ON ABSOLUTE CONTINUITY

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We prove that in any dimension a variational measure associated with an additive continuous function is σ-finite whenever it is absolutely continuous. The one-dimensional version of our result was obtained in [1] by a different technique. As an application, we establish a simple and transparent relationship between the Lebesgue integral and the generalized Riemann integral defined in [7, Chapter 12]. In the process, we obtain a result (Theorem 4.1) involving Hausdorff measures and Baire category, which is of independent interest. As variations defined by BV sets coincide with those defined by figures [8], we restrict our attention to figures.

The set of all real numbers is denoted by $\mathbb{R}$, and the ambient space of this paper is $\mathbb{R}^m$ where $m \geq 1$ is a fixed integer. In $\mathbb{R}^m$ we use exclusively the metric induced by the maximum norm $|\cdot|$. The usual inner product of $x, y \in \mathbb{R}^m$ is denoted by $x \cdot y$, and $\mathbf{0}$ denotes the zero vector of $\mathbb{R}^m$. For an $x \in \mathbb{R}^m$ and $\varepsilon > 0$, we let

$$B_\varepsilon(x) = \left\{ y \in \mathbb{R}^m : |x - y| < \varepsilon \right\}.$$

The closure, interior, boundary, and diameter of a set $E \subset \mathbb{R}^m$ are denoted by $\mathrm{cl}\, E$, $\mathrm{int}\, E$, ∂E, and $d(E)$, respectively.

The Lebesgue measure in $\mathbb{R}^m$ is denoted by λ; however, for $E \subset \mathbb{R}^m$, we write $|E|$ instead of $\lambda(E)$. A set $E \subset \mathbb{R}^m$ with $|E| = 0$ is called *negligible*. We say sets $A, B \subset \mathbb{R}^m$ *overlap* if $|A \cap B| > 0$. Unless specified otherwise, the words "measure" and "measurable" as well as the expressions "almost all," "almost everywhere," and "absolutely continuous" always refer to the Lebesgue measure λ.

The s-dimensional Hausdorff measure in $\mathbb{R}^m$ is denoted by $\mathcal{H}^s$. A set $T \subset \mathbb{R}^m$ of σ-finite measure $\mathcal{H}^{m-1}$ is called *thin*. Thin sets will play an important role in our exposition.

Unless specified otherwise, a number is an extended real number, and a function is an extended real-valued function.

1 Charges

A *cell* is a compact nondegenerate subinterval of $\mathbb{R}^m$, and a *figure* is a finite (possibly empty) union of cells. If A and B are figures, then so are $A \cup B$,

$$A \odot B = \mathrm{cl}\,(\mathrm{int}\, A \cap \mathrm{int}\, B) \quad \text{and} \quad A \ominus B = \mathrm{cl}\,(A - B)\,.$$

Each figure A has the *perimeter* $\|A\| = \mathcal{H}^{m-1}(\partial A)$, and the *unit exterior normal* ν_A defined, in the obvious way, $\mathcal{H}^{m-1}$-almost everywhere on ∂A.

The family $\mathcal{F}$ of all figures is topologized as follows. For $n = 1, 2, \ldots$, topologize the families

$$\mathcal{F}_n = \bigl\{A \in \mathcal{F} : A \subset B_n(\mathbf{0}) \text{ and } \|A\| < n\bigr\}$$

by the metric $\rho(A, B) = \bigl|(A - B) \cup (B - A)\bigr|$, and give $\mathcal{F}$ the largest topology τ for which all inclusion maps $\mathcal{F}_n \hookrightarrow \mathcal{F}$ are continuous. It is easy to verify that the topology τ is Hausdorff, sequential [3, p. 78], but not metrizable. Moreover, τ is induced by a uniformity, and the completion of $(\mathcal{F}, \tau)$ is the space of all bounded BV sets [4, Section 5.1].

A *charge* is an additive τ-continuous real-valued function defined on the family $\mathcal{F}$. Explicitly, $F : \mathcal{F} \to \mathbb{R}$ is a charge if it satisfies the following two conditions.

Additivity: $F(A \cup B) = F(A) + F(B)$ for each pair of nonoverlapping figures A and B.

Continuity: Given $\varepsilon > 0$, there is an $\eta > 0$ such that $\bigl|F(C)\bigr| < \varepsilon$ for each figure $C \subset B(\mathbf{0}, 1/\varepsilon)$ with $\|C\| < 1/\varepsilon$ and $|C| < \eta$.

To illustrate the concept of charge, we give two important examples.

1. If f is a locally integrable function on $\mathbb{R}^m$, let $F(A) = \int_A f\, d\lambda$ for each figure A. Then F is a charge by the absolute continuity of the Lebesgue integral.

2. If v is a continuous vector field on $\mathbb{R}^m$, let $F(A) = \int_{\partial^* A} v \cdot \nu_A\, d\mathcal{H}^{m-1}$ for each figure A. Then F is a charge, called the *flux* of v [7, Proposition 11.2.8].

2 Derivates

Given an $\eta > 0$, we say a figure A is η-*regular* if

$$\frac{|A|}{[d(A)\|A\|} > \eta\,.$$

The closer η is to $1/(2m)$, the more cube-like are η-regular figures. In particular, no η-regular figure is empty.

Let $x \in \mathbb{R}^m$, and let F be a real-valued function defined on $\mathcal{F}$. For a positive $\eta < 1/(2m)$, let

$$\overline{D}_\eta F(x) = \inf_{\delta>0} \left[\sup_B \frac{F(B)}{|B|} \right]$$

where B is an η-regular figure with $x \in B$ and $d(B) < \delta$. The number

$$\overline{D} F(x) = \sup_{0<\eta<\frac{1}{2m}} \overline{D}_\eta F(x)$$

is called the *upper derivate* of F at x. Using an argument similar to [10, Chapter IV, Theorem 4.2], it is easy to show that the functions $\overline{D}_\eta F$ and $\overline{D}F$, defined on $\mathbb{R}^m$ in the obvious way, are measurable.

