VARIATIONS OF ADDITIVE FUNCTIONS

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Abstract

We study the relationship between derivates and variational measures of additive functions defined on families of figures or bounded sets of finite perimeter. Our results, valid in all dimensions, include a generalization of Ward's theorem, a necessary and sufficient condition for derivability, and full descriptive definitions of certain conditionally convergent integrals.

It has been recognized for a long time that there are important classes of additive functions whose usual variation is infinite. In dimension one, these include the classical families of VBG_* and ACG_* functions [12, Chapter VII, Sections 7 and 8]. Unfortunately, their definitions, closely tied to the order structure of the real line, give no indication on how to proceed in higher dimensions. In his recent memoir [14], Thomson defined the BVG_* and ACG_* functions by means of variational measures, which are multidimensional objects. Nonetheless, many arguments of [14] still employ in essential way the ordering of the reals. Based on Ward's theorem [12, Chapter IV, Section 11], a natural generalization of BVG_* and ACG_* classes was obtained for additive functions of intervals in any dimension [3].

In the present paper, we pass from intervals to figures (i.e., finite unions of intervals), and to the sets of finite perimeter [6, Section 5.1], referred to as BV sets. This is an important step which has a direct bearing on the coordinant

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invariance of our concepts (see Proposition 4.14 below, and [11, Chapters 11 and 12]). We prove a version of Ward's theorem for additive functions of figures and, proceeding along the lines established in [3], give a necessary and sufficient condition for derivability in terms of variational measures. We show that essentially everything we proved for figures remain valid for bounded BV sets.

Applying these results to the conditionally convergent integrals of [9] and [11, Chapter 12] produces their full descriptive definitions — a significant improvement over partial descriptive definitions established in [10] and [11]. Immediate corollaries are the divergence theorem for discontinuously differentiable vector fields and the change of variable theorems for lipeomorphic transformations (Theorems 5.6 and 5.7 below).

The organization of the paper is sufficiently indicated by the section titles. The authors are obliged to B. Bongiorno for valuable suggestions concerning Theorem 4.7 below.

1 Preliminaries

The set of all real numbers is denoted by $\mathbf{R}$, and the ambient space of this paper is $\mathbf{R}^m$ where m is a fixed positive integer. In $\mathbf{R}^m$ we use exclusively the metric induced by the maximum norm $|\cdot|$. The origin of $\mathbf{R}^m$ is denoted by $\mathbf{0}$. For an $x \in \mathbf{R}^m$ and $\varepsilon > 0$, we let

$$U(x,\varepsilon) = \{y \in \mathbf{R}^m : |x-y| < \varepsilon\} \quad \text{and} \quad U[x,\varepsilon] = \{y \in \mathbf{R}^m : |x-y| \le \varepsilon\}.$$

For $x = (\xi_1, \dots, \xi_m)$ and $y = (\eta_1, \dots, \eta_m)$ in $\mathbf{R}^m$, we let $x \cdot y = \sum_{i=1}^m \xi_i \eta_i$. Note that $|x \cdot y| \le m^2 |x| \cdot |y|$ is the Schwartz inequality with the maximum norm.

The closure, interior, boundary, and diameter of a set $E \subset \mathbf{R}^m$ are denoted by $\operatorname{cl} E$, $\operatorname{int} E$, ∂E, and $d(E)$, respectively. If $A, B \subset \mathbf{R}^m$ and $x \in \mathbf{R}^m$, we let

$$A \triangle B = (A - B) \cup (B - A) \quad \text{and} \quad \operatorname{dist}(x, A) = \inf\{|x-y| : y \in A\}.$$

Unless specified otherwise, a number is an extended real number, and a function is an extended real-valued function.

The Lebesgue measure in $\mathbf{R}^m$ is denoted by λ; however, for $E \subset \mathbf{R}^m$, we write $|E|$ instead of $\lambda(E)$. A set $E \subset \mathbf{R}^m$ with $|E| = 0$ is called *negligible*. Sets $A, B \subset \mathbf{R}^m$ are called, respectively, *equivalent* or *nonoverlapping* according to whether the set $A \triangle B$ or $A \cap B$ is negligible. Unless specified otherwise, the words "measure" and "measurable" as well as the expresions "almost all," "almost everywhere," and "absolutely continuous" always refer to the Lebesgue measure λ.

Let $E \subset \mathbf{R}^m$, and for $0 \le \alpha \le 1$, set

$$E(\alpha) = \left\{ x \in \mathbf{R}^m : \lim_{\varepsilon \to 0+} \frac{|U[x,\varepsilon] \cap E|}{(2\varepsilon)^m} = \alpha \right\}.$$

The sets $\mathrm{cl}^* E = \mathbf{R}^m - E(0)$, $\mathrm{int}^* E = E(1)$, and $\partial^* E = \mathrm{cl}^* E - \mathrm{int}^* E$ are called, respectively, the *essential closure, essential interior*, and *essential boundary* of E. If $E = \mathrm{cl}^* E$, we say that E is *essentially closed*.

The $(m-1)$-dimensional Hausdorff measure in $\mathbf{R}^m$ is denoted by $\mathcal{H}$, and a set $T \subset \mathbf{R}^m$ of σ-finite measure $\mathcal{H}$ is called *thin*. The symbol $\int$ always denotes the Lebesgue integral, with respect to λ or $\mathcal{H}$ as the case may be.

A BV set (BV for *bounded variation*) is a bounded set $A \subset \mathbf{R}^m$ for which the number $\|A\| = \mathcal{H}(\partial^* A)$, called the *perimeter* of A, is finite. By [7, Section 2.10.6 and Theorem 4.5.11], the family of all BV sets coincides with the collection of all bounded measurable subsets of $\mathbf{R}^m$ whose DeGiorgi perimeter [6, Section 5.1] is finite. For a BV set A, the essential boundary $\partial^* A$ is $\mathcal{H}$-measurable [6, Sections 5.7.3 and 5.8]. Moreover, there is an $\mathcal{H}$-measurable vector field ν_A, defined $\mathcal{H}$-almost everywhere on $\partial^* A$, such that $\nu_A \cdot \nu_A = 1$ and

$$\int_A \operatorname{div} v \, d\lambda = \int_{\partial^* A} v \cdot \nu_A \, d\mathcal{H}$$

for every vector field v continuously differentiable in a neighborhood of $\mathrm{cl}\, A$ [6, Sections 5.1 and 5.8].

