{k,d} \neq \emptyset$ it follows that $Q_j \subseteq Q_i$. Denote by $\mathcal{J}_k$ the set of those $j \neq i$ for which $B_j^{k,u} \neq \emptyset$, or $B_j^{k,d} \neq \emptyset$.

It is clear that for $j \in \mathcal{J}_k \cup \{i\}$

$$B_j^{k,d} \times I_k = (\mathbb{R}^m \times I_k) \odot A_j^{i,r}, \text{ and}$$

$$B_j^{k,u} \times J_k = (\mathbb{R}^m \times J_k) \odot A_j^{i,r}.$$

Hence our assumption about $\mathcal{P}_<^{i,r}$ implies that for $j \in \mathcal{J}_k$, $B_j^{k,u}$ or $B_j^{k,d}$ are η^*-regular m-dimensional figures.

We set $\mathcal{J}_k^u = \{j \in \mathcal{J}_k : B_j^{k,u} \neq \emptyset\}$ and $\mathcal{J}_k^d = \{j \in \mathcal{J}_k : B_j^{k,d} \neq \emptyset\}$. Clearly, $\{(B_j^{k,u}, Q_j^*)\}_{j \in \mathcal{J}_k^u}$ and $\{(B_j^{k,d}, Q_j^*)\}_{j \in \mathcal{J}_k^d}$ are m-dimensional 64^ν-partitions.

Next we split J_k into L_k many non-overlapping closed intervals of equal length. Denote these intervals by $L_{k,h}$, $h = 1, ..., L_k$. Observe that $|L_{k,h}|_1 = |I_k|_1$ for all h.

Our next purpose is to choose m-dimensional figures in Q_i^*, denoted by $B_j^{k,h}$ for $j \in \mathcal{J}_k \cup \{i\}$ and $h = 0, 1, ..., L_k$, such that if we let

$$A_j^{k,swap} = (B_j^{k,0} \times I_k) \cup (\cup_{k=1}^{L_k} B_j^{k,h} \times J_{k,h})$$

then the h-partition $\mathcal{P}^{k,swap} = \{(A_j^{k,swap}, Q_j^*)\}_{j\in\mathcal{J}_k\cup\{i\}}$ is $\otimes$-equivalent to the partition

$$\mathcal{P}^{i,r,k} = \{((B_j^{k,d} \times I_k) \cup (B_j^{k,u} \times J_k), Q_j^*)\}_{j\in\mathcal{J}_k\cup\{i\}}.$$

Where one can observe that

$$(B_j^{k,d} \times I_k) \cup (B_j^{k,u} \times J_k) = A_j^{i,r} \odot (Q_i^* \times (I_k \cup J_k)).$$

We will furthermore assume that $\mathcal{P}_t^{k,swap}$ will be η^*-regular for almost all t. It is clear that if we can achieve the above goal for all k's then we can define

$$A_j^i = (A_j^{i,r} \ominus (Q_i^* \times \cup_k (I_k \cup J_k))) \cup \cup_k A_j^{k,swap}$$

for all $j = 1, ..., i$, and set $A_j^i = A_j^{i,r}$ for $j = i+1, ..., p$. Then $\mathcal{P}^i = \{(A_j^i, Q_j^*)\}_{j=1}^p$ will be the required partition. Indeed, by (20) we have (16) with i replaced by $i+1$. We need to check whether we have (23) and (24)-(26) for $i' \in \mathcal{I}_{\max}^i$ when in these formulae $i-1$ is replaced by i. Observe that $i \in \mathcal{I}_{\max}^i$ and $i' \neq i$ is in $\mathcal{I}_{\max}^i$ if and only if $i' \in \mathcal{I}_{\max}^{i-1}$ and $Q_{i'} \not\subseteq Q_i$. Our spreading step in Section 5.7 implies that we have (23) and (24)-(26) if $i' = i$. On the other hand, $A_i^i \cap \mathrm{int}(Q_{i'}) = \emptyset$ if $i' \in \mathcal{I}_{\max}^i \setminus \{i\}$ and $A_j^{i-1} \cap \mathrm{int}(Q_{i'}) = A_j^i \cap \mathrm{int}(Q_{i'})$ for $j > i$ easily implies (23) and (24)-(26) with $i-1$ replaced by i. It also follows that $\mathcal{P}^i$ is $\otimes$-equivalent to $\mathcal{P}^{i-1}$ and $\mathcal{P}_{\leq}^i$ is an η^*-regular, 64^ν-net h-partition. Thus to complete our proof we need to see how to do the swapping.

5.9 Swapping algorithm for a fixed k

We assume that k is fixed and need to define $B_j^{k,h}$ and $B_j^{k,0,h}$ for $j \in \mathcal{J}_k \cup \{i\}$ and $h = 0, ..., L_k$.

We define the sets $B_j^{k,h}$ by an algorithm, we call this algorithm the swapping algorithm.

We set $B_j^{k,0,0} = B_j^{k,d}$.

Our algorithm has L_k steps and provided $B_j^{k,0,h-1}$ is defined we define $B_j^{k,h}$ and $B_j^{k,0,h}$ at step h. In fact, we do a sort of swapping betwen $B_j^{k,0,h-1}$ and $B_j^{k,u}$ and the name comes from this procedure.

In the end we set $B_j^{k,0} \stackrel{\text{def}}{=} B_j^{k,0,L_k}$.

We assume that $B_j^{k,0,h-1}$ is being defined for $j \in \mathcal{J}_k \cup \{i\}$. We denote by $\mathcal{J}_k^{0,h-1}$, or by $\mathcal{J}_k^u$ the set of those $j \in \mathcal{J}_k$ for which $B_j^{k,0,h-1} \neq \emptyset$, or $B_j^{k,u} \neq \emptyset$, respectively. We assume that $(B_j^{k,0,h-1}, Q_j^*)$ is η^*- regular for each $j \in \mathcal{J}_k^{0,h-1}$ and $\{(B_j^{k,0,h-1}, Q_j^*)\}_{j\in\mathcal{J}_k^{0,h-1}}$ forms an m-dimensional 64^ν-net partition. This property is obviously true if $h = 1$.

We separate two cases. Case A is when

$$\mathcal{H}^{m-1}(\partial B_i^{k,0,h-1} \cap \mathrm{int}(Q_i^*)) \geq 3||Q_i^*||_m \tag{41}$$

and we deal with case B if the above inequality does not hold.

First we discuss case A, which will also be split into two subcases.

Since (41) holds we can choose an m-dimensional 4^ν-net subcube $Q^h \subseteq Q_i^*$ such that

$$\mathcal{H}^{m-1}(\partial B_i^{k,0,h-1} \cap \mathrm{int}(Q^h)) \geq 3||Q_i^*||_m,$$

but for any $Q \in 4^\nu CHDRN(Q^h)$ we have

$$\mathcal{H}^{m-1}(\partial B_i^{k,0,h-1} \cap \mathrm{int}(Q)) < 3||Q_i^*||_m.$$

Since Q^h has $4^{\nu m}$ children the above inequality implies

$$\mathcal{H}^{m-1}(\partial B_i^{k,0,h-1} \cap \mathrm{int}(Q^h)) < 4 \cdot 4^{\nu m}||Q_i^*||_m. \tag{42}$$

Since $B_i^{k,0,h-1}$ is a figure one can see that such a Q^h exists.

Next we separate two subcases.

If $Q^h \neq Q_i^*$ then we deal with case AA. If $Q^h = Q_i^*$ then we deal with case AB.