If $\overline{D}F(x) = -\overline{D}(-F)(x) \neq \pm\infty$, we denote this common value by $DF(x)$, and say F is *derivable* at x; the number $DF(x)$ is called the *derivate* of F at x. When $\overline{D}_\eta |F|(x) < +\infty$ for all positive $\eta < 1/(2m)$, we say that F is *almost derivable* at x (cf. [7, Section 11.7]); in particular, F is almost derivable at x whenever $\overline{D}|F|(x) < +\infty$. The term "almost derivable" is justified by the following result proved in [2, Theorem 3.3].

Theorem 2.1. *Let F be a charge, and let E be the set of all $x \in \mathbb{R}^m$ at which F is almost derivable. Then F is derivable at almost all $x \in E$.*

3 Variations

A *partition* is a collection (possibly empty) $P = \{(A_1, x_1), \dots, (A_p, x_p)\}$ where $A_1, \dots, A_p$ are nonoverlapping figures and $x_i \in A_i$ for $i = 1, \dots, p$. Given $\eta > 0$ and a nonnegative function δ on $E \subset \mathbb{R}^m$, the partition P is called

η-regular if each figure A_i is η-regular,

δ-fine if $x_i \in E$ and $d(A_i) < \delta(x_i)$ for $i = 1, \dots, p$,

in E if $\bigcup_{i=1}^p A_i \subset E$.

If P is δ-fine, then each x_i belongs to the set $\{x \in E : \delta(x) > 0\}$. A nonnegative real-valued function δ defined on a set $E \subset \mathbb{R}^m$ is called a *gage* on E if the *zero set* $Z_\delta = \{x \in E : \delta(x) = 0\}$ of δ is thin. The following proposition, established in [6], connects partitions with charges.

Proposition 3.1. *Let F be a charge, and let $0 < \varepsilon < 1/(2m)$. For any gage δ on a figure A, there is an ε-regular δ-fine partition $\{(A_1, x_1), \dots, (A_p, x_p)\}$ in A such that*

$$\left| F\left(A \ominus \bigcup_{i=1}^p A_i \right) \right| < \varepsilon\,.$$

Let F be a function on $\mathcal{F}$, and let $E \subset \mathbb{R}^m$. For $0 < \eta < 1/(2m)$, let

$$V_\eta F(E) = \inf_\delta \sup_P \sum_{i=1}^{p} \bigl|F(A_i)\bigr|$$

where δ is a gage on E and $P = \bigl\{(A_1, x_1), \ldots, (A_p, x_p)\bigr\}$ is an η-regular δ-fine partition, and

$$V_* F(E) = \sup_{0<\eta<\frac{1}{2m}} V_\eta F(E) .$$

It is easy to verify the functions $V_\eta F$ and $V_* F$, defined in the obvious way, are Borel regular measures in $\mathbb{R}^m$ [2, Lemma 4.6] that vanish on thin sets. The measure $V_* F$ is called the *critical variation* of F. For charges, the critical variation $V_* F$ extends the *classical variation* VF [10, Chapter III, Section 4] from figures to arbitrary subsets of $\mathbb{R}^m$. We state this explicitly in the following proposition proved in [2, Lemma 4.4].

Proposition 3.2. *If F is a charge, then $V_* F(A) = VF(A)$ for each figure A.*

If F is a function on $\mathcal{F}$ and $A \in \mathcal{F}$, we define a function $F \mathbin{\mathbf{L}} A$ on $\mathcal{F}$ by the formula $(F \mathbin{\mathbf{L}} A)(B) = F(A \odot B)$ for each $B \in \mathcal{F}$. Similarly, if μ is a measure in $\mathbb{R}^m$ and $A \subset \mathbb{R}^m$, we define a measure $\mu \mathbin{\mathbf{L}} A$ by the formula $(\mu \mathbin{\mathbf{L}} A)(B) = \mu(A \cap B)$ for each $B \subset \mathbb{R}^m$. Since the boundary of any figure is thin, it is easy to verify that

$$V_\eta(F \mathbin{\mathbf{L}} A) = (V_\eta F) \mathbin{\mathbf{L}} A \qquad \text{and} \qquad V_*(F \mathbin{\mathbf{L}} A) = (V_* F) \mathbin{\mathbf{L}} A$$

for each figure A. A *charge in a figure* A is a charge F with $F = F \mathbin{\mathbf{L}} A$. Similarly, a *measure in a set* $E \subset \mathbb{R}^m$ is a measure μ with $\mu = \mu \mathbin{\mathbf{L}} E$.

Recall from [10, Chapter III, Section 12] a charge F in a figure A is called *absolutely continuous* (abbreviated as AC) if given $\varepsilon > 0$, there is a $\Delta > 0$ such that

$$\sum_{B \in \mathcal{B}} \bigl|F(B)\bigr| < \varepsilon$$

for each finite collection $\mathcal{B}$ of nonoverlapping subfigures of A. Also recall the classical variation VF of an AC charge F is finite

Proposition 3.3. *A charge F in a figure A is AC if and only if $V_* F$ is AC and finite.*

Proof. Let F be AC and $\varepsilon > 0$. Choose a $\Delta > 0$ as in the definition above. Given a negligible set $E \subset A$, find an open set U so that $E \subset U$ and $|U| < \Delta$. There is a positive function δ on E such that $B_{\delta(x)}(x) \subset U$ for each $x \in E$. If $P = \bigl\{(A_1, x_1), \ldots, (A_p, x_p)\bigr\}$ is a δ-fine partition, then P is a partition in U by the choice of δ. Thus

$$\sum_{i=1}^{p} |A \odot A_i| \le \sum_{i=1}^{p} |A_i| \le |U| < \Delta ,$$

and we have

$$\sum_{i=1}^{p} \bigl|F(A_i)\bigr| = \sum_{i=1}^{p} \bigl|F(A \odot A_i)\bigr| < \varepsilon .$$