Let A be a BV set. The set of all $x \in \mathrm{int}^* A$ such that

$$\lim_{\varepsilon \to 0+} \frac{\mathcal{H}(U[x,\varepsilon] \cap \partial^* A)}{(2\varepsilon)^{m-1}} = 0$$

is called the *critical interior* of A, denoted by $\mathrm{int}^c A$. According to [15, Section 4], $\mathcal{H}(\mathrm{int}^* A - \mathrm{int}^c A) = 0$.

The *regularity* of a BV set A is the number

$$r(A) = \begin{cases} \frac{|A|}{d(A)\|A\|} & \text{if } d(A)\|A\| > 0, \\ 0 & \text{otherwise.} \end{cases}$$

The usual concept of regularity introduced in [12, Chapter IV, Section 2] is related to $r(A)$ by the inequality $[2r(A)]^m \le |A|/[d(A)]^m$ [9, Proposition 2.1]. If $r(A) > \eta > 0$, we say the BV set A is *η-regular*.

A *cell* is a compact nondegenerate subinterval of $\mathbf{R}^m$, and a *figure* is a finite (possibly empty) union of cells. Each figure is a BV set, and a figure C of maximal regularity, i.e., with $r(C) = 1/(2m)$, is a cell called a *cube*.

The family $\mathcal{F}$ of all figures is contained in the family $\mathcal{BV}$ of all essentially closed BV sets. Since $\mathrm{cl}^* A \in \mathcal{BV}$ for each BV set A, every class of equivalent BV sets contains precisely one representative in $\mathcal{BV}$. It is easy to see that $\mathcal{F} = \mathcal{BV}$ when $m = 1$. For $m \ge 2$, however, we only have the approximation result of DeGiorgi.

Proposition 1.1 *For each BV set A there is a sequence $\{A_n\}$ of figures such that*

1. $\lim |A_n \triangle A| = 0$;

2. $\sup \|A_n\| \leq \kappa \|A\|$, *where* $\kappa \geq 1$ *is a constant depending only on the dimension* m;

3. $\sup d(A_n) \leq d(A)$.

If $A \in \mathcal{BV}$, *then* $\lim \operatorname{dist}(x, A_n) = 0$ *for each* $x \in A$*; in particular,* $\lim d(A_n) = d(A)$.

PROOF. Avoiding a triviality, assume $\|A\| > 0$. By [8, Theorem 1.24], there is a sequence $\{C_n\}$ in $\mathcal{BV}$ such that $\lim |C_n \triangle A| = 0$, $\sup \|C_n\| < 2\|A\|$, and $\partial C_n = \partial^* C_n$ for $n = 1, 2, \ldots$. In view of [11, Proposition 12.6.4], for each integer $n \geq 1$, we can find a figure $B_n \subset C_n$ with $|C_n - B_n| < 1/n$ and $\|B_n\| < 2\gamma\|A\|$, where γ is a constant depending only on the dimension m. Now it suffices to select a cube K of diameter $d(A)$ containing A, and let $A_n = B_n \cap K$ for $n = 1, 2, \ldots$. If $A \in \mathcal{BV}$, then $|A \cap U(x, \varepsilon)| > 0$ for each $\varepsilon > 0$. Since $\lim |A_n \triangle A| = 0$, the proposition follows.

We say that a sequence $\{A_n\}$ of BV sets *converges* to a set $A \subset \mathbf{R}^m$, and write $A_n \to A$, whenever $\lim |A_n \triangle A| = 0$ and $\sup \|A_n\| < +\infty$. Note that if $A_n \to A$ and A is bounded, then A is already a BV set [6, Section 5.2.1, Theorem 1].

Remark 1.2 The above described convergence of BV sets defines a sequential Hausdorff topology τ in the family $\mathcal{BV}$, which is induced by a nonmetrizable uniformity ν. Indeed, let

$$\mathcal{BV}_n = \{B \in \mathcal{BV} : B \subset U[\mathbf{0}, n] \text{ and } \|B\| \leq n\}$$

for $n = 1, 2, \ldots$, and observe that on each $\mathcal{BV}_n$ the convergence of BV sets is the same as the convergence with respect to the metric $\rho(B, C) = |B \triangle C|$. Thus τ is induced by the largest uniformity ν in $\mathcal{BV}$ for which all imbeddings $(\mathcal{BV}_n, \rho) \hookrightarrow (\mathcal{BV}, \nu)$ are uniformly continuous. It follows from Proposition 1.1 that $\mathcal{F}$ is dense in the spaces $(\mathcal{BV}, \tau)$, and consequently $(\mathcal{BV}, \nu)$ is the *completion* of $(\mathcal{F}, \nu)$.

In what follows, we often make identical definitions and statements about the families $\mathcal{F}$ and $\mathcal{BV}$. To avoid repetitions in such circumstances, we shall use the letter $\mathcal{A}$ that stands either for $\mathcal{F}$ or $\mathcal{BV}$. For an $A \in \mathcal{A}$, we let $\mathcal{A}_A = \{B \in \mathcal{A} : B \subset A\}$. If sets B and C belong to $\mathcal{A}$, then so do the sets $B \cup C$,

$$B \odot C = \operatorname{cl}^*(B \cap C) \qquad \text{and} \qquad B \ominus C = \operatorname{cl}^*(B - C),$$

and the following inequality holds:

$$\max\{\|B \cup C\|, \|B \odot C\|, \|B \ominus C\|\} \leq \|B\| + \|C\|.$$

2 Additive functions

Definition 2.1 A real-valued function F defined on the family $\mathcal{A}$ is called

1. *additive* if $F(B \cup C) = F(B) + F(C)$ for each pair of nonoverlapping sets $B, C \in \mathcal{A}$;

2. *continuous* if given $\varepsilon > 0$, there is an $\eta > 0$ such that $|F(B)| < \varepsilon$ for each $B \in \mathcal{A}$ with $B \subset U(\mathbf{0}, 1/\varepsilon)$, $\|B\| < 1/\varepsilon$, and $|B| < \eta$.

Remark 2.2 An additive function F on $\mathcal{A}$ is continuous according to the above definition if and only if it is *uniformly* ν-continuous, where ν is the uniformity in $\mathcal{A}$ introduced in Remark 1.2. A distribution function of an additive continuous function on $\mathcal{F}$ is continuous, but the converse is true only in dimension one [5].

Let F be a real-valued function on $\mathcal{BV}$. If F is, respectively, additive or continuous, then so is the restriction $F \lceil \mathcal{F}$ of F to $\mathcal{F}$. However, the converse is false: it is possible that $F \lceil \mathcal{F}$ is both additive and continuous, while F is neither.

Example 2.3 We give two important examples of additive continuous functions on $\mathcal{BV}$.