In case of subcase AA we denote by $\mathcal{J}_k^{h,u}$ the set of those $j \in \mathcal{J}_k^u$ for which $Q_j^* \supset Q^h$. Due to the net partition property the sizes of the Q_j^*'s are different for different $j \in \mathcal{J}_k^{h,u}$ and each $Q_j^* \subsetneq Q_i^*$. We also recall that $(B_j^{k,u}, Q_j^*)$ are η^*-regular for all $j \in \mathcal{J}_k^{h,u}$. Hence, by (7)

$$\sum_{j \in \mathcal{J}_k^{h,u}} ||B_j^{k,u}||_m \leq \frac{1}{\eta^*} \sum_{j \in \mathcal{J}_k^{h,u}} ||Q_j^*||_m \leq \frac{1}{\eta^*} \sum_{j=1}^{\infty} ||Q_i^*||_m 64^{-j\nu(m-1)} < \frac{||Q_i^*||_m}{4}. \tag{43}$$

Similarly, we denote by $\mathcal{J}_k^{h,d}$ the set of those $j \in \mathcal{J}_k^{0,h-1}$ for which $Q_j^* \supset Q^h$. Arguing as above we obtain

$$\sum_{j \in \mathcal{J}_k^{h,d}} ||B_j^{k,u}||_m \leq \frac{1}{\eta^*} \sum_{j=1}^{\infty} ||Q_i^*||_m 64^{-j\nu(m-1)} < \frac{||Q_i^*||_m}{4}. \tag{44}$$

Set $\mathcal{J}_k^h = \mathcal{J}_k^{h,u} \cup \mathcal{J}_k^{h,d}$. For $j \in \mathcal{J}_k \setminus \mathcal{J}_k^h$ we swap the contents of Q^h, that is, we set

$$B_j^{k,h} = (B_j^{k,u} \ominus Q^h) \cup (B_j^{k,0,h-1} \odot Q^h),$$

and

$$B_j^{k,0,h} = (B_j^{k,0,h-1} \ominus Q^h) \cup (B_j^{k,u} \odot Q^h).$$

Observe that by our choice of $\mathcal{J}_k^h$ for $j \in \mathcal{J}_k \setminus \mathcal{J}_k^h$ the cubes Q_j^* and Q^h are either non-overlapping, or $Q_j^* \subseteq Q^h$, this means that in the first case $B_j^{k,h} = B_j^{k,h-1}$ and $B_j^{k,0,h} = B_j^{k,0,h-1}$, or in the second case, we swap the contents of Q^h, that is, $B_j^{k,h} = B_j^{k,0,h-1}$ and $B_j^{k,0,h} = B_j^{k,h-1}$. Thus $(B_j^{k,h}, Q_j^*)$ and $(B_j^{k,0,h}, Q_j^*)$ are η^*-regular.

If $j \in \mathcal{J}_k^h$ then we set $B_j^{k,h} = B_j^{k,0,h} = \emptyset$. The pieces $B_j^{k,u}$, or $B_j^{k,0,h-1}$ will be added to $B_i^{k,u}$, or to $B_i^{k,0,h-1}$, respectively. Then we do the swapping. Thus we first set

$$B_{i,swap}^{k,u} \stackrel{\text{def}}{=} B_i^{k,u} \cup \bigcup_{j \in \mathcal{J}_k^h} B_j^{k,u} = B_i^{k,u} \cup \bigcup_{j \in \mathcal{J}_k^{h,u}} B_j^{k,u},$$

and

$$B_{i,swap}^{k,0,h-1} \stackrel{\text{def}}{=} B_i^{k,0,h-1} \cup \bigcup_{j \in \mathcal{J}_k^h} B_j^{k,0,h-1} = B_i^{k,0,h-1} \cup \bigcup_{j \in \mathcal{J}_k^{h,d}} B_j^{k,0,h-1},$$

observe that for $j \in \mathcal{J}_k^h \setminus \mathcal{J}_k^{h,u}$ we have $B_j^{k,u} = \emptyset$ and for $j \in \mathcal{J}_k^h \setminus \mathcal{J}_k^{h,d}$ we have $B_j^{k,0,h-1} = \emptyset$; hence the second equality holds in the above definitions. Let

$$B_i^{k,h} = (B_{i,swap}^{k,u} \ominus Q^h) \cup (B_{i,swap}^{k,0,h-1} \odot Q^h),$$

and

$$B_i^{k,0,h} = (B_{i,swap}^{k,0,h-1} \ominus Q^h) \cup (B_{i,swap}^{k,u} \odot Q^h).$$

It is clear that by our choices the h-partitions

$$\{((B_j^{k,0,h-1} \times I_k) \cup (B_j^{k,u} \times J_{k,h}), Q_j^*)\}_{j \in \mathcal{J}_k \cup \{i\}}, \tag{45}$$

and

$$\{((B_j^{k,0,h} \times I_k) \cup (B_j^{k,h} \times J_{k,h}), Q_j^*)\}_{j \in \mathcal{J}_k \cup \{i\}} \tag{46}$$

are $\otimes$-equivalent. It is also obvious that almost all horizontal sections of the elements of the above partitions for $j \in \mathcal{J}_k$ are η^*-regular. We recall that by (14) of the choice of the heart system $||B_i^{k,u}||_m < ||Q_i^*||_m$. We need to estimate the regularity properties of $(B_i^{k,h}, Q_i^*)$ and obtain a perimeter estimate for $B_i^{k,0,h-1}$:

$$||B_i^{k,h}||_m \leq ||Q^h||_m + ||B_{i,swap}^{k,u}||_m + \sum_{j \in \mathcal{J}_k^{h,d}} ||B_j^{k,0,h-1}||_m +$$

$$\mathcal{H}^{m-1}(B_i^{k,0,h-1} \cap \operatorname{int}(Q^h)) \leq \frac{||Q_i^*||_m}{4^{\nu(m-1)}} + ||B_i^{k,u}||_m + \sum_{j \in \mathcal{J}_k^{h,u}} ||B_j^{k,u}||_m +$$

$$\sum_{j \in \mathcal{J}_k^{h,d}} ||B_j^{k,0,h-1}||_m + 4 \cdot 4^{\nu m} ||Q_i^*||_m < 8 \cdot 4^{\nu m} ||Q_i^*||_m = \frac{||Q_i^*||}{\eta^*}.$$

Since Q^h is a 4^ν netsubcube of Q_i^* and $\widetilde{Q}_i^*$ is a 2^ν child of Q_i^* we have $|Q^h|_m < |\widetilde{Q}_i^*|_m/4$ and hence

$$|B_i^{k,h}|_m > |B_i^{k,u}|_m - |Q^h|_m > \frac{1}{2}|\widetilde{Q}_i^*|_m - \frac{|\widetilde{Q}_i^*|_m}{4} = \frac{|\widetilde{Q}_i^*|_m}{4} = \frac{|Q_i^*|_m}{4 \cdot 2^{\nu m}} > \eta^* |Q_i^*|_m.$$

Next we estimate $||B_i^{k,0,h}||_m$.

Clearly,

$$||B_i^{k,0,h}||_m \leq ||B_{i,swap}^{k,0,h-1}||_m - \mathcal{H}^{m-1}(\partial B_i^{k,0,h-1} \cap \operatorname{int}(Q^h)) + ||Q^h||_m + ||B_i^{k,u}||_m +$$

$$\sum_{j \in \mathcal{J}_k^{h,u}} ||B_j^{k,u}||_m \leq ||B_i^{k,0,h-1}||_m + \sum_{j \in \mathcal{J}_k^{h,d}} ||B_j^{k,0,h-1}||_m - 3||Q_i^*||_m + ||Q^h||_m +$$

$$||Q_i^*||_m + \sum_{j \in \mathcal{J}_k^{h,u}} ||B_j^{k,u}||_m \leq ||B_i^{k,0,h-1}||_m - ||Q_i^*||_m,$$

that is, at this step $||B_i^{k,0,h-1}||_m$ is strictly decreased by at least $||Q_i^*||_m$.

In case we arrive to an h when we have case AB, or case B then we terminate the algorithm in two steps. Our choice of L_k implies that this will happen since at each step $||B_i^{k,0,h}||_m$ is decreased by at least $||Q_i^*||_m$.