This implies $V_* F(E) \leq \varepsilon$ and, as ε is arbitrary, $V_* F(E) = 0$. Since $V_* F$ is a measure in A, it is AC. In view of Proposition 3.2,

$$V_* F(\mathbb{R}^m) = V_* F(A) = VF(A) < +\infty .$$

Conversely, assume $V_* F$ is AC and finite. Given $\varepsilon > 0$, a standard argument [9, Theorem 6.11] shows there is a $\Delta > 0$ such that $V_* F(E) < \varepsilon$ for each Borel set E with $|E| < \Delta$. Select a finite collection $\mathcal{B}$ of nonoverlapping subfigures of A with $|\bigcup \mathcal{B}| = \sum_{B \in \mathcal{B}} |B| < \Delta$. Proposition 3.2 implies

$$\sum_{B \in \mathcal{B}} \bigl|F(B)\bigr| \leq \sum_{B \in \mathcal{B}} VF(B) = \sum_{B \in \mathcal{B}} V_* F(B) = V_* F\left(\bigcup \mathcal{B}\right) < \varepsilon .$$

□

Corollary 3.4. *A charge F in a figure A is the indefinite Lebesgue integral of a function f on A if and only if $V_* F$ is AC and finite.*

4 The main result and its application

A *support* of a measure μ in $\mathbb{R}^m$ is the set

$$\operatorname{supp} \mu = \mathbb{R}^m - \bigcup \bigl\{ G \subset \mathbb{R}^m : G \text{ is open and } \mu(G) = 0 \bigr\} .$$

Clearly $\mu(U) > 0$ for each open set $U \subset \mathbb{R}^m$ with $U \cap \operatorname{supp} \mu \neq \emptyset$. Since $\mathbb{R}^m$ is Lindelöf, $\mu(\mathbb{R}^m - \operatorname{supp} \mu) = 0$. We say $E \subset \mathbb{R}^m$ is a *supporting set* if $|E| > 0$ and $E = \operatorname{supp} (\lambda \mathbin{\mathbf{L}} E)$.

In order to prove our main theorem, we need a technical result concerning Hausdorff measures. It reconciles a particular problem arising from the testy relationship between measure and category. We postpone its proof to Section 5.

Theorem 4.1. *Let $0 < s < m$, and let $E \subset \mathbb{R}^m$ be a measurable set with $|E| > 0$. There is a supporting compact set $C \subset E$ and a G_δ set $H \subset C$ satisfying the following conditions:*

1. *H is dense in C and $|H| = 0$;*
2. *given $Z \subset \mathbb{R}^m$ with $\mathcal{H}^s(Z) = 0$, we can find a supporting compact set $C_Z \subset C - Z$ so that $H \cap C_Z$ is dense in C_Z.*

Proposition 4.2. *Let F be a charge, and let $0 < \eta < 1/(2m)$. If $V_\eta F$ is absolutely continuous, then*

$$E = \bigl\{ x \in \mathbb{R}^m : \overline{D}_\eta |F|(x) = +\infty \bigr\}$$

is a negligible set.

Proof. Proceeding towards a contradiction, suppose $|E| > 0$. As E is measurable, there is a supporting compact set $C \subset E$ and a negligible G_δ set $H \subset C$ satisfying the conditions of Theorem 4.1 with $m-1 < s < m$. Choose a gage δ on H, and use condition 2 of Theorem 4.1 to find a supporting compact set $K \subset C - Z_\delta$ so that $D = H \cap K$ is dense in K; this is possible as $\mathcal{H}^s(Z_\delta) = 0$. Applying the Baire category theorem to the G_δ set D, find a $t > 0$ and an open set $U \subset \mathbb{R}^m$ so that $D \cap U \neq \emptyset$ and the set $D_t = \{x \in D \cap U : \delta(x) > t\}$ is dense in $D \cap U$ and hence in $K \cap U$.

The family $\mathcal{B}$ of all η-regular figures B with $|F(B)| \geq |B|/|K \cap U|$ and $d(B) < t$ is a Vitali's cover of $K \cap U$. Thus up to a negligible set, $K \cap U$ is covered by a disjoint sequence $\{B_i\}$ in $\mathcal{B}$. We may assume the interior of each B_i meets $K \cap U$, and select an $x_i \in B_i \cap D_t$. Now

$$\sum_{i=1}^{\infty} |F(B_i)| > \frac{1}{|K \cap U|} \sum_{i=1}^{\infty} |B_i| \geq 1\,,$$

and there is an integer $p \geq 1$ with $\sum_{i=1}^{p} |F(B_i)| > 1$. As $\{(B_1, x_1), \dots, (B_p, x_p)\}$ is an η-regular δ-fine partition, the arbitrariness of δ implies $V_\eta F(H) \geq 1$, a contradiction. $\square$

Corollary 4.3. *Let F be a charge. If $V_* F$ is AC, then F is derivable almost everywhere.*

Proof. If $V_* F$ is AC, then so is $V_\eta F$ for each positive $\eta < 1/(2m)$. Observe

$$E = \bigcup_{n=2m+1}^{\infty} \{x \in \mathbb{R}^m : \overline{D}_{1/n} F(x) = +\infty\}$$

is the set of all $x \in \mathbb{R}^m$ at which F is *not* almost derivable. Since E is negligible by Proposition 4.2, the corollary follows from Theorem 2.1. $\square$

Theorem 4.4. *Let F be a charge. If $V_* F$ is AC, then up to a thin set, $\mathbb{R}^m$ is the union of compact sets $K_1, K_2, \dots$ with $V_* F(K_n) < +\infty$ for $n = 1, 2, \dots$. In particular, $V_* F$ is σ-finite.*

Proof. By [2, Proposition 4.2 and Corollary 4.8], the equality

$$V_* F(E) = \int_E \overline{D}|F|(x)\, d\lambda(x)$$

holds for each measurable set $E \subset \mathbb{R}^m$. Corollary 4.3 implies $\overline{D}|F|(x) = |DF(x)| < +\infty$ almost everywhere. Thus $\mathbb{R}^m$ is the union of a negligible set and the sets

$$E_{n,k} = \{x \in B_n(\mathbf{0}) : \overline{D}|F|(x) \leq k\}\,.$$

As $V_* F(E_{n,k}) \leq k(2n)^m$ for $k, n = 1, 2, \dots$, the measure $V_* F$ is σ-finite.