1. Let $f \in L^1_{\mathrm{loc}}(\mathbf{R}^m, \lambda)$ [6, Section 1.3], and let $F(A) = \int_A f \, d\lambda$ for each $A \in \mathcal{BV}$. Then F is an additive continuous function on $\mathcal{BV}$ by the absolute continuity of the Lebesgue integral.

2. Let v be a continuous vector field on $\mathbf{R}^m$, and let $F(A) = \int_{\partial^* A} v \cdot \nu_A \, d\mathcal{H}$ for each $A \in \mathcal{BV}$. Then F is an additive continuous function on $\mathcal{BV}$, called the *flux* of v [9, Example 4.2].

Proposition 2.4 *Each additive continuous function F on $\mathcal{F}$ has a unique extension to an additive continuous function $\hat{F}$ on $\mathcal{BV}$.*

PROOF. The proposition follows immediately from Remarks 1.2 and 2.2. Nonetheless, extending an additive continuous function F on $\mathcal{F}$ directly is instructive.

By Proposition 1.1, to each $A \in \mathcal{BV}$ converges a sequence $\{A_i\}$ of figures. Since

$$\begin{aligned} |A_i \triangle A_j| &\le |A_i \triangle A| + |A \triangle A_j| \,, \\ |F(A_i) - F(A_j)| &= |F(A_i \ominus A_j) - F(A_j \ominus A_i)| \\ &\le |F(A_i \ominus A_j)| + |F(A_j \ominus A_i)| \,, \end{aligned}$$

the sequnce $\{F(A_i)\}$ is Cauchy. Let $\hat{F}(A) = \lim F(A_i)$, and show by a standard argument that $\hat{F}(A)$ depends only on A, and not on the choice of a sequence $\{A_i\}$ of figures converging to A. Thus we have defined a real-valued function $\hat{F}$ on $\mathcal{BV}$ that extends F. If $\{A_i\}$ and $\{B_i\}$ are sequences of figures converging to nonoverlapping sets A and B, respectively, then $\{A_i \cup (B_i \ominus A_i)\}$ converges to $A \cup B$. From this we infer $\hat{F}$ is additive.

Let κ be the constant from Proposition 1.1, and choose an $\varepsilon > 0$. There is an $\eta > 0$ such that $|F(B)| < \varepsilon$ for each figure $B \subset U[\mathbf{0}, 1/\varepsilon]$ with $\|B\| < \kappa/\varepsilon$ and $|B| < \eta$. Let $A \in \mathcal{BV}$ be such that $A \subset U(\mathbf{0}, 1/\varepsilon)$, $\|A\| < 1/\varepsilon$, and $|A| < \eta$. If $\{A_i\}$ is a sequence of figures associated with A according to Proposition 1.1, then $|\hat{F}(A)| = |\lim F(A_i)| \leq \varepsilon$ and the continuity of $\hat{F}$ is established.

If G is an additive continuous function on $\mathcal{BV}$, then it is easy to see that $G(A) = \lim G(A_i)$ whenever $\{A_i\}$ is a sequence of figures converging to a set $A \in \mathcal{BV}$. The uniqueness of $\hat{F}$ follows.

Let $A \in \mathcal{A}$ and let F be a function defined on $\mathcal{A}_A$. As anticipated, we say F is, respectively, *additive* or *continuous* whenever it satisfies condition 1 or 2 of Definition 2.1 on the family $\mathcal{A}_A$. For instance, the flux of a *uniformly* continuous vector field on $A \in \mathcal{BV}$ is an additive continuous function on $\mathcal{BV}_A$, since v has a continuous extension to $\operatorname{cl} A$ and hence to $\mathbf{R}^m$. Observe that in the definition of continuity, the restriction $B \subset U(\mathbf{0}, 1/\varepsilon)$ is satisfied for all $B \in \mathcal{A}_A$ whenever $\varepsilon > 0$ is sufficiently small. Setting

$$(F\lfloor A)(B) = F(A \odot B)$$

for each $B \in \mathcal{A}$ defines a function $F\lfloor A$ on $\mathcal{A}$ that extends F. We call $F\lfloor A$ the *canonical extension* of F, and note that it is additive or continuous if and only if F is additive or continuous, respectively. If $\mathcal{A} = \mathcal{F}$ and F is additive and continuous, let

$$\hat{F} = (\widehat{F\lfloor A})\lceil \mathcal{BV}_A$$

where $\widehat{F\lfloor A}$ denotes the unique extension of $F\lfloor A$ to $\mathcal{BV}$ (Proposition 2.4). Using Proposition 1.1, it is easy to verify that $\hat{F}\lfloor A = \widehat{F\lfloor A}$ and conclude that $\hat{F}$ is the unique additive continuous extension of F to $\mathcal{BV}_A$.

Convention 2.5 We shall always assume that each additive continuous function F on $\mathcal{F}$ or $\mathcal{F}_A$, where $A \in \mathcal{F}$, has been extended to an additive continuous function on $\mathcal{BV}$ or $\mathcal{BV}_A$, respectively, still denoted by F (instead of $\hat{F}$). Consequently, we shall consider only additive continuous functions on $\mathcal{BV}$ or $\mathcal{BV}_A$ with $A \in \mathcal{BV}$, to which we refer as additive continuous functions in $\mathbf{R}^m$ or A, respectively.

3 Derivatives

Let $x \in \mathbf{R}^m$, and let F be a real-valued function defined on $\mathcal{A}$. For a positive $\eta < 1/(2m)$, set

$$\underline{D}_\eta^{\mathcal{A}} F(x) = \sup_{\delta>0} \left[\inf_B \frac{F(B)}{|B|} \right] \quad \text{and} \quad \overline{D}_\eta^{\mathcal{A}} F(x) = \inf_{\delta>0} \left[\sup_B \frac{F(B)}{|B|} \right]$$

where the infimum and supremum in the brackets are taken over all η-regular $B \in \mathcal{A}$ with $x \in B$ and $B \subset U(x, \delta)$. The numbers

$$\underline{D}^{\mathcal{A}} F(x) = \inf_{0<\eta<\frac{1}{2m}} \underline{D}^{\mathcal{A}}_{\eta} F(x) \quad \text{and} \quad \overline{D}^{\mathcal{A}} F(x) = \sup_{0<\eta<\frac{1}{2m}} \overline{D}^{\mathcal{A}}_{\eta} F(x)$$

are called, respectively, the *lower* and *upper* $\mathcal{A}$-*derivate* of F at x.