So after this step we will define $B_j^{k,h'}$, and $B_j^{k,0,h'}$ for $h' = h, h+1, ..., L_k$ and $j \in \mathcal{J}_k \cup \{i\}$. In fact, we define $B_j^{k,h}$, $B_j^{k,0,h}$, $B_j^{k,h+1}$, $B_j^{k,0,h+1}$ and then we set $B_j^{k,h'} = B_j^{k,u}$, $B_j^{k,0,h'} = B_j^{k,0,h+1}$ for all $h' = h+2, ..., L_k$ and $j \in \mathcal{J}_k \cup \{i\}$.

Dealing with Case AB from $Q^h = Q_i^*$ and from (42) it follows that

$$\mathcal{H}^{m-1}(\partial B_i^{k,0,h-1} \cap \operatorname{int}(Q_i^*)) < 4 \cdot 4^{\nu m} ||Q_i^*||_m. \tag{47}$$

It is also clear that in Case B the negation of (41) implies (47), in this Case we also set $Q^h = Q_i^*$.

Next, by a hyperplane perpendicular to the m'th unit vector split Q^h into two nonoverlapping intervals R^h and R^{h+1}, so that $|R^h \cap \widetilde{Q}_i^*|_m = |\widetilde{Q}_i^*|_m/2$ and $|R^{h+1} \cap \widetilde{Q}_i^*|_m = |\widetilde{Q}_i^*|_m/2$, that is, both R^h and R^{h+1} contain exact halves of $\widetilde{Q}_i^*$ and $Q^h = Q_i^* = R^h \cup R^{h+1}$. Since $\widetilde{Q}_i^*$ was a 2^ν-child of $\widetilde{Q}_i$ it is clear that R^h and R^{h+1} are both unions of 64^ν-children of Q_i^* and hence for $j \in \mathcal{J}_k$ if they overlap with a Q_j^* then they contain it.

For $j \in \mathcal{J}_k \cup \{i\}$ set

$$B_j^{k,h} = (B_j^{k,u} \ominus R^h) \cup (B_j^{k,0,h-1} \odot R^h),$$

$$B_j^{k,0,h} = (B_j^{k,0,h-1} \ominus R^h) \cup (B_j^{k,u} \odot R^h),$$

and

$$B_j^{k,h+1} = (B_j^{k,u} \ominus R^{h+1}) \cup (B_j^{k,0,h} \odot R^{h+1}),$$

$$B_j^{k,0,h+1} = (B_j^{k,0,h} \ominus R^{h+1}) \cup (B_j^{k,u} \odot R^{h+1});$$

again in this case $(B_j^{k,h}, Q_j^*)$, $(B_j^{k,h+1}, Q_j^*)$, $(B_j^{k,0,h}, Q_j^*)$, $(B_j^{k,0,h+1}, Q_j^*)$ are all η^*-regular for $j \in \mathcal{J}_k$. Also observe that

$$B_i^{k,0,h+1} = (B_i^{k,u} \odot R^h) \cup (B_i^{k,u} \odot R^{h+1}) = B_i^{k,u}.$$

Hence $(B_i^{k,0,h+1}, Q_i^*)$ is also η^*-regular.

Now,

$$||B_i^{k,h}||_m \leq ||R^h||_m + ||B_i^{k,u}||_m + ||B_i^{k,0,h-1}||_m \leq$$
$$3||Q_i^*||_m + \mathcal{H}^{m-1}(\partial B_i^{k,0,h-1} \cap \mathrm{int}(Q_i^*)) < 8 \cdot 4^{\nu m}||Q_i^*||_m$$

and a similar estimation shows $||B_i^{k,h+1}||_m < 8 \cdot 4^{\nu m}||Q_i^*||_m$.

On the other hand, $B_i^{k,h} \supset B_i^{k,u} \ominus R^h$. Since R^h cuts $\widetilde{Q}_i^*$ in half and $B_i^{k,u} \subseteq \widetilde{Q}_i^*$ and $|B_i^{k,u}|_m \geq 0.9|\widetilde{Q}_i^*|_m$ we have

$$|B_i^{k,u} \ominus R^h|_m \geq 0.4|\widetilde{Q}_i^*|_m = \frac{0.4|Q_i^*|_m}{2^{\nu m}}.$$

Thus $|B_i^{k,h}|_m > |Q_i^*|_m/2^{\nu m+2}$. A similar estimation shows that $|B_i^{k,h+1}|_m > |Q_i^*|_m/2^{\nu m+2}$. This implies that $(B_i^{k,h}, Q_i^*)$ and $(B_i^{k,h+1}, Q_i^*)$ are both η^*-regular.

This completes our discussion of the swapping algorithm.

6 Details of the proof of Lemma 3 when $m = 1$

Why do we need a different argument?
The main problem is that in this case the surface measure is atomic and we cannot use estimates like (43) or (7). Indeed, in these estimates $m - 1 = 0$ yields an infinite sum. On the other hand, to resolve this problem we can take advantage of the very simple structure of one-dimensional BV sets.

6.1 One-dimensional preliminaries

One-dimensional BV sets coincide (modulo sets of zero Lebesgue measure) with finite unions of closed intervals, that is, with one-dimensional figures. Recalling that the zero-dimensional Hausdorff measure is the counting measure if the BV set $A \subseteq \mathbb{R}$ is equivalent to $\cup_{j=1}^k I_j$ (where the I_j's are disjoint closed intervals) then the essential boundary of A coincides with the endpoints of the intervals I_j and $||A||_1 = 2k$ (the number of the endpoints of these intervals).

If a one-dimensional BV set $A \subseteq \mathbb{R}$ is η-regular then

$$d(A) \geq |A|_1 > \eta d(A)||A||_1 \text{ and hence } 1/\eta \geq ||A||_1, \tag{48}$$

that is, A is equivalent to a finite union of less than $[1/2\eta]$ many closed intervals. Since $||A||_1 \geq 2$ we also have

$$|A|_1 > \eta d(A)||A||_1 \geq 2\eta d(A). \tag{49}$$

6.2 First steps

Arguing as in the m-dimensional case ($m > 1$) we choose η_0, ϵ^*, δ^*, δ, and γ as in Section 5.2 and obtain a two-dimensional partition $\mathcal{P}_{0,0}$ for which $(F \otimes \mathcal{L})(\mathcal{P}_{0,0}) > \gamma/2$. Like in Sections 5.2 and 5.4 using dyadic nets in the plane we can again choose $\eta_1 > 0$ and an η_1-regular, δ/η_1-fine, two-dimensional partition

$\mathcal{P}_{0,1}$ such that $(|F|\otimes\mathcal{L})(\mathcal{P}_{0,1}) > \gamma\eta_1/4$ and $Q_j \neq Q_{j'}$ for $j \neq j'$, again from now on if we talk about net-partitions we assume this property.

As we remarked earlier (7) does not apply when $m = 1$. However, the other statements of Theorem 9 hold in this case as well. And choose ν according to this Theorem.

In the one-dimensional case we do not have to use 64^ν-net partitions, instead we select like in Section 5.4 a subpartition $\mathcal{P} = \{(A_j, Q_j)\}_{j=1}^p$ of $\mathcal{P}_{0,1}$ such that it is a fixed η_1-regular, δ/η_1-fine, 4^ν-net two-dimensional partition, satisfying $(|F| \otimes \mathcal{L})(\mathcal{P}) > \gamma\eta_1/8\nu$. Again, without limiting generality we assume that $Q_j \subseteq Q^{down} = [0,1]^2$.

Next we choose the heart system.

We choose A'_j and A_j^H as in Section 5.5 but now from (11) it follows that for $t \in T_{j,H}$

$$\mathcal{H}^0(\mathrm{int}(\widetilde{Q}_{j,t}) \cap \partial A_{j,t}^H) = \mathcal{H}^0(\mathrm{int}(\widetilde{Q}_{j,t}) \cap \partial A'_{j,t}) < \frac{|T(\widetilde{Q}_j)|_1^0}{2} < \frac{1}{2}.$$

This means that for these t's $\mathrm{int}(\widetilde{Q}_{j,t}) \cap \partial A_{j,t}^H = \emptyset$ and (13) implies $|A_{j,t}^H|_1 > 0$. Hence

$$A_{j,t}^H = \widetilde{Q}_{j,t} \tag{50}$$

and we will also have property (15).