We complete the argument by showing that, up to a thin set, each set $E \subset \mathbb{R}^m$ with $V_* F(E) < +\infty$ can be covered by closed sets $C_1, C_2, \dots$ with

$V_*F(C_n) < +\infty$. To this end, choose a positive $\eta < 1/(2m)$, and find a gage δ on E so that

$$\sum_{i=1}^{p} \bigl|F(A_i)\bigr| < V_*F(E) + 1$$

for each η-regular δ-fine partition $\bigl\{(A_1, x_1), \dots, (A_p, x_p)\bigr\}$. For $n = 1, 2, \dots$, let $E_n = \bigl\{x \in E : d(x) > 1/n\bigr\}$ and $C_n = \operatorname{cl} E_n$. Since $E = Z_\delta \cup \bigcup_{n=1}^{\infty} E_n$, it remains to show $V_*F(C_n) < +\infty$. Define a positive function δ_n on C_n by letting

$$\delta_n(x) = \begin{cases} \delta(x) & \text{if } x \in E_n, \\ \frac{1}{n} & \text{if } x \in C_n - E_n, \end{cases}$$

and choose an η-regular δ_n-fine partition $\bigl\{(B_1, y_1), \dots, (B_p, y_p)\bigr\}$. If $y_i \in C_n - E_n$, select an $x_i \in E_n$ and a cube K_i containing x_i. For $i = 1, ;p$, let

$$(A_i, x_i) = \begin{cases} (B_i, y_i) & \text{if } y_i \in E_n, \\ (B_i \cup K_i, x_i) & \text{if } y_i \in C_n - E_n. \end{cases}$$

Clearly, it is possible to choose x_i so close to y_i and K_i so small that $\bigl\{(A_1, x_1), \dots, (A_p, x_p)\bigr\}$ is an η-regular δ-fine partition, and

$$\sum_{i=1}^{p} \bigl|F(B_i)\bigr| < \sum_{i=1}^{p} \bigl|F(A_i)\bigr| + 1 < V_*F(E) + 2\,.$$

As η is arbitrary, we conclude $V_*F(C_n) \le V_*F(E) + 1$. □

We recall the definition of the *generalized Riemann integral* (abbreviated as *R-integral*) introduced in [7, Chapter 12, particularly Theorem 12.2.2].

Definition 4.5. A function f on a figure A is *R-integrable* whenever there is a charge F in A satisfying the following condition: given $\varepsilon > 0$, we can find a gage δ on A so that

$$\sum_{i=1}^{p} \bigl|f(x_i)|A_i| - F(A_i)\bigr| < \varepsilon$$

for each ε-regular δ-fine partition $\bigl\{(A_1, x_1), \dots, (A_p, x_p)\bigr\}$ in A. The charge F, which is uniquely determined by f, is called the *indefinite R-integral* of f.

Note that neither the R-integrability of f nor the indefinite R-integral of f depend on the values f takes on a negligible set. If f and g are R-integrable and have the same indefinite R-integral, then $f = g$ almost everywhere.

Theorem 4.6. *A charge F in a figure A is an indefinite R-integral of a function f on A if and only if V_*F is AC. In which case $DF = f$ almost everywhere in A, the function f is measurable, and $V_*F(E) = \int_E |f|\, d\lambda$ for each measurable set $E \subset A$.*

Proof. Suppose V_*F is AC, and denote by N the negligible set of all $x \in A$ at which F is not derivable (Corollary 4.3). Let

$$f(x) = \begin{cases} 0 & \text{if } x \in N, \\ DF(x) & \text{if } x \in A - N, \end{cases}$$

and choose a positive $\varepsilon < 1/(2m)$. There is a gage δ_N on N such that

$$\sum_{i=1}^{p} \left|F(A_i)\right| < \varepsilon$$

for each ε-regular δ_N-fine partition $\{(A_1, x_1), \dots, (A_p, x_p)\}$. Making N larger and δ_N smaller, we may assume $\partial A \subset Z_{\delta_N}$ and $B_{\delta_N(x)}(x) \subset A$ for each $x \in N \cap \operatorname{int} A$. Given $x \in A - N$, find a $\Delta_x > 0$ so that $B_{\Delta_x}(x) \subset A$, and

$$\left|\frac{F(B)}{|B|} - f(x)\right| < \varepsilon$$

for each ε-regular figure B with $x \in B$ and $d(B) < \Delta_x$. Define a gage δ on A by the formula

$$\delta(x) = \begin{cases} \delta_N(x) & \text{if } x \in N, \\ \Delta_x & \text{if } x \in A - N. \end{cases}$$

Let $P = \{(A_1, x_1), \dots, (A_p, x_p)\}$ be an ε-regular δ-fine partition, and note P is a partition in A by the choice of δ. From the inequality

$$\begin{aligned} \sum_{i=1}^{p} \left|f(x_i)|A_i| - F(A_i)\right| &= \sum_{x_i \in N} \left|F(A_i)\right| + \sum_{x_i \in A-N} \left|f(x_i)|A_i| - F(A_i)\right| \\ &< \varepsilon + \sum_{x_i \in A-N} \varepsilon |A_i| \le \varepsilon\big(1 + |A|\big) \end{aligned}$$

we deduce F is the indefinite R-integral of f. Since $DF = f$ almost everywhere, f is measurable by [10, Chapter IV, Theorem 4.2]. Finally [2, Theorem 4.7] implies $V_*F(E) = \int_E |f|$ for each measurable set $E \subset A$.