Using an argument similar to [12, Chapter IV, Theorem 4.2], it is easy to show that the functions $\underline{D}^{\mathcal{A}}_{\eta} F$, $\overline{D}^{\mathcal{A}}_{\eta} F$, $\underline{D}^{\mathcal{A}} F$, and $\overline{D}^{\mathcal{A}} F$ defined on $\mathbf{R}^m$ in the obvious way are measurable, and that

$$\underline{D}^{\mathcal{A}} F \leq \underline{D}^{\mathcal{A}}_{\eta} F \leq \underline{D}^{\mathcal{A}}_{\theta} \leq \overline{D}^{\mathcal{A}}_{\theta} F \leq \overline{D}^{\mathcal{A}}_{\eta} F \leq \overline{D}^{\mathcal{A}} F$$

for all η, θ with $0 < \eta < \theta < 1/(2m)$.

If $\underline{D}^{\mathcal{A}} F(x) = \overline{D}^{\mathcal{A}} F(x) \neq \pm\infty$, we denote this common value by $D^{\mathcal{A}} F(x)$, and say that F is $\mathcal{A}$-*derivable* at x; the number $D^{\mathcal{A}} F(x)$ is called the $\mathcal{A}$-*derivate* of F at x. If $\overline{D}^{\mathcal{A}}_{\eta} |F|(x) < +\infty$ for all positive $\eta < 1/(2m)$, we say that F is *almost* $\mathcal{A}$-*derivable* at x (cf. [11, Section 11.7]); in particular, F is almost $\mathcal{A}$-derivable at x whenever $\overline{D}^{\mathcal{A}} |F|(x) < +\infty$. The term "almost derivable" will be justified by a variant of *Ward's theorem* [12, Chapter IV, Theorem 11.15] proved below (Theorem 3.3).

Lemma 3.1 *Let $x \in \mathbf{R}^m$, and let F be an additive continuous function in $\mathbf{R}^m$. Then*

$$\underline{D}^{\mathcal{F}} F(x) = \underline{D}^{\mathcal{BV}} F(x) \qquad \text{and} \qquad \overline{D}^{\mathcal{F}} F(x) = \overline{D}^{\mathcal{BV}} F(x)\,.$$

Moreover, at x the function F is almost $\mathcal{F}$-derivable if and only if it is almost $\mathcal{BV}$-derivable.

Proof. Suppose $\underline{D}^{\mathcal{BV}} F(x) < \underline{D}^{\mathcal{F}} F(x)$, and find a positive $\eta < 1/(2m)$ so that $\underline{D}^{\mathcal{BV}}_{\eta} F(x) < \underline{D}^{\mathcal{F}} F(x) \leq \underline{D}^{\mathcal{F}}_{\eta/\kappa} F(x)$, where κ is the constant introduced in Proposition 1.1. There is a $\delta > 0$ such that

$$\underline{D}^{\mathcal{BV}}_{\eta} F(x) < \inf \frac{F(C)}{|C|} = c\,,$$

where the infimum is taken over all (η/κ)-regular figures $C \subset U(x, \delta)$ with $x \in C$. Select a positive $\gamma < \delta$ and an η-regular $B \in \mathcal{BV}$ with $x \in B$, $B \subset U(x, \gamma)$, and $F(B)/|B| < c$. Using Proposition 1.1, it is easy to construct an (η/κ)-regular figure $C \subset U(x, \delta)$ with $x \in C$ and $F(C)/|C| < c$, a contradiction. Since clearly $\underline{D}^{\mathcal{BV}} F(x) \leq \underline{D}^{\mathcal{F}} F(x)$, we conclude $\underline{D}^{\mathcal{BV}} F(x) = \underline{D}^{\mathcal{F}} F(x)$. The rest of the lemma is proved similarly.

In view of Lemma 3.1, for additive continuous functions in $\mathbf{R}^m$, we now say "derivate" and "derivablity" instead of "$\mathcal{A}$-derivate" and "$\mathcal{A}$-derivability," and write $\underline{D}F(x)$, $\overline{D}F(x)$, and $DF(x)$ instead of $\underline{D}^{\mathcal{A}}F(x)$, $\overline{D}^{\mathcal{A}}F(x)$, and $D^{\mathcal{A}}F(x)$, respectively.

Let v be a vector field defined on $\mathbf{R}^m$. We say v is *almost differentiable* at $x \in \mathbf{R}^m$ if there are positive numbers c and δ such that

$$|v(y) - v(x)| \leq c|y - x|$$

for each $y \in \mathbf{R}^m$ with $|x - y| < \delta$. If E is the set of all $x \in \mathbf{R}^m$ at which v is almost differentiable, then v is differentiable almost everywhere in E [7, Stepanoff's Theorem 3.1.9]. Note that E is measurable whenever v is continuous.

Example 3.2 Let F be the flux of a continuous vector field v on $\mathbf{R}^m$, and let $x \in \mathbf{R}^m$. The following facts are easy to prove [11, Lemma 11.7.4].

1. If v is almost differentiable at x, then F is almost derivable at x.

2. If v is differentiable at x, then F is derivable at x and $DF(x) = \operatorname{div} v(x)$.

Theorem 3.3 *Let F be an additive continuous function in $\mathbf{R}^m$, and let E be the set of all $x \in \mathbf{R}^m$ at which F is almost derivable. Then F is derivable at almost all $x \in E$.*

Proof. If $x \in \mathbf{R}^m$ and a finite limit $\lim[F(C_n)/|C_n|]$ exists for every sequence $\{C_n\}$ of cubes with $\lim d(C_n) = 0$ and $x \in C_n$ for $n = 1, 2, \ldots$, then all these limits have the same value denoted by $F'(x)$. When $m = 1$, a simple argument reveals that an additive function F defined on $\mathcal{F}$ is derivable at $x \in \mathbf{R}$ if and only if $F'(x)$ exists, in which case $F'(x) = DF(x)$ (cf. [11, Proposition 5.3.3 and Lemma 8.3.4]). Hence for $m = 1$, the theorem is a direct consequence of Ward's theorem, and it holds for any additive function F defined on $\mathcal{F}$, whether it is continuous or not.