Since $\widetilde{Q}_j$ is a two-dimensional 2^ν-net children of Q_j it is clear that, for $t \in T(A_j^H)$, $A_{j,t}^H = \widetilde{Q}_{j,t} = \widetilde{Q}_j^*$ consists of the union of $4^\nu/2^\nu = 2^\nu$-many one-dimensional 4^ν-net children of $Q_{j,t} = Q_j^*$.

6.3 Preparation to one-dimensional Spreading and Swapping

We say that $\mathcal{P} = \{(A_j, Q_j^*)\}_{j=1}^p$ is an h-partition if for almost all $t \in \mathbb{R}$ the horizontal section $\mathcal{P}_t = \{(A_{j,t}, Q_j^*)\}_{j=1}^p$ is a one-dimensional 4^ν-net partition, again we can assume that from $j' \neq j$ and $Q_{j'}^* = Q_j^*$ it follows that for almost all t at least one of $A_{j',t} = \emptyset$, or $A_{j,t} = \emptyset$.

For our one-dimensional swapping algorithm we will need a property which is stronger than regularity. Assume that $(A_{j,t}, Q_j^*)$ is a given partition element. We denote by $\Gamma^*(A_{j,t})$ the number of those $Q \in 4^\nu CHDRN(Q_j^*)$ for which $Q \subseteq A_{j,t}$. The union of these components will be denoted by $HEART(A_{j,t})$. We also put $\Gamma(A_{j,t}) = \min\{\Gamma^*(A_{j,t}), 2^\nu\}$.

We say that a one-dimensional partition element $(A_{j,t}, Q_j^*)$ is *normal* if either $A_{j,t} = \emptyset$, or $\Gamma(A_{j,t}) \geq 1$ and

$$||A_{j,t}||_1 \leq 64 \cdot 2^\nu \cdot \Gamma(A_{j,t}) \leq 64 \cdot 4^\nu. \tag{51}$$

From $\Gamma(A_{j,t}) \geq 1$ it follows

$$|A_{j,t}|_1 \geq |Q_j^*|/4^\nu.$$

The above inequalities imply that $(A_{j,t}, Q_j^*)$ is $\eta^* \stackrel{\text{def}}{=} 1/64 \cdot 4^\nu$-regular. We say that $\mathcal{P}$ is normal if for each j and for almost every t the partition element $(A_{j,t}, Q_j^*)$ is normal.

In our one-dimensional spreading and swapping we will "essentially" replace our original two-dimensional, η_1-regular 4^ν-net partition $\mathcal{P}$ with another $\otimes$-equivalent h-partition $\mathcal{P}'$ such that for almost all t the horizontal sections $\mathcal{P}'_t$ are one-dimensional normal 4^ν-net partitions. This will make it possible for us to estimate $|F(\mathcal{P}'_t)|$ from above for almost all t's.

To do our swapping algorithm we need the one-dimensional heart system defined previously. We also use an integer constant $c_{aff} = 2c_{sw}$, where the integer constant c_{sw} is chosen in (39) and depends only on η_0 (in fact, it depends on η_1, but η_1 depends on η_0).

Denote by Ψ_{aff} the affine transformation which maps $Q^{down} = [0,1]^2$ onto $Q^{up} \stackrel{\text{def}}{=} [0,1] \times [1, 1 + c_{aff}]$ and equals the identity on the first coordinate axis. Choose the heart system $\{(A_j, Q_j)\}_{j=1}^p$ as it was given in Section 6.2 and set

$$A_j^0 \stackrel{\text{def}}{=} A_j \cup \Psi_{aff}(A_j^H), \text{ and } Q_j^0 = Q_j \cup \Psi_{aff}(Q_j)$$

for each j and consider the h-partition $\mathcal{P}^0 = \{(A_j^0, Q_j^*)\}_{j=1}^p$. Clearly,

$$\frac{\gamma\eta_1}{8\nu} < (|F| \otimes \mathcal{L})(\mathcal{P}^0). \tag{52}$$

Without limiting generality we assume that our partition is indexed in a nondecreasing order of the size of the Q_j's, that is, for $j < j'$ we have $\ell(Q_j) \leq \ell(Q_{j'})$.

In the sequel we will define the h-partitions $\mathcal{P}^i = \{(A_j^i, Q_j^*)\}_{j=1}^p$ by induction so that the A_j^i's are two-dimensional figures, $\mathcal{P}^i$ is $\otimes$-equivalent to $\mathcal{P}^0$ and if we let $\mathcal{P}^i_{\leq} = \{(A_j^i, Q_j^*)\}_{j=1}^i$ then it is a normal 4^ν-net h-partition.

Assume that we have defined $\mathcal{P}^p = \{(A_j^p, Q_j^*)\}_{j=1}^p$ such that it is a normal 4^ν-net h-partition.

Since $\mathcal{P} = \{(A_j, Q_j)\}_{j=1}^p$ was a δ/η_1-fine two-dimensional partition each Q_j contains a point x_j such that $Q_j \subseteq B_2(x_j, \delta(x_j)/\eta_1)$. Then each Q_j^* contains the point $x_j^* = \pi_1(x_j)$ such that $Q_j \subseteq B_1(x_j^*, \delta^*(x_j^*))$ and hence $A_{j,t} \subseteq B_1(x_j^*, \delta^*(x_j^*))$ for all t. Again for η^* we can choose $\eta_0^* > 0$ such that $(A_{j,t} \cup \{x_j^*\}, x_j^*)_{j=1}^p$ will be an η_0^*-regular m-dimensional BV-partition for almost every t and by (3) $|F|(\mathcal{P}_t^p) < \epsilon^*$ holds. This implies

$$\frac{\gamma\eta_1}{8\nu} < (|F| \otimes \mathcal{L})(\mathcal{P}^p) \leq \int_0^{1+c_{aff}} |F|(\mathcal{P}_t^p)dt < \epsilon^*(1 + c_{aff})$$

where the constant c_{aff} depends only on m and η_0. Thus choosing ϵ^* so that $\epsilon^*(1 + c_{aff}) < \gamma\eta_1/8\nu$ we obtain a contradiction to (52) which concludes the proof of Lemma 5.

Next we show how to obtain the partitions $\mathcal{P}^i$. Before turning to the details we say a few words about the construction of these partitions.

We assume again that (16)-(19) hold.

Given $\mathcal{P}^{i-1} = \{(A_j^{i-1}, Q_j^*)\}_{j=1}^p$ we do the spreading step as it was done in Section 5.6 and obtain $\mathcal{P}^{i,r} = \{(A_j^{i,r}, Q_j^*)\}_{j=1}^p$, the spreaded partition. We have $A_j^{i,r} = A_j^{i-1}$ for $j < i$, $\mathcal{P}^{i,r}$ and $\mathcal{P}^{i-1}$ are $\otimes$-equivalent and for almost all t $A_{j,t}^{i,r} \odot Q_i^* \neq \emptyset$ can hold for at most one $j \geq i$.

We can start to argue for the general plan of swapping like in Section 5.8.

We also know that $\widetilde{Q}_i \in 2^\nu CHDRN(Q_i)$, $A_i^H \subseteq \widetilde{Q}_i$ if $\widetilde{Q}_i^u \stackrel{\text{def}}{=} \Psi_{aff}(\widetilde{Q}_i)$ then $A_i^{i,r,u} = \Psi_{aff}(A_i^H) \subseteq \widetilde{Q}_i^u$. For $t \in T_{up}$, $A_{i,t}^{i,r,u} = \widetilde{Q}_{i,t}^u$ which means that $|A_{i,t}^{i,r,u}|_1 = |Q_i^*|_1$ and $||A_{i,t}^{i,r,u}||_1 = 2$. Set

$$L_k = \frac{||B_i^{k,d}||_1}{||Q_i^*||_1} = \frac{||B_i^{k,d}||_1}{2}$$

and choose the nonoverlapping intervals $I_1, ..., I_\kappa$ and $J_1, ..., J_\kappa$ as in Section 5.8 only the word η^*-regular should be replaced by normal and we use 4^ν-net instead of 64^ν-net.