Conversely, suppose F is the indefinite R-integral of a function f on A, and select a negligible set $N \subset A$. With no loss of generality, we may assume $f(x) = 0$ for each $x \in N$. Given a positive $\varepsilon < 1/(2m)$, there is a gage δ on A such that

$$\sum_{i=1}^{p} \left|f(x_i)|A_i| - F(A_i)\right| < \varepsilon$$

for each ε-regular δ-fine partition $\{(A_1, x_1), \dots, (A_p, x_p)\}$ in A. Choose a gage Δ on N so that $\Delta \le \delta \restriction N$, $N \cap \partial A \subset Z_\Delta$, and $B_{\Delta(x)}(x) \subset A$ for each $x \in N \cap \operatorname{int} A$. Now if $Q = \{(B_1, y_1), \dots, (B_q, y_q)\}$ is an ε-regular Δ-fine partition, then Q is a δ-fine partition in A. Hence

$$\sum_{j=1}^{q} \left|F(B_j)\right| = \sum_{j=1}^{q} \left|f(y_j)|B_j| - F(B_j)\right| < \varepsilon\,,$$

and we conclude $V_*F(N) \leq \varepsilon$. The arbitrariness of ε implies V_*F is AC. $\square$

Theorem 4.6 in conjunction with Corollary 3.4 provides an elegant comparison of the generalized Riemann and Lebesgue integrals.

Corollary 4.7. *Let F be a charge in a figure A.*

- *If V_*F is AC, then F is the indefinite R-integral of DF.*
- *If V_*F is AC and finite, then F is the indefinite Lebesgue integral of DF.*

5 Proof of Theorem 4.1

Throughout this section a number s with $0 < s < m$, and an integer N with $N^{m-s} > 2$ remain fixed. For $n = 0, 1, \ldots$, the product

$$\prod_{i=1}^{m} \left[\frac{k_i}{N^n}, \frac{k_i + 1}{N^n} \right],$$

where $k_1, \ldots, k_m$ are integers, is called an *n-cube*. Denote by $\mathcal{Q}(n)$ the family of all n-cubes, and let $\mathcal{Q} = \bigcup_{n=0}^{\infty} \mathcal{Q}(n)$. For $i = 1, 2, \ldots$, select an integer

$$p_i \geq \frac{(m-s) + is}{m-s} \qquad \text{and let} \qquad \eta_i = \frac{2N^{m(p_i+i)}}{2N^{m(p_i+i)} - 1}$$

A set $C \subset \mathbb{R}^m$ is called *amiable* if it is compact and supporting, and if there are positive integers n_i such that for $i = 1, 2, \ldots$, we have $n_{i+1} > n_i + p_i + i$ and $|C \cap Q| \geq |Q|/\eta_i$ whenever $Q \in \mathcal{Q}(n_i)$ meets C. We prove Theorem 4.1 by showing that each measurable set of positive measure has an amiable subset, and that amiable sets have the desired properties.

Lemma 5.1. *Let $K \subset R^m$ be a compact set with $|K| > 0$ and let $k \geq 0$ be an integer. Suppose $Q_j \in \mathcal{Q}$ and $\nu_j > 1$ are chosen so that $|K \cap Q_j| > |Q_j|/\nu_j$ for $j = 1, \ldots, k$. If $\nu > 1$, then for each sufficiently large integer $n \geq 1$ there is a nonempty compact set $K_n \subset K$ such that $|K_n \cap Q_j| > |Q_j|/\nu_j$ for $j = 1, \ldots, k$, and $|K_n \cap Q| > |Q|/\nu$ for each $Q \in \mathcal{Q}(n)$ which meets K_n.*

Proof. If $x \in K$ is a density point of K, there is an integer $n_x \geq 1$ such that $|K \cap Q| > |Q|/\nu$ for each $Q \in \bigcup_{n \geq n_x} \mathcal{Q}(n)$ containing x. Letting

$$E_n = \{x \in K : n_x \leq n\},$$

the Lebesgue density theorem implies $\lim |E_n| = |K|$, and hence $\lim |E_n \cap Q_j| = |K \cap Q_j|$ for $j = 1, \ldots, k$. Thus for each sufficiently large integer $n \geq 1$ we have $|E_n| > 0$ and $|E_n \cap Q_j| > |Q_j|/\nu_j$ for $j = 1 \ldots, k$. Fix such an n, and let

$$\mathcal{K} = \{Q \in \mathcal{Q}(n) : E_n \cap Q \neq \emptyset\} \quad \text{and} \quad K_n = K \cap \bigcup \mathcal{K}.$$

As the family $\mathcal{K}$ is finite, K_n is compact, and since $E_n \subset K_n$, the set K_n is not empty and $|K_n \cap Q_j| \geq |E_n \cap Q_j| > |Q_j|/\nu_j$ for $j = 1, \ldots, k$. If $Q \in \mathcal{Q}(n)$ meets K_n, then $Q \in \mathcal{K}$. Hence $K_n \cap Q = K \cap Q$ and $E_n \cap Q \neq \emptyset$. In particular, Q contains an $x \in K$ with $n_x \leq n$, which implies $|K_n \cap Q| = |K \cap Q| > |Q|/\nu$. $\square$

Corollary 5.2. *Each measurable set $E \subset \mathbb{R}^m$ with $|E| > 0$ has an amiable subset.*