Thus we assume $m \geq 2$ and, proceeding towards a contradiction, we also assume $|\{x \in E : \underline{D}F(x) < \overline{D}F(x)\}| > 0$. The identity

$$\{x \in E : \underline{D}F(x) < \overline{D}F(x)\} = \bigcup_{n=2m+1}^{\infty} \{x \in E : \underline{D}^{\mathcal{F}}_{1/n}F(x) < \overline{D}^{\mathcal{F}}_{1/n}F(x)\}$$

implies that $|\{x \in E : \underline{D}^{\mathcal{F}}_{\eta}F(x) < \overline{D}^{\mathcal{F}}_{\eta}F(x)\}| > 0$ for all sufficiently small $\eta > 0$. We fix such an η so that

$$0 < \eta \leq \frac{1}{2(1+4m)} < \frac{1}{2m},$$

and denote by E' the set of all $x \in E$ for which $F'(x)$ exists. By [12, Chapter IV, Section 11; particularly the small print on p. 139], the set $E - E'$ is negligible; for

$$\max\{|\underline{D}^{\mathcal{F}}_{\eta}F|, |\overline{D}^{\mathcal{F}}_{\eta}F|\} \leq \overline{D}^{\mathcal{F}}_{\eta}|F|$$

whenever $0 < \eta < 1/(2m)$. Since $\underline{D}_\eta^{\mathcal{F}} F(x) \leq F'(x) \leq \overline{D}_\eta^{\mathcal{F}} F(x)$ for each $x \in E'$, by symmetry, we may assume

$$|\{x \in E' : \underline{D}_\eta^{\mathcal{F}} F(x) \leq F'(x) < \overline{D}_\eta^{\mathcal{F}} F(x)\}| > 0\,.$$

Using standard techniques, find rational numbers M, a, and t so that the set

$$E_\circ = \{x \in E' : t - M < \underline{D}_\eta^{\mathcal{F}} F(x) \leq F'(x) < t < t + a < \overline{D}_\eta^{\mathcal{F}} F(x) < t + M\}$$

has positive measure. The function $G = F - t\lambda$ is an additive continuous function in $\mathbf{R}^m$, and

$$E_\circ = \{x \in E' : -M < \underline{D}_\eta^{\mathcal{F}} G(x) \leq G'(x) < 0 < a < \overline{D}_\eta^{\mathcal{F}} G(x) < M\}\,.$$

For each $x \in E_\circ$ select the *largest* $\delta_x > 0$ so that the following conditions hold:

1. $G(K) \leq 0$ for each cube $K \subset U(x, \delta_x)$ with $x \in K$;
2. $-M \leq G(A)/|A| \leq M$ for each η-regular figure $A \subset U(x, \delta_x)$ with $x \in A$.

Since $E_\circ = \bigcup_{k=1}^\infty \{x \in E_\circ : \delta_x \geq 1/k\}$, there is a $\delta > 0$ such that the set $E_\delta = \{x \in E_\circ : \delta_x \geq \delta\}$ has positive measure.

The set

$$E = \bigcap_{n=2m+1}^{\infty} \{x \in \mathbf{R}^m : \overline{D}_{1/n}|F|(x) < +\infty\}$$

is measurable, and so is E'. Using [12, Chapter IV, Section 4], we verify that the set $E_\circ$ is also measurable. From the additivity and continuity of G, it is easy to deduce that $x \mapsto \delta_x$ is an upper semicontinuous function on $E_\circ$. It follows that E_δ is a relatively closed subset of $E_\circ$; in particular, E_δ is a measurable set.

Select an $\varepsilon > 0$ and a point $z \in E_\delta \cap \operatorname{int}^* E_\delta$. As E_δ is measurable, we can find a positive $\Delta \leq (\delta/2)\eta^{1/(m-1)} < \delta/2$ so that

$$\frac{|U[z,h] - E_\delta|}{|U[z,h]|} < \varepsilon \tag{1}$$

for each positive $h < \Delta$.

As $\overline{D}_\eta G(z) > a$, there is an η-regular figure A with $z \in A$, $d(A) < \Delta$, and $G(A)/|A| > a$. If $d = d(A)$ and $K = U[z,d]$, then $A \subset K$. For $i = 1, 2, \ldots$, let $\mathcal{D}_i$ be the dyadic division of K into 2^i cubes of diameters equal to $2d/2^i$. In view of the additivity and continuity of G, we may assume $A = \bigcup \mathcal{D}$, where $\mathcal{D} \subset \mathcal{D}_n$ for an integer $n \geq 0$.

Let $\mathcal{C} = \{D \in \mathcal{D} : D \cap E_\delta \neq \emptyset\}$ and $C = \bigcup \mathcal{C}$. For a $D \in \mathcal{D} - \mathcal{C}$, find an $x \in D \cap E_\delta$ and observe that

$$D \subset U[x, 2d] \subset U(x, 2\Delta) \subset U(x, \delta) \subset U(x, \delta_x)\,.$$

Thus $G(D) \leq 0$ for each $D \in \mathcal{D} - \mathcal{C}$ (condition 1 above), and we have

$$a|A| < G(A) = \sum_{D \in \mathcal{D}} G(D) \leq \sum_{D \in \mathcal{C}} G(D) = G(C)\,. \tag{2}$$

With each $D \in \mathcal{C}$ associate the *largest* cube $D^* \in \bigcup_{i=1}^{n} \mathcal{D}_i$ containing D and still disjoint from E_δ. If $B = \bigcup\{D^* : D \in \mathcal{C}\}$, then $B \subset K - E_\delta$ and $C = A \cap B$. Since $d < \Delta$, inequality (1) implies

$$\frac{|B|}{|K|} \leq \frac{|K - E_\delta|}{|K|} < \varepsilon\,. \tag{3}$$

The isoperimetric inequality [11, Proposition 12.1.6] and $r(A) > \eta$ yield

$$|A| > \eta d \|A\| \geq \eta d (2m) |A|^{\frac{m-1}{m}}\,,$$

and hence $|A| > (m\eta)^m (2d)^m = (m\eta)^m |K|$. Combining the last inequality with (2), we get

$$G(A \cap B) = G(C) > a|A| > 2\beta |K| \tag{4}$$

where $\beta = a(m\eta)^m/2$. Let $\mathcal{K}$ be a nonoverlapping subfamily of $\{D^* : D \in \mathcal{C}\}$ such that $B = \bigcup \mathcal{K}$, and let $\mathcal{L} = \{L \in \mathcal{K} : G(A \cap L) \geq (\beta/\varepsilon)|L|\}$. By (3),

$$\sum_{L \in \mathcal{K} - \mathcal{L}} G(A \cap L) < \frac{\beta}{\varepsilon} \sum_{L \in \mathcal{K} - \mathcal{L}} |L| \leq \frac{\beta}{\varepsilon} |B| < \beta |K|$$

and so by (4),

$$\sum_{L \in \mathcal{L}} G(A \cap L) = G(A \cap B) - \sum_{L \in \mathcal{K} - \mathcal{L}} G(A \cap L) > \beta |K| = \beta d^m\,. \tag{5}$$