6.4 Swapping algorithm for a fixed k when $m = 1$

We assume that k is fixed and need to define $B_j^{k,0,h}$ and $B_j^{k,h}$ for $j \in \mathcal{J}_k \cup \{i\}$ and $h = 0, ..., L_k$.

We set $B_j^{k,0,0} = B_j^{k,d}$. Our algorithm has L_k steps and provided $B_j^{k,0,h-1}$ is defined we define $B_j^{k,h}$ and $B_j^{k,0,h}$ at step h.

In the end we set $B_j^{k,0} \stackrel{\text{def}}{=} B_j^{k,0,L_k}$.

We assume that $B_j^{k,0,h-1}$ is being defined for $j \in \mathcal{J}_k \cup \{i\}$ and $(B_j^{k,0,h-1}, Q_j^*)$ is normal for each $j \in \mathcal{J}_k$.

If Q is a 4^ν-net subinterval of Q_i^* we define

$$\text{Exc}_{0,h-1}(Q) \stackrel{\text{def}}{=} ||B_i^{k,0,h-1} \odot Q||_1 + \sum_{j \in \mathcal{J}_k,\, Q \subsetneq Q_j} ||B_j^{k,0,h-1} \odot Q||_1,$$

and

$$\text{Exc}_u(Q) \stackrel{\text{def}}{=} ||B_i^{k,u} \odot Q||_1 + \sum_{j \in \mathcal{J}_k,\, Q \subsetneq Q_j} ||B_j^{k,u} \odot Q||_1.$$

Before we get into the details we say a few informal words about our procedure. Our notation Exc comes from the word excess and this is the "excess" surface in Q which needs to be "kept alive" as we do the swapping. Of course if $\text{Exc}_{0,h-1}(Q)$ is bigger than $\text{Exc}_u(Q)$ then we can "move up" to $B_i^{k,h}$ some surface of $B_i^{k,0,h-1}$. We will be interested in the difference $\text{Exc}_{0,h-1} - \text{Exc}_u$ since this is the surface which we can move up. Of course we cannot move up too much surface because the heart of $B_i^{k,u}$ has a "capacity" of surface not exceeding $64 \cdot 4^\nu$, at least if we want to get a normal partition element $(B_i^{k,h}, Q_i^*)$. We need to handle some difficulties, for example, by swapping the contents of the interval Q we might destroy some heart intervals of some normal elements $(B_j^{k,0,h-1}, Q_j^*)$, or of $(B_j^{k,u}, Q_j^*)$. Here it will be useful that by such a move we are decreasing only by one $\Gamma(B_j^{k,0,h-1})$, or $\Gamma(B_j^{k,u})$. The components, which we cannot "keep alive" by $B_j^{k,0,h-1}$ or by $B_j^{k,u}$ will be "reassigned," and kept alive by the heart of $B_i^{k,h}$.

What will we mean by reassignment of components? Assume that we have two nonoverlapping figures B and B'. We say that the nonoverlapping figures $\widetilde{B}$ and $\widetilde{B}'$ are obtained from B and B' by a reassignment of components, if $\widetilde{B} \cup \widetilde{B}' = B \cup B'$ and the connected components of $\widetilde{B}$, or $\widetilde{B}'$ are unions of some (possibly one) components of B, or of B' (for example we also allow that some adjacent components of B and B' are united to form one component of B', or a component of B' may become part of $\widetilde{B}$). Clearly $||\widetilde{B}||_1 + ||\widetilde{B}'||_1 \leq ||B||_1 + ||B'||_1$.

If the 4^ν-net children of Q are denoted by Q_s, $s = 1, ..., 4^\nu$, then

$$\begin{aligned}\mathrm{Exc}_{0,h-1}(Q) + \sum_{j \in \mathcal{J}_k, Q_j = Q} ||B_j^{k,0,h-1} \odot Q||_1 \leq \sum_{s=1}^{4^\nu} \mathrm{Exc}_{0,h-1}(Q_s) \leq \\ \mathrm{Exc}_{0,h-1}(Q) + 4 \cdot 4^\nu + \sum_{j \in \mathcal{J}_k, Q_j = Q} ||B_j^{k,0,h-1} \odot Q||_1\end{aligned} \tag{53}$$

where at the upper estimate the term $4 \cdot 4^\nu$ takes into consideration the endpoints of the intervals Q_s, $s = 1, ..., 4^\nu$. Also observe that there is at most one $j \in \mathcal{J}_k$ for which $Q_j = Q$. A similar estimate holds for Exc_u:

$$\begin{aligned}\mathrm{Exc}_u(Q) + \sum_{j \in \mathcal{J}_k, Q_j = Q} ||B_j^{k,u} \odot Q||_1 \leq \sum_{s=1}^{4^\nu} \mathrm{Exc}_u(Q_s) \leq \\ \mathrm{Exc}_u(Q) + 4 \cdot 4^\nu + \sum_{j \in \mathcal{J}_k, Q_j = Q} ||B_j^{k,u} \odot Q||_1.\end{aligned} \tag{54}$$

Lemma 10. *Using the above notation assume that $R_0 = Q_i^*$ and for $q = 1, ..., \alpha$ the nested intervals R_q belong to $4^\nu CHDRN(R_{q-1})$ and $\mathrm{Exc}_{0,h-1}(R_q) \geq 6$ for $q = 0, ..., \alpha$. Then we can redefine $\{(B_j^{k,0,h-1}, Q_j^*)\}_{j \in \mathcal{J}_k \cup \{i\}}$ to obtain the 4^ν-net partition $\{(B_j^{k,0,h-1,\alpha}, Q_j^*)\}_{j \in \mathcal{J}_k \cup \{i\}}$ such that*

- *i)* $||B_j^{k,0,h-1,\alpha}||_1 \leq ||B_j^{k,0,h-1}||_1$ *for all* $j \in \mathcal{J}_k$,
- *ii)* $HEART(B_j^{k,0,h-1,\alpha}) \supseteq HEART(B_j^{k,0,h-1})$ *for all* $j \in \mathcal{J}_k$,
- *iii)* $B_j^{k,0,h-1,\alpha} = B_j^{k,0,h-1}$ *if* $R_\alpha \not\subseteq Q_j^*$, *or* $R_\alpha = Q_j^*$,
- *iv)* $\displaystyle\bigcup_{j \in \mathcal{J}_k \cup \{i\},\ R_\alpha \subseteq Q_j^*} B_j^{k,0,h-1,\alpha} = \bigcup_{j \in \mathcal{J}_k \cup \{i\},\ R_\alpha \subseteq Q_j^*} B_j^{k,0,h-1}$,
- *v) either* $||B_i^{k,0,h-1,\alpha}||_1 \leq ||B_i^{k,0,h-1}||_1 - 2$, *or* $B_i^{k,0,h-1,\alpha}$ *has at least two components fully contained in* R_α.

Properties iii) and iv) of the above lemma will obviously imply that $\{(B_j^{k,0,h-1}\times I_k, Q_j^*)\}_{j\in\mathcal{J}_k\cup\{i\}}$ is $\otimes$-equivalent to $\{(B_j^{k,0,h-1,\alpha}\times I_k, Q_j^*)\}_{j\in\mathcal{J}_k\cup\{i\}}$. From i) and ii) it follows that if the m-dimensional net-partition element $(B_j^{k,0,h-1}, Q_j^*)$ is normal then $(B_j^{k,0,h-1,\alpha}, Q_j^*)$ will also be normal for $j\in\mathcal{J}_k$.

Proof. We do induction on α. If $\alpha=0$ set $B_j^{k,0,h-1,\alpha}=B_j^{k,0,h-1}$ and the Lemma obviously holds.