Proof. Select a compact set $K \subset E$ with $|K| > 0$. Using Lemma 5.1, construct inductively positive integers n_i and nonempty compact sets $K_i \subset K$ so that for $i = 1, 2, \ldots$, we have

$$n_{i+1} > n_i + p_i + i\,, \qquad K_{i+1} \subset K_i\,,$$

and $|K_i \cap Q| > |Q|/\eta_k$ whenever $k \leq i$ and $Q \in \mathcal{Q}(n_k)$ meets K_i. Having done this, let $K_0 = \bigcap_{i=1}^{\infty} K_i$, and observe

$$|K_0 \cap Q| = \lim_{i \to \infty} |K_i \cap Q| \geq \frac{|Q|}{\eta_k}$$

for each $Q \in \mathcal{Q}(n_k)$ which meets K_0. Now $|K_0| > 0$ since $K_0 \neq \emptyset$, and it suffices to let $C = \operatorname{supp}(\lambda \mathbin{\mathbf{L}} K_0)$.

To justify the inductive construction, first apply Lemma 5.1 to $k = 0$ and $\nu = \eta_1$, and obtain an integer $n_1 \geq 1$ and a nonempty compact set $K_1 \subset K$ such that $|K_1 \cap Q| > |Q|/\eta_1$ for each $Q \in \mathcal{Q}(n_1)$ which meets K_1. Next assume positive integers $n_2, \ldots, n_i$ and nonempty compact sets $K_2, \ldots, K_i$ have been constructed so that for $j = 2, \ldots, i$, we have

$$n_j > n_{j-1} + p_{j-1} + (j-1)\,, \qquad K_j \subset K_{j-1}\,,$$

and $|K_j \cap Q| > |Q|/\eta_k$ whenever $Q \in \mathcal{Q}(n_k)$ meets K_j and $1 \leq k \leq j$. Since only finitely many $Q \in \bigcup_{k=1}^{i} \mathcal{Q}(n_k)$ meet K_i, Lemma 5.1 implies there is a an integer $n_{i+1} > n_i + p_i + i$ and a nonempty compact set $K_{i+1} \subset K_i$ such that $|K_{i+1} \cap Q| > |Q|/\eta_k$ whenever $Q \in \mathcal{Q}(n_k)$ meets K_{i+1} and $1 \leq k \leq i+1$. $\square$

Let $C \subset \mathbb{R}^m$ be an amiable set, and let $\{n_i\}$ be the associated sequence of positive integers. For each integer $i \geq 1$ and each $Q \in \mathcal{Q}(n_i + p_i)$ select a $Q^* \in \mathcal{Q}(n_i + p_i + i)$, and let

$$H_i = \bigcup_{Q \in \mathcal{Q}(n_i + p_i)} \operatorname{int} Q^* \quad \text{and} \quad H_C = \bigcap_{j=1}^{\infty} \bigcup_{i=j}^{\infty} H_i\,.$$

Each $\bigcup_{i=j}^{\infty} H_i$ is a dense open subset of $\mathbb{R}^m$, and so H_C is a dense G_δ subset of $\mathbb{R}^m$ by the Baire category theorem. Every $Q \in \mathcal{Q}(0)$ contains $N^{m(n_i+p_i)}$ connected components of H_i, and each connected component of H_i has measure $N^{-m(n_i+p_i+i)}$. Thus $|Q \cap H_i| = N^{-mi} < 2^{-i}$ by our choice of N, and

$$|Q \cap H_C| = \lim_{j \to \infty} \sum_{i=j}^{\infty} |H_i| = 0\,.$$

We conclude $H_C = \bigcup_{Q \in \mathcal{Q}(0)} (Q \cap H_C)$ is a negligible set.

Observation 5.3. *If $Q \in \mathcal{Q}(n_i)$ meets C, then $|C \cap Q^*| \geq |Q^*|/2$ for each connected component Q^* of $H_i \cap Q$.*

Proof. As C is an amiable set,

$$|Q - C| = |Q| - |Q \cap C| \leq \left(1 - \frac{1}{\eta_i}\right)|Q| = \frac{|Q|}{2N^{m(p_i+i)}} = \frac{|Q^*|}{2},$$

and hence

$$|Q^* \cap C| = |Q^*| - |Q^* - C| \geq |Q^*| - |Q - C| \geq \frac{|Q^*|}{2}.$$

□

Corollary 5.4. *The set $H = C \cap H_C$ is dense in C.*

Proof. Let $j \geq 1$ be an integer, and select a $Q \in \mathcal{Q}(n_i)$ with $i \geq j$ and $Q \cap C \neq \emptyset$. Then $Q \cap (C \cap H_i) \neq \emptyset$ by Observation 5.3. We infer each $Q \in \bigcup_{i=j}^{\infty} \mathcal{Q}(n_i)$ that meets C meets also $C \cap \bigcup_{i=j}^{\infty} H_i$. Since $\bigcup_{i=j}^{\infty} \mathcal{Q}(n_i)$ is a base for the topology of $\mathbb{R}^m$, the set $C \cap \bigcup_{i=j}^{\infty} H_i$ is a relatively open dense subset of C. The corollary follows from the Baire category theorem. □

If $\mathcal{A}$ is a family of sets and B is a set, we let $\mathcal{A}_B = \{A \in \mathcal{A} : A \subset B\}$. For the purposes of this section only, we introduce the following terminology.

Let $Z \subset \mathbb{R}^m$ and $\varepsilon > 0$. An *ε-cover* of Z is a nonoverlapping family $\mathcal{Z} \subset \mathcal{Q}$ satisfying the following conditions:

1. $Z \subset \operatorname{int} \bigcup \mathcal{Z}$;
2. $\sum_{B \in \mathcal{Z}} d(B)^s < \varepsilon$;
3. $\sum_{B \in \mathcal{Z}_Q} d(B)^s < d(Q)^s$ for each $Q \in \mathcal{Q} - \mathcal{Z}$.