Now fix an $L \in \mathcal{L}$ and let $l = d(L)$. By construction, each $D \in \bigcup_{i=0}^{n} \mathcal{D}_i$ that properly contains L meets E_δ. It follows that $L \subset U[y, 2l]$ for a $y \in E_\delta$. Choose an integer $p \geq 1$ and, using the additivity and continuity of G, subdivide L into nonoverlapping cells $L_1, \ldots, L_p$ so that

$$G(A \cap L_i) = \frac{1}{p} G(A \cap L) \geq \frac{\beta}{p\varepsilon} |L| = \frac{\beta}{p\varepsilon} l^m$$

for $i = 1, \ldots, p$. Making ε sufficiently small, we may assume

$$\frac{G(A \cap L)}{p(2l)^m} \geq \frac{\beta}{p\varepsilon 2^m} > 4M\,. \tag{6}$$

If $h = \min\{r, s\}$ where $r = [G(A \cap L)/(4pM)]^{1/m}$ and $s = \Delta/\eta^{1/(m-1)}$, then $2l < h \leq \delta/2$. Indeed, the choice of Δ implies $s \leq \delta/2$; moreover, $2l < r$ by (6), and $2l \leq d < \Delta < s$ since $L \cap E_\delta = \emptyset$.

As $L \subset U[y, 2l] \subset U[y, h]$ and $y \notin L$, there is a cube $Q \subset U[y, h] - L$ with $y \in Q$ and $d(Q) = h$. By condition 2 above, $G(Q)/|Q| > -M$; for $h < \delta \leq \delta_y$ and Q is η-regular. For $i = 1, \ldots, p$, let $S_i = (A \cap L_i) \cup Q$. Then

$$\begin{aligned} |S_i| &= |A \cap L_i| + |Q| \leq |L| + |Q| = h^m\Big[(l/h)^m + 1\Big] \\ &< h^m(2^{-m} + 1) < 2h^m \leq 2r^m = \frac{G(A \cap L_i)}{2M} \end{aligned}$$

and consequently

$$\frac{G(S_i)}{|S_i|} = \frac{G(A \cap L_i)}{|S_i|} + \frac{G(Q)}{|Q|} \cdot \frac{|Q|}{|S_i|} > 2M - M\frac{|Q|}{|S_i|} > M\,.$$

As $y \in S_i$ and $S_i \subset U[y, h] \subset U(y, \delta) \subset U(y, \delta_y)$, condition 2 implies $r(S_i) \leq \eta$ or equivalently

$$\|S_i\| \geq \frac{|S_i|}{\eta d(S_i)} > \frac{|Q|}{\eta(2h)} = \frac{h^{m-1}}{2\eta}\,.$$

On the other hand,

$$\begin{aligned} \|S_i\| &= \|A \cap L_i\| + \|Q\| \leq \|L_i\| + \|Q\| + \mathcal{H}(\partial A \cap \operatorname{int} L_i) \\ &\leq 2ml^{m-1} + 2mh^{m-1} + \mathcal{H}(\partial A \cap \operatorname{int} L_i) \\ &< 4mh^{m-1} + \mathcal{H}(\partial A \cap \operatorname{int} L_i) \end{aligned}$$

and so

$$\mathcal{H}(\partial A \cap \operatorname{int} L_i) > \frac{h^{m-1}}{2\eta} - 4mh^{m-1} = \left(\frac{1}{2\eta} - 4m\right) h^{m-1} \geq h^{m-1} \qquad (7)$$

by the choice of η.

We conclude the proof by infering from (7) that $\|A\| > d^{m-1}/\eta$ and hence $r(A) \leq d^m/(d\|A\|) < \eta$, contrary to the choice of A. Indeed for $h = s$,

$$\|A\| \geq \mathcal{H}(\partial A \cap \operatorname{int} L_i) > s^{m-1} = \frac{\Delta^{m-1}}{\eta} > \frac{d^{m-1}}{\eta}$$

by (7) applied to any integer i with $1 \leq i \leq p$. For $h = r$, applying (7) to $i = 1, \ldots, p$ yields

$$\begin{aligned} \mathcal{H}(\partial A \cap \operatorname{int} L) &\geq \sum_{i=1}^{p} \mathcal{H}(\partial A \cap \operatorname{int} L_i) > pr^{m-1} \\ &= p\left[\frac{G(A \cap L)}{4pM}\right]^{\frac{m-1}{m}} = \frac{p^{\frac{1}{m}}}{\gamma}[G(A \cap L)]^{\frac{m-1}{m}} \end{aligned}$$

where $\gamma = (4M)^{(m-1)/m}$. Adding up these inequalities over all $L \in \mathcal{L}$ and using (5), we obtain

$$\begin{aligned} \|A\| &\geq \sum_{L \in \mathcal{L}} \mathcal{H}(\partial A \cap \operatorname{int} L) > \frac{p^{\frac{1}{m}}}{\gamma} \sum_{L \in \mathcal{L}} [G(A \cap L)]^{\frac{m-1}{m}} \\ &\geq \frac{p^{\frac{1}{m}}}{\gamma} \left[\sum_{L \in \mathcal{L}} G(A \cap L) \right]^{\frac{m-1}{m}} > p^{\frac{1}{m}} \frac{\beta^{\frac{m-1}{m}}}{\gamma} d^{m-1}. \end{aligned}$$

Thus for a sufficiently large p, the inequality $\|A\| > d^{m-1}/\eta$ holds again (note that neither β nor γ depends on p).

It will be convenient to *relativize* the concept of derivates. Let $A \in \mathcal{A}$, $x \in A$, and let F be a real-valued function on $\mathcal{A}$. For a positive $\eta < 1/(2m)$, set

$$\underline{D}_{\eta}^{\mathcal{A}} F_A(x) = \sup_{\delta > 0} \inf_{B} \frac{F(B)}{|B|}$$

where the infimum is taken over all η-regular $B \in \mathcal{A}_A$ with $x \in B$ and $B \subset U(x, \delta)$. The number

$$\underline{D}^{\mathcal{A}} F_A(x) = \inf_{0 < \eta < \frac{1}{2m}} \underline{D}_{\eta}^{\mathcal{A}} F_A(x)$$

is called the *lower $\mathcal{A}$-derivate* of F at x *relative* to A. The numbers $\overline{D}_{\eta}^{\mathcal{A}} F_A(x)$, $\overline{D}^{\mathcal{A}} F_A(x)$, and $D^{\mathcal{A}} F_A(x)$ are defined similarly; the meaning of derivability and almost derivability *relative* to A is obvious. Since the derivates of F relative to A do not depend on the values F takes on $\mathcal{A} - \mathcal{A}_A$, they remain meaningful when F is defined only on $\mathcal{A}_A$.