Assume that for $\alpha-1$ and for each $j\in\mathcal{J}_k\cup\{i\}$ we can choose $B_j^{k,0,h-1,\alpha-1}$ which satisfies i)-v) with $\alpha-1$ used instead of α.

If $||B_i^{k,0,h-1,\alpha-1}||_1\le||B_i^{k,0,h-1}||_1-2$ then we set $B_j^{k,0,h-1,\alpha}=B_j^{k,0,h-1,\alpha-1}$ for all $j\in\mathcal{J}_k\cup\{i\}$ and we are done.

Next we assume $||B_i^{k,0,h-1,\alpha-1}||_1=||B_i^{k,0,h-1}||_1$. From iii) and iv) applied with $\alpha-1$ it follows that

$$\bigcup_{j\in\mathcal{J}_k\cup\{i\},\ R_\alpha\subsetneq Q_j^*} B_j^{k,0,h-1,\alpha-1}=\bigcup_{j\in\mathcal{J}_k\cup\{i\},\ R_\alpha\subsetneq Q_j^*} B_j^{k,0,h-1},$$

and from $\mathrm{Exc}_{0,h-1}(R_\alpha)\ge 6$ it follows that R_α contains in its interior at least 2 components of some of the figures $B_j^{k,0,h-1,\alpha-1}$, $j\in\mathcal{J}_k\cup\{i\}$, $R_\alpha\subsetneq Q_j^*$. From v) applied with $\alpha-1$ and from $||B_i^{k,0,h-1,\alpha-1}||_1=||B_i^{k,0,h-1}||_1$ it follows that $B_i^{k,0,h-1,\alpha-1}$ has at least 2 components fully contained in $R_{\alpha-1}$. One can easily see that we can reassign components of $\bigcup_{j\in\mathcal{J}_k\cup\{i\},\ R_\alpha\subsetneq Q_j^*} B_j^{k,0,h-1,\alpha-1}$ so that the new system $B_j^{k,0,h-1,\alpha}$ satisfies i)-v). We only remark that from $\mathrm{Exc}_{0,h-1}(R_\alpha)\ge 6$ it follows that R_α is not a subset of a $HEART(B_j^{k,0,h-1})$ for a $j\in\mathcal{J}_k\cup\{i\}$. $\square$

Now we return to our main discussion of the one-dimensional swapping. Clearly, $\mathrm{Exc}_{0,h-1}(Q_i^*)=||B_i^{k,0,h-1}||_1$ and $\mathrm{Exc}_u(Q_i^*)=||B_i^{k,u}||_1=2$.

First we assume

$$\mathrm{Exc}_{0,h-1}(Q_i^*)-\mathrm{Exc}_u(Q_i^*)<16\cdot 4^\nu, \tag{55}$$

we call this Case A.

From (55) it follows that

$$||B_i^{k,0,h-1}||_1<16\cdot 4^\nu+2. \tag{56}$$

Recall that $B_i^{k,u}=\widetilde{Q}_i^*$. Choose a closed interval $R_1^{k,h}$

$$|R_1^{k,h}\cap\widetilde{Q}_i^*|_1=|\widetilde{Q}_i^*|_1/2 \text{ and } |\widetilde{Q}_i^*\setminus R_1^{k,h}|_1=|\widetilde{Q}_i^*|_1/2. \tag{57}$$

Now we swap the contents of $R_1^{k,h}$. Set

$$B_j^{k,0,h}=(B_j^{k,0,h-1}\ominus R_1^{k,h})\cup(B_j^{k,u}\odot R_1^{k,h}),\text{and}$$

$$B_j^{k,h} = (B_j^{k,u} \ominus R_1^{k,h}) \cup (B_j^{k,0,h-1} \odot R_1^{k,h}) \text{ for } j \in \mathcal{J}_k \cup \{i\}.$$

From (56) and (57) it follows that $(B_i^{k,0,h}, Q_i^*)$ and $(B_i^{k,h}, Q_i^*)$ are both normal. Since for $j \in \mathcal{J}_k$, Q_j^* is a 4^ν-net subinterval of Q_i^* we can easily see that $Q_j^* \subseteq R_1^{k,h}$, or $Q_j^* \subseteq Q_i^* \ominus R_1^{k,h}$ and hence $(B_j^{k,0,h}, Q_j^*)$ and $(B_j^{k,h}, Q_j^*)$ will be normal for $j \in \mathcal{J}_k$. This means that in this case we can terminate our algorithm by setting $B_j^{k,0,h'} = B_j^{k,0,h}$, and $B_j^{k,h'} = B_j^{k,h}$ for all $h' = h+1, ..., L_k$. This concludes Case A.

Now we turn to Case B by assuming that the negation of (55) holds, that is,

$$\mathrm{Exc}_{0,h-1}(Q_i^*) - \mathrm{Exc}_u(Q_i^*) \geq 16 \cdot 4^\nu. \tag{58}$$

This case will be split into subcases. In the end of each of these subcases we have the followig goals

- α) to have the $\otimes$-equivalence of the net partitions (45) and (46),
- β) to obtain $(B_j^{k,0,h}, Q_j^*)$ and $(B_j^{k,h}, Q_j^*)$ which will be normal for each $j \in \mathcal{J}_k$,
- γ) to obtain a normal $(B_i^{k,h}, Q_i^*)$,
- δ) to have $||B_i^{k,0,h}||_1 \leq ||B_i^{k,0,h-1}||_1 - 2$.

Property δ) implies that in less than L_k steps we end up with Case A and then terminate our swapping algorithm.

From (58) it follows that $\mathrm{Exc}_{0,h-1}(Q_i^*) \geq 16 \cdot 4^\nu$ and this clearly implies that $Q_i^* \not\subseteq B_j^{k,0,h-1}$ for any $j \in \mathcal{J}_k \cup \{i\}$.

Since $B_j^{k,0,h-1}$ and $B_j^{k,u}$ are figures for all $j \in \mathcal{J}_k \cup \{i\}$ it is clear that there exists a large $n_{h,k}$ such that if Q is a 4^ν-net subinterval with $|Q|_1 < 4^{-n_{h,k}\nu}|Q_i^*|_1$ then $\mathrm{Exc}_{0,h-1}(Q) \leq 4$ and $\mathrm{Exc}_u(Q) \leq 4$. This means that we can choose a minimal 4^ν-net subinterval $Q_{\min}$ in Q_i^* such that

- a) $\mathrm{Exc}_{0,h-1}(Q_{\min}) - \mathrm{Exc}_u(Q_{\min}) \geq 16 \cdot 4^\nu$,
- b) if Q is a 4^ν-net child of $Q_{\min}$ then $\mathrm{Exc}_{0,h-1}(Q) - \mathrm{Exc}_u(Q) < 16 \cdot 4^\nu$,
- c) if $Q_{\min} \subseteq Q' \subseteq Q_i^*$ and Q' is a 4^ν-net subinterval of Q_i^* then
$$\mathrm{Exc}_{0,h-1}(Q') - \mathrm{Exc}_u(Q') \geq 16 \cdot 4^\nu. \tag{59}$$

Denote by $Q_{\min,s}$, $s = 1, ..., 4^\nu$ the 4^ν-net children of $Q_{\min}$.

Now we separate some cases.

- Case BA: There exists s such that
$$\mathrm{Exc}_{0,h-1}(Q_{\min,s}) - \mathrm{Exc}_u(Q_{\min,s}) \geq 6. \tag{60}$$

- Case BB: For every $s = 1, ..., 4^\nu$

$$\mathrm{Exc}_{0,h-1}(Q_{\min,s}) - \mathrm{Exc}_u(Q_{\min,s}) < 6. \tag{61}$$

If Case BA holds then setting $R_\alpha = Q_{\min,s}$ we can apply Lemma 10 and obtain a system $B_j^{k,0,h-1,\alpha}$ satisfying i)-v).

If $||B_i^{k,0,h-1,\alpha}||_1 \leq ||B_i^{k,0,h-1}||_1 - 2$ then we set $B_j^{k,0,h} = B_j^{k,0,h-1,\alpha}$ and $B_j^{k,h} = B_j^{k,u}$ for all $j \in \mathcal{J}_k \cap \{i\}$.