Lemma 5.5. *If $Z \subset \mathbb{R}^m$ and $\mathcal{H}^s(Z) = 0$, then Z has an ε-cover for each $\varepsilon > 0$.*

Proof. The existence of a nonoverlapping family $\mathcal{C}(0)$ that satisfies conditions 1 and 2 of an ε-cover of Z follows from [5, Theorem 5.1]. Proceeding inductively, for $n = 0, 1, \ldots$, construct families $\mathcal{C}(n)$ and $\mathcal{D}(n)$ so that $\mathcal{D}(n)$ consists of all $D \in \mathcal{Q}(n) - \mathcal{C}(n)$ for which $d(D)^s \leq \sum_{B \in \mathcal{C}(n)_D} d(B)^s$, and

$$\mathcal{C}(n+1) = \mathcal{C}(n) - \bigcup_{D \in \mathcal{D}(n)} \mathcal{C}(n)_D.$$

Let $\mathcal{C} = \bigcap_{n=1}^{\infty} \mathcal{C}(n)$ and $\mathcal{D} = \bigcup_{n=1}^{\infty} \mathcal{D}(n)$, and observe that $\mathcal{Z} = \mathcal{C} \cup \mathcal{D}$ is a nonoverlapping family with $\bigcup \mathcal{C}(0) \subset \bigcup \mathcal{Z}$; in particular, $Z \subset \operatorname{int} \bigcup \mathcal{Z}$. Furthermore,

$$\begin{aligned}\sum_{B\in\mathcal{Z}} d(B)^s &= \sum_{B\in\mathcal{C}} d(B)^s + \sum_{n=1}^{\infty}\sum_{D\in\mathcal{D}(n)} d(D)^s \\ &\le \sum_{B\in\mathcal{C}} d(B)^s + \sum_{n=1}^{\infty}\sum_{D\in\mathcal{D}(n)}\sum_{B\in\mathcal{C}(n)_D} d(B)^s \\ &= \sum_{B\in\mathcal{C}} d(B)^s + \sum_{B\in\mathcal{C}(0)-\mathcal{C}} d(B)^s = \sum_{B\in\mathcal{C}(0)} d(B)^s < \varepsilon\,.\end{aligned}$$

Select a $Q \in \mathcal{Q}(n) - \mathcal{Z}$, and observe that $Q \notin \mathcal{C}(n)$. Indeed, if $k \ge n$ is an integer, then $\mathcal{C}(k)$ and $\mathcal{C}(n)$ contain the same n-cubes; thus $Q \in \mathcal{C}(n)$ implies $Q \in \mathcal{C}$, a contradiction. Since $Q \notin \mathcal{D}(n)$, we have

$$\begin{aligned}d(Q)^s &> \sum_{B\in\mathcal{C}(n)_Q} d(B)^s = \sum_{B\in\mathcal{C}_Q} d(B)^s + \sum_{k=n}^{\infty}\sum_{D\in\mathcal{D}(k)_Q}\sum_{B\in\mathcal{C}(k)_D} d(B)^s \\ &\ge \sum_{B\in\mathcal{C}_Q} d(B)^s + \sum_{k=n}^{\infty}\sum_{D\in\mathcal{D}(k)_Q} d(D)^s \\ &= \sum_{B\in\mathcal{C}_Q} d(B)^s + \sum_{D\in\mathcal{D}_Q} d(D)^s = \sum_{B\in\mathcal{Z}_Q} d(B)^s\,.\end{aligned}$$

□

Observation 5.6. *Let $Q \in \mathcal{Q}$ and let $\mathcal{B} \subset \mathcal{Q}$ be a nonoverlapping family such that $|Q \cap \bigcup \mathcal{B}| \ge |Q|/2$. Then $d(Q)^s \le \sum_{B\in\mathcal{B}} d(B)^s$, and the equality occurs if and only if $\mathcal{B} = \{Q\}$.*

Proof. As the other cases are clear, assume $\mathcal{B}$ consists of proper subsets of Q, and observe that $d(Q)/d(B) \ge N$ for each $B \in \mathcal{B}$. As $N^{m-s} > 2$, we obtain

$$\begin{aligned}d(Q)^s &= \frac{|Q|}{d(Q)^{m-s}} \le \frac{2}{d(Q)^{m-s}}\left|\bigcup\mathcal{B}\right| = \frac{2}{d(Q)^{m-s}}\sum_{B\in\mathcal{B}}|B| \\ &\le 2\left[\frac{d(B)}{d(Q)}\right]^{m-s}\sum_{B\in\mathcal{B}} d(B)^s \le \frac{2}{N^{m-s}}\sum_{B\in\mathcal{B}} d(B)^s < \sum_{B\in\mathcal{B}} d(B)^s\,.\end{aligned}$$

□

Lemma 5.7. *Let $Q \in \mathcal{Q}(n_i)$ and let $\mathcal{B} \subset \mathcal{Q}$ be a nonoverlapping family of proper subcubes of Q such that $|Q^* \cap \bigcup \mathcal{B}| \ge |Q^*|/2$ for each connected component Q^* of $H_i \cap Q$. Then $d(Q)^s < \sum_{B\in\mathcal{B}} d(B)^s$.*