Let v be a vector field defined on a measurable set $E \subset \mathbf{R}^m$. We say v is *almost differentiable* at $x \in E$ *relative* to E if there are positive numbers c and δ such that

$$|v(y) - v(x)| \leq c|y - x|$$

for each $y \in E$ with $|x - y| < \delta$. If $C \subset E$ and $x \in C$, then v may be almost differentiable at x relative to C but not relative to E. However, using an argument similar to [13, Chapter VI, Theorem 3], one can show that a uniformly continuous vector field v on E can be extended to a continuous vector field w on $\mathbf{R}^m$ so that w is almost differentiable at each $x \in E$ at which v is almost differentiable relative to E. If S is the set of all $x \in E$ at which v is almost differentiable relative to E, then w is differentiable almost everywhere in S, and we let $\operatorname{div} v(x) = \operatorname{div} w(x)$ for each $x \in S$ at which w is differentiable. Up to a negligible set, $\operatorname{div} v$ is determined uniquely by v and does not depend on the extension w [11, Lemma 10.5.5].

Example 3.4 Let F be the flux of a uniformly continuous vector field v defined on $A \in \mathcal{BV}$. If v is almost diferentiable at x relative to A, then F is almost $\mathcal{BV}$-derivable at $x \in A$ relative to A [11, Lemma 11.7.4].

For figures, the connection between derivates and relative derivates is simple. Let $A \in \mathcal{F}$ and let F be a real-valued function on $\mathcal{F}_A$. If $x \in \operatorname{int} A$, then

$$\underline{D}_{\eta}^{\mathcal{F}} F_A(x) = \underline{D}_{\eta}^{\mathcal{F}} (F \lfloor A)(x)$$

for each positive $\eta < 1/(2m)$. Moreover, $D(F \lfloor A)(x) = 0$ for every $x \in \mathbf{R}^m - A$.

Lemma 3.5 *Let $A \in \mathcal{BV}$, $x \in \operatorname{int}^c A$, and let F be an additive continuous function in A. Then*

$$\underline{D}^{\mathcal{BV}} F_A(x) = \underline{D}(F \lfloor A)(x) \qquad \text{and} \qquad \overline{D}^{\mathcal{BV}} F_A(x) = \overline{D}(F \lfloor A)(x) .$$

Moreover, F is almost $\mathcal{BV}$-derivable at x relative to A if and only if $F \lfloor A$ is almost derivable at x.

Proof. Let $G = F \lfloor A$. [10, Lemma 1.2] implies that

$$\underline{D}^{\mathcal{BV}} F_A(x) = \inf_{0<\eta<\frac{1}{2m}} \sup_{\delta>0} \inf_{B} \left[\frac{G(B)}{|B|} \cdot \frac{|B|}{|A \cap B|} \right] ,$$

where the second infimum is taken over all η-regular $B \in \mathcal{BV}$ with $x \in B$ and $B \subset U(x, \delta)$. Choose positive numbers $\varepsilon < 1$ and $\eta < 1/(2m)$. If $B \in \mathcal{BV}$ is η-regular with $x \in B$ and $d = d(B)$, then $B \subset U[x, 2d]$ and

$$|B| > (2\eta)^m d^m = \eta^m |U[x, 2d]| .$$

By [10, Lemma 1.1], there is a $\delta > 0$ such that $\varepsilon \leq |B|/|A \cap B| \leq 1$ for each η-regular $B \in \mathcal{BV}$ with $x \in B$ and $B \subset U(x, \delta)$. Consequently

$$\varepsilon \underline{D} G(x) \leq \underline{D}^{\mathcal{BV}} F_A(x) \leq \underline{D} G(x) ,$$

and the equality $\underline{D} G(x) = \underline{D}^{\mathcal{BV}} F_A(x)$ follows from the arbitrariness of ε and the convention following Lemma 3.1. The rest of the lemma is established similarly.

Let $A \in \mathcal{A}$, and let F be an additive continuous function in A. Following our previous convention, for each $x \in \operatorname{int}^c A$, we write $\underline{D} F_A(x)$, $\overline{D} F_A(x)$, and $D F_A(x)$ instead of $\underline{D}^{\mathcal{A}} F_A(x)$, $\overline{D}^{\mathcal{A}} F_A(x)$, and $D^{\mathcal{A}} F_A(x)$, respectively. Moreover, relative to A, we say "derivable" instead of "$\mathcal{A}$-derivable."

Remark 3.6 If $A \in \mathcal{BV}$ and F is a real-valued function on $\mathcal{BV}_A$, then clearly $D(F \lfloor A)(x) = 0$ for each $x \in \mathbf{R}^m - \operatorname{cl} A$. However, if A is not a figure and $x \in \operatorname{cl} A - A$, we may have $\underline{D}(F \lfloor A)(x) \neq 0$ even when F is an additive continuous function in A (cf Example 5.8).

Let $A \in \mathcal{BV}$, let $\Phi : A \to \mathbf{R}^m$ be a lipeomorphism, and let $B^\bullet = \Phi(B)$ for each $B \in \mathcal{BV}_A$. By [9, Lemmas 6.5 and 6.6], the map $B \mapsto B^\bullet$, still denoted by Φ, is a bijection from $\mathcal{BV}_A$ onto $\mathcal{BV}_{A^\bullet}$. According to Kirszbraun's theorem [7, Theorems 2.10.43], the map Φ can be extended to a Lipschitz map $\Psi : \mathbf{R}^m \to \mathbf{R}^m$. If Ψ is differentiable at $x \in A$, let $J_\Phi(x) = |\det D\Psi(x)|$ where $D\Psi(x)$ is the *derivative* of Ψ at x (in spite of the same notation, there is no danger of confusing the *derivative of a map* with the *derivate of a function*). In view of Rademacher's theorem [6, Section 3.1.2, Theorem 2], $J_\Phi(x)$ is defined for almost all $x \in A$, and by [11, Lemma 10.5.5], up to a negligible set, the function $J_\Phi : x \mapsto J_\Phi(x)$ is determined uniquely by Φ and does not depend on the extension Ψ. The area theorem [6, Section 3.3.2, Theorem 1], the Lebesgue derivation theorem [12, Chapter IV, Theorem 6.3], and Lemma 3.5 imply

$$D(\lambda \circ \Phi)_A(x) = J_\Phi(x)$$

for almost all $x \in A$. The following proposition is proved by a straightforward calculation.