If $||B_i^{k,0,h-1,\alpha}||_1 = ||B_i^{k,0,h-1}||_1$ then $Q_{\min,s}$ contains in its interior at least two components of $B_i^{k,0,h-1,\alpha}$. From b) it follows that

$$\mathrm{Exc}_{0,h-1}(Q_{\min,s}) - \mathrm{Exc}_u(Q_{\min,s}) < 16 \cdot 4^\nu. \tag{62}$$

There is at most one such $j' \in \mathcal{J}_k \cup \{i\}$ for which $Q_{\min,s} \subseteq HEART(B_{j'}^{k,u})$. First we assume that there is no such j', this case will be denoted by BAA. The other one will be case BAB.

In Case BAA we swap *and reassign* the contents of $Q_{\min,s}$, this means that we choose $B_j^{k,0,h}$ and $B_j^{k,h}$ such that

$$B_j^{k,0,h} \ominus Q_{\min,s} = B_j^{k,0,h-1,\alpha} \ominus Q_{\min,s}, \tag{63}$$

$$B_j^{k,h} \ominus Q_{\min,s} = B_j^{k,u} \ominus Q_{\min,s}; \text{ for all } j \in \mathcal{J}_k \cup \{i\};$$

$$\bigcup_{j \in \mathcal{J}_k \cup \{i\}} B_j^{k,0,h} \odot Q_{\min,s} = \bigcup_{j \in \mathcal{J}_k \cup \{i\}} B_j^{k,u} \odot Q_{\min,s}, \tag{64}$$

$$\bigcup_{j \in \mathcal{J}_k \cup \{i\}} B_j^{k,h} \odot Q_{\min,s} = \bigcup_{j \in \mathcal{J}_k \cup \{i\}} B_j^{k,0,h-1,\alpha} \odot Q_{\min,s},$$

finally, if $Q_j^* \subseteq Q_{\min,s}$ then $B_j^{k,h} = B_j^{k,0,h-1,\alpha}$ and $B_j^{k,0,h} = B_j^{k,u}$.

From (60), (62) and from the existence of the two components of $B_i^{k,0,h-1,\alpha}$ completely inside $Q_{\min,s}$ it follows that we can define $B_j^{k,0,h}$ and $B_j^{k,h}$ so that

- $||B_i^{k,0,h}||_1 \leq ||B_i^{k,0,h-1,\alpha}||_1 - 2 = ||B_i^{k,0,h-1}||_1 - 2,$
- $||B_j^{k,0,h}||_1 \leq ||B_j^{k,0,h-1,\alpha}||_1, \ ||B_j^{k,h}||_1 \leq ||B_j^{k,u}||_1, \ j \in \mathcal{J}_k, \ Q_j^* \not\subseteq Q_{\min,s},$
- $HEART(B_j^{k,0,h}) = HEART(B_j^{k,0,h-1,\alpha})$ and $HEART(B_j^{k,h}) = HEART(B_j^{k,u})$ for $j \in \mathcal{J}_k, \ Q_j^* \not\subseteq Q_{\min,s}$.
- $B_i^{k,h}$ equals the union of $B_i^{k,u} \ominus Q_{\min,s}$ and some components of

  $$\cup_{j \in \mathcal{J}_k \cup \{i\}} B_j^{k,0,h-1,\alpha} \odot Q_{\min,s}.$$

  Since $B_i^{k,h} \ominus Q_{\min,s} = \widetilde{Q}_i^* \setminus Q_{\min,s}$, we have $\Gamma(B_i^{k,h}) \geq 2^\nu - 1$ and hence more than $(2^\nu - 2)64 \cdot 2^\nu$ components of $\cup_{j \in \mathcal{J}_k \cup \{i\}} B_j^{k,0,h-1,\alpha} \odot Q_{\min,s}$ can be added to $B_i^{k,u} \ominus Q_{\min,s}$ to obtain $B_i^{k,u}$.

It is not difficult to see that the above properties imply that we can satisfy α)-δ). This shows that we can deal with Case BAA.

Next we turn to case BAB. That is, we assume that there exists a $j' \in \mathcal{J}_k \cup \{i\}$ such that $Q_{\min,s} \subseteq HEART(B_{j'}^{k,u})$. We argue as in Case BAA and obtain (63)-(64). Then we only need to change our argument due to the fact that it may happen that $HEART(B_{j'}^{k,h}) \neq HEART(B_{j'}^{k,u})$, since we are swapping the contents of $Q_{\min,s}$, and we can destroy a heart interval. However we have $\Gamma(B_{j'}^{k,h}) \geq \Gamma(B_{j'}^{k,u}) - 1$. This means that we might need to reassign at most $64 \cdot 2^\nu$ components of $B_{j'}^{k,u}$ to $B_i^{k,h}$.

From (60), (62) and from the existence of the two components of $B_i^{k,0,h-1,\alpha}$ completely inside $Q_{\min,s}$ it follows that we can define $B_j^{k,0,h}$ and $B_j^{k,h}$ so that

- $||B_i^{k,0,h}||_1 \leq ||B_i^{k,0,h-1,\alpha}||_1 - 2 = ||B_i^{k,0,h-1}||_1 - 2$,
- $||B_j^{k,0,h}||_1 \leq ||B_j^{k,0,h-1,\alpha}||_1$, $||B_j^{k,h}||_1 \leq ||B_j^{k,u}||_1$, $j \in \mathcal{J}_k$, $Q_j^* \not\subseteq Q_{\min,s}$,
- $HEART(B_j^{k,0,h}) = HEART(B_j^{k,0,h-1,\alpha})$,
  $HEART(B_j^{k,h}) = HEART(B_j^{k,u})$ for $j \in \mathcal{J}_k \setminus \{j'\}$, $Q_j^* \not\subseteq Q_{\min,s}$, and $\Gamma(B_{j'}^{k,h}) \geq \Gamma(B_{j'}^{k,u}) - 1$,
- $B_i^{k,h}$ equals the union of $B_i^{k,u} \ominus Q_{\min,s}$, together with some components of
  $$\bigcup_{j \in \mathcal{J}_k \cup \{i\}} B_j^{k,0,h-1,\alpha} \odot Q_{\min,s}$$
  and some components of $B_{j'}^{k,u}$. Since $B_i^{k,h} \ominus Q_{\min,s} = \widetilde{Q}_i^* \setminus Q_{\min,s}$, we have $\Gamma(B_i^{k,h}) \geq 2^\nu - 1$ and hence more than $(2^\nu - 2)64 \cdot 2^\nu$ components of $\cup_{j \in \mathcal{J}_k \cup \{i\}} B_j^{k,0,h-1,\alpha} \odot Q_{\min,s}$ and $B_{j'}^{k,u}$ can be added to $B_i^{k,u} \ominus Q_{\min,s}$ to obtain $B_i^{k,u}$.

This concludes Case BAB.

If we deal with Case BB then adding (61) for all $s = 1, ..., 4^\nu$ and using (53), (54), and a) we obtain

$$6 \cdot 4^\nu > \sum_{s=1}^{4^\nu} (\mathrm{Exc}_{0,h-1}(Q_{\min,s}) - \mathrm{Exc}_u(Q_{\min,s})) \geq \mathrm{Exc}_{0,h-1}(Q_{\min}) +$$
$$\sum_{j \in \mathcal{J}_k,\ Q_j = Q_{\min}} ||B_j^{k,0,h-1} \odot Q_{\min}||_1 - \mathrm{Exc}_u(Q_{\min}) -$$
$$4 \cdot 4^\nu - \sum_{j \in \mathcal{J}_k,\ Q_j = Q_{\min}} ||B_j^{k,u} \odot Q_{\min}||_1 \geq$$
$$12 \cdot 4^\nu + \sum_{j \in \mathcal{J}_k,\ Q_j = Q_{\min}} (||B_j^{k,0,h-1} \odot Q_{\min}||_1 - ||B_j^{k,u} \odot Q_{\min}||_1).$$

Thus,

$$\sum_{j\in\mathcal{J}_k,\ Q_j=Q_{\min}} (||B_j^{k,u}\odot Q_{\min}||_1 - ||B_j^{k,0,h-1}\odot Q_{\min}||_1) > 6\cdot 4^\nu. \qquad (65)$$

From the net property it follows that there is at most one $j'\in\mathcal{J}_k$ for which $Q^*_{j'}=Q_{\min}$ and $B_{j'}^{k,u}\neq\emptyset$ and (65) implies that there is at least one such j'.