Proof. Enumerate $\mathcal{Q}(n_i + p_i)_Q$ as $\{Q_1, \dots, Q_r\}$ where $r = N^{m(p_i+i)}$, and let $Q_j^* = Q_j \cap H_i$ for $j = 1, \dots, r$. We may assume that for an integer k with $0 \le k \le r$, each $Q_1, \dots, Q_k$ is contained in a $B \in \mathcal{B}$, and no $Q_{k+1}, \dots, Q_r$ is contained in any $B \in \mathcal{B}$. Denote by $\mathcal{C}$ the family of all $B \in \mathcal{B}$ which contain one of the $Q_1, \ldots, Q_k$, and observe that each $B \in \mathcal{B} - \mathcal{C}$ is properly contained in one of the $Q_{k+1}, \ldots, Q_r$. If $k+1 \leq j \leq r$, then

$$|Q_j^*| \leq 2 \left| Q_j^* \cap \bigcup \mathcal{B} \right| = \left| Q_j^* \cap \bigcup \mathcal{B}_{Q_j} \right| ,$$

and Observation 5.6 implies $d(Q_j^*)^s \leq \sum_{B \in \mathcal{B}_{Q_j}} d(B)^s$. Now Q_j^* is the interior of an $(n_i + p_i + i)$-cube, and hence $d(Q_j^*) = N^{-(p_i+i)} d(Q)$ for $j = 1, \ldots$. Since $(m-s)p_i - si \geq m - s$ by our choice of p_i, we obtain

$$\begin{aligned}
\sum_{B \in \mathcal{B}-\mathcal{C}} d(B)^s &= \sum_{j=k+1}^{r} \sum_{B \in \mathcal{B}_{Q_j}} d(B)^s \geq \sum_{j=k+1}^{r} d(Q_j^*)^s = (r-k) d(Q)^s N^{-s(p_i+i)} \\
&= (r-k) d(Q)^s N^{-mp_i} N^{(m-s)p_i - si} \geq \frac{r-k}{r} d(Q)^s N^{m-s} .
\end{aligned}$$

On the other hand, $\bigcup_{j=1}^{k} Q_j \subset \bigcup \mathcal{C}$, $|Q_j| = |Q|/r$ for $j = 1, \ldots, k$, and $d(B) \leq d(Q)/N$ for each $B \in \mathcal{B}$. Thus

$$\begin{aligned}
\frac{k}{r}|Q| &= \sum_{j=1}^{k} |Q_j| \leq \sum_{B \in \mathcal{C}} |B| = \sum_{B \in \mathcal{C}} d(B)^s d(B)^{m-s} \\
&\leq \frac{d(Q)^{m-s}}{N^{m-s}} \sum_{B \in \mathcal{C}} d(B)^s = \frac{|Q|}{d(Q)^s N^{m-s}} \sum_{B \in \mathcal{C}} d(B)^s ,
\end{aligned}$$

which yields $\sum_{B \in \mathcal{C}} d(B)^s \geq (k/r) d(Q)^s N^{m-s}$. We conclude

$$\sum_{B \in \mathcal{B}} d(B)^s = \sum_{B \in \mathcal{C}} d(B)^s + \sum_{B \in \mathcal{B}-\mathcal{C}} d(B)^s \geq d(Q)^s N^{m-s} > d(Q)^s .$$

$\square$

Proof of Theorem 4.1. By Corollary 5.2, the set E has an amiable subset C. If $\{n_i\}$ is the sequence of positive integers associated with C, define the set H_C as in the paragraph preceding Observation 5.3. Then $|H_C| = 0$, and $H = C \cap H_C$ is a G_δ subset of C, which is dense in C by Corollary 5.4. Choose a set $Z \subset \mathbb{R}^m$ with $\mathcal{H}^s(Z) = 0$, and a positive $\varepsilon < |C|$. In view of Lemma 5.5 there is an ε-cover $\mathcal{Z}$ of Z, and we let $G = \operatorname{int} \bigcup \mathcal{Z}$ and $C_Z = C - G$.

Since $\mathcal{Q}$ is a base for the topology of $\mathbb{R}^m$, the set C_Z is supporting whenever $|Q \cap C_Z| > 0$ for each $Q \in \mathcal{Q}$ that meets C_Z. Proceeding towards a contradiction, suppose there is a $Q \in \mathcal{Q}(n_i)$ with $Q \cap C_Z \neq \emptyset$ and $|Q \cap C_Z| = 0$; in particular $Q \notin \mathcal{Z}$. As C is an amiable set, $|Q \cap C| \geq |Q|/\eta_i > |Q|/2$, and hence

$$\left| Q \cap \bigcup \mathcal{Z}_Q \right| = \left| Q \cap \bigcup \mathcal{Z} \right| \geq |Q \cap G| \geq |Q \cap C| > \frac{|Q|}{2} .$$

A contradiction $d(Q)^s < \sum_{B \in \mathcal{Z}_Q} d(B)^s$ follows from Observation 5.6.

It remains to show that $H \cap C_Z$ is a dense subset of C_Z. To this end, let $j \geq 1$ be an integer, and select a $Q \in \mathcal{Q}(n_i)$ with $i \geq j$ and $Q \cap C_Z \neq \emptyset$; in particular $Q \notin \mathcal{Z}$. Proceeding towards a contradiction, assume $Q \cap (C_Z \cap H_i) = \emptyset$. Thus if Q^* is a connected component of $Q \cap H_i$, then $Q^* \cap C \subset Q^* \cap G$, and Observation 5.3 yields

$$\left| Q^* \cap \bigcup \mathcal{Z}_Q \right| = \left| Q^* \cap \bigcup \mathcal{Z} \right| \geq |Q^* \cap G| \geq |Q^* \cap C| \geq \frac{|Q|}{2}.$$

A contradiction $d(Q)^s < \sum_{B \in \mathcal{Z}_Q} d(B)^s$ follows from Lemma 5.7. We conclude each $Q \in \bigcup_{i=j}^{\infty} \mathcal{Q}(n_i)$ that meets C_Z meets also $C_Z \cap \bigcup_{i=j}^{\infty} H_i$. Since $\bigcup_{i=j}^{\infty} \mathcal{Q}(n_i)$ is a base for the topology of $\mathbb{R}^m$, the set $C_Z \cap \bigcup_{i=j}^{\infty} H_i$ is a relatively open dense subset of C_Z. The proposition follows from the Baire category theorem. $\square$

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