From a) it follows that for all $j\in\mathcal{J}_k\cup\{i\}$, $Q_{\min}\not\subseteq HEART(B_j^{k,0,h-1})$.

Assume there exists $j''\in\mathcal{J}_k$ such that $Q^*_{j''}=Q_{\min}$ and $\Gamma(B_{j'}^{k,u})\leq\Gamma(B_{j''}^{k,0,h-1})$ we call this Case BBA when this does not hold we have Case BBB.

Since $(B_j^{k,u},Q_j^*)$ is normal for all $j\in\mathcal{J}_k$ using (51) and (65)

$$\begin{aligned}&64\cdot 2^\nu\Gamma(B_{j''}^{k,0,h-1})\geq 64\cdot 2^\nu\Gamma(B_{j'}^{k,u})\geq ||B_{j'}^{k,u}\odot Q_{\min}||_1 = \\ &||B_{j'}^{k,u}||_1\geq ||B_{j''}^{k,0,h-1}||_1+6\cdot 4^\nu.\end{aligned} \qquad (66)$$

Now we apply Lemma 10 with $R_\alpha=Q_{\min}$ and obtain $B_j^{k,0,h-1,\alpha}$, satisfying i)-v). Observe that from iii) it follows that $B_{j''}^{k,0,h-1,\alpha}=B_{j''}^{k,0,h-1}$. If $||B_i^{k,0,h-1,\alpha}||_1=||B_i^{k,0,h-1}||_1$ then $||B_i^{k,0,h-1,\alpha}||_1$ has at least two components in the interior of $Q_{\min}$. Then these two components can be reassigned to $B_{j''}^{k,0,h-1,\alpha}$ to obtain $B_{j''}^{k,0,h}$. From (66) it follows that $B_{j''}^{k,0,h}$ will be normal. Since we reassigned two components of $B_i^{k,0,h-1,\alpha}$ the set $B_i^{k,0,h}$ we obtained satisfies

$$||B_i^{k,0,h}||_1\leq ||B_i^{k,0,h-1,\alpha}||_1-4=||B_i^{k,0,h-1}||_1-4.$$

For $j\in\mathcal{J}_k\setminus\{j''\}$ we set $B_j^{k,0,h}=B_j^{k,0,h-1,\alpha}$, and for all $j\in\mathcal{J}_k\cup\{i\}$ we set $B_j^{k,h}=B_j^{k,u}$.

If $||B_i^{k,0,h-1,\alpha}||_1\leq ||B_i^{k,0,h-1}||_1-2$ then we set $B_j^{k,0,h}=B_j^{k,0,h-1,\alpha}$ and $B_j^{k,h}=B_j^{k,u}$ for $j\in\mathcal{J}_k\cup\{i\}$. This shows how to deal with Case BBA.

Now we turn to Case BBB, that is we assume that there does not exist $j''\in\mathcal{J}_k$ such that $B_{j''}^{k,0,h-1}\neq\emptyset$, and $Q^*_{j''}=Q_{\min}$, or if there exists such a $j''\in\mathcal{J}_k$ then

$$\Gamma(B_{j'}^{k,u})>\Gamma(B_{j''}^{k,0,h-1}). \qquad (67)$$

Then we can choose $Q_{\min,s}\in 4^\nu CHDRN(Q_{\min})$ such that $Q_{\min,s}\subseteq HEART(B_{j'}^{k,u})$ but $Q_{\min,s}\not\subseteq HEART(B_{j''}^{k,0,h-1})$. (In case there is no $j''\in\mathcal{J}_k$ such that $B_{j''}^{k,0,h-1}\neq\emptyset$ and $Q^*_{j''}=Q_{\min}$ then we use here and later in this argument $j''=j'$ and $B_{j'}^{k,0,h-1}=\emptyset$, this is true since $Q^*_{j'}=Q_{\min}$ implies $B_{j'}^{k,0,h-1}=\emptyset$.)

Now $Q_{\min,s}\subseteq HEART(B_{j'}^{k,u})$ implies $\mathrm{Exc}_u(Q_{\min,s})=2$. This and (61) imply $\mathrm{Exc}_{0,h-1}(Q_{\min,s})<8$.

Next we apply Lemma 10 with $R_\alpha=Q_{\min}$.

Again if $||B_i^{k,0,h-1,\alpha}||_1 \leq ||B_i^{k,0,h-1}||_1 - 2$ then we set $B_j^{k,0,h} = B_j^{k,0,h-1,\alpha}$, and $B_j^{k,h} = B_j^{k,u}$ for $j \in \mathcal{J}_k \cup \{i\}$.

If $||B_i^{k,0,h-1,\alpha}||_1 = ||B_i^{k,0,h-1}||_1$ then $||B_i^{k,0,h-1,\alpha}||_1$ has at least two components in the interior of $Q_{\min}$, denote them by $COMP_1$ and $COMP_2$.

For $j \in \mathcal{J}_k \setminus \{j''\}$ set $B_j^{k,0,h} = B_j^{k,0,h-1,\alpha} \ominus Q_{\min,s}$.

We also put $B_i^{k,0,h} = (B_i^{k,0,h-1,\alpha} \ominus Q_{\min,s}) \setminus (COMP_1 \cup COMP_2)$. Observe that iii) of Lemma 10 implies that $B_{j''}^{k,0,h-1,\alpha} = B_{j''}^{k,0,h-1}$. Set $B_{j''}^{k,0,h} = B_{j''}^{k,0,h-1,\alpha} \cup Q_{\min,s} \cup COMP_1 \cup COMP_2$. Since $Q_{\min,s} \subseteq HEART(B_{j''}^{k,0,h})$ we have by (67) and (51) $\Gamma(B_{j''}^{k,0,h}) > \Gamma(B_{j''}^{k,0,h-1})$. Hence $(B_{j''}^{k,0,h}, Q_{j''}^*)$ is a normal partition element.

Now $\mathrm{Exc}_{0,h-1}(Q_{\min,s}) < 8$ implies that in $Q_{\min,s}$ we can choose a figure $E_{\min,s} \subseteq Q_{\min,s}$ such that $||E_{\min,s}||_1 < 8$ and

$$E_{\min,s} = \bigcup_{j \in \mathcal{J}_k,\ Q_{\min,s} \subsetneq Q_j} B_j^{k,0,h-1} \odot Q_{\min,s}.$$

For $j \in \mathcal{J}_k \setminus \{j'\}$ set

$$B_j^{k,h} = (B_j^{k,u} \ominus Q_{\min,s}) \cup ((B_j^{k,0,h-1} \odot Q_{\min,s}) \ominus E_{\min,s}).$$

Now $Q_{\min,s}$ is moved out of $B_{j'}^{k,u}$ but we also have $B_i^{k,u}$ with $||B_i^{k,u}||_1 = 2$, $\Gamma(B_i^{k,u}) = 2^\nu$. It is easy to see that we can choose $B_i^{k,h}$ and $B_{j'}^{k,h}$ such that $B_i^{k,h} \cup B_{j'}^{k,h} = ((B_i^{k,u} \cup B_{j'}^{k,u}) \ominus Q_{\min,s}) \cup E_{\min,s}$ and $(B_i^{k,h}, Q_i^*)$, and $(B_j^{k,h}, Q_{j'}^*)$ are both normal. This completes the one-dimensional swapping algorithm.

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