Proposition 3.7 *Let $A \in \mathcal{BV}$, let $\Phi : A \to \mathbf{R}^m$ be a lipeomorphism, and let F be an additive continuous function in $\Phi(A)$. Then $F \circ \Phi$ is an additive continuous function in A, and*

$$D(F \circ \Phi)_A(x) = DF_{\Phi(A)}(\Phi(x)) \cdot D(\lambda \circ \Phi)_A(x)$$

at each $x \in A$ at which any two of the derivates in the formula exist.

4 Variations

Recall again that $\mathcal{A}$ stands either for $\mathcal{F}$ or $\mathcal{BV}$. An *$\mathcal{A}$-partition* is a collection (possibly empty) $P = \{(A_1, x_1), \dots, (A_p, x_p)\}$ where $A_1, \dots, A_p$ are nonoverlapping sets from $\mathcal{A}$, and $x_i \in A_i$ for $i = 1, \dots, p$. Given a positive $\eta < 1/(2m)$, a set $E \subset \mathbf{R}^m$, and a nonnegative function on E, we say that P is

1. *η-regular* if each A_i is η-regular;
2. *in* E if $\bigcup_{i=1}^p A_i \subset E$;
3. *anchored* in E if $\{x_1, \dots, x_p\} \subset E$;
4. *δ-fine* if it is anchored in E and $d(A_i) < \delta(x_i)$ for $i = 1, \dots, p$.

Remark 4.1 Notwithstanding DeGiorge's approximation (Proposition 1.1), for $m \geq 2$, there are significant differences between $\mathcal{F}$-partitions and $\mathcal{BV}$-partitions. For instance, if $P = \{(A_1, x_1), \dots, (A_p, x_p)\}$ is an $\mathcal{F}$-partition, then at most 2^m of the points $x_1, \dots, x_p$ can coalesce, while if P is a $\mathcal{BV}$-partition, then any number of the points $x_1, \dots, x_p$ can coalesce.

A nonnegative real-valued function defined on a set $E \subset \mathbf{R}^m$ is called a *gage* or an *essential gage* (abbreviated as *e-gage*) on E whenever its *null set* $N_\delta = \{x \in E : \delta(x) = 0\}$ is thin or negligible, respectively.

Let $E \subset \mathbf{R}^m$, and let F be a real-valued function defined on $\mathcal{A}$. Given a positive $\eta < 1/(2m)$ and a nonnegative function δ defined on E, set

$$V^{\mathcal{A}}_{\eta,\delta} F(E) = \sup_P \sum_{i=1}^{p} |F(A_i)|$$

where the supremum is taken over all η-regular $\mathcal{A}$-partitions $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ anchored in E that are δ-fine. The *$\mathcal{A}$-variation* or *essential $\mathcal{A}$-variation* (abbreviated as *e-$\mathcal{A}$-variation*) of F on E is the number

$$\sup_{0<\eta<\frac{1}{2m}} \inf_{\delta} V^{\mathcal{A}}_{\eta,\delta} F(E)$$

where the infimum is taken over all gages or e-gages on E, respectively; it is denoted by $V^{\mathcal{A}}_{*} F(E)$ or $V^{\mathcal{A}}_{e*} F(E)$, respectively. An easy verification reveals that the functions

$$V^{\mathcal{A}}_{*} F : E \mapsto V^{\mathcal{A}}_{*} F(E) \qquad \text{and} \qquad V^{\mathcal{A}}_{e*} F : E \mapsto V^{\mathcal{A}}_{e*} F(E)$$

are metric measures in $\mathbf{R}^m$ (cf. [14, Theorem 3.7] and [11, Section 3.2]), and that the measure $V^{\mathcal{A}}_{e*} F$ is absolutely continuous. The following inequalities are obvious:

$$V^{\mathcal{A}}_{e*} F \leq V^{\mathcal{A}}_{*} F\,, \quad V^{\mathcal{F}}_{*} F \leq V^{\mathcal{BV}}_{*} F\,, \quad V^{\mathcal{F}}_{e*} F \leq V^{\mathcal{BV}}_{e*} F\,.$$

In dimension one, a concept similar to $\mathcal{A}$-variation has been introduced in [14]. Versions of the e-$\mathcal{A}$-variation were studied previously in the real line (see [4] and [2]) and in an abstract measure space (see [1]).

Proposition 4.2 *If F is a real-valued function on $\mathcal{A}$, then $V^{\mathcal{A}}_{*} F = V^{\mathcal{A}}_{e*} F$ whenever $V^{\mathcal{A}}_{*} F$ is absolutely continuous.*

PROOF. Assume $V^{\mathcal{A}}_{*} F(N) = 0$ for every negligible set $N \subset \mathbf{R}^m$, and seeking a contradiction suppose $V^{\mathcal{A}}_{e*} F(E) < V^{\mathcal{A}}_{*} F(E)$ for an $E \subset \mathbf{R}^m$. There is a positive $\eta < 1/(2m)$ and an e-gage σ on E such that $V^{\mathcal{A}}_{\eta,\sigma} F(E) < V^{\mathcal{A}}_{\eta,\delta} F(E)$ for every gage δ on E. Since the null set N_δ of δ is negligible, $V^{\mathcal{A}}_{*} F(N_\delta) = 0$. Thus given $\varepsilon > 0$, we can find a gage ρ on N_δ so that $V^{\mathcal{A}}_{\eta,\rho} F(N_\delta) < \varepsilon$. Define a gage δ on E by setting

$$\delta(x) = \begin{cases} \sigma(x) & \text{if } x \in E - N_\delta, \\ \rho(x) & \text{if } x \in N_\delta, \end{cases}$$

and observe that

$$V^{\mathcal{A}}_{\eta,\delta} F(E) \leq V^{\mathcal{A}}_{\eta,\sigma} F(E - N_\delta) + V^{\mathcal{A}}_{\eta,\rho} F(N_\delta) < V^{\mathcal{A}}_{\eta,\sigma} F(E) + \varepsilon\,.$$

A contradiction follows from the arbitrariness of ε. Thus $V_*^{\mathcal{A}}F \leq V_{e*}^{\mathcal{A}}F$ and, as the reverse inequality always holds, the proposition is proved.

Let F be a real-valued function defined on $\mathcal{A}$. The *standard $\mathcal{A}$-variation* of F on $A \in \mathcal{A}$ is the number

$$V^{\mathcal{A}}F(A) = \sup \sum_{k=1}^{n} |F(A_k)|$$

where the supremum is taken over all nonoverlapping collections $\{A_1, \dots, A_n\} \subset \mathcal{A}_A$. If F is additive, a routine argument shows that the function $V^{\mathcal{A}}F$, defined on $\mathcal{A}$ in the obvious way, is additive whenever it is real-valued. Note that if F is a real-valued function defined only on $\mathcal{A}_A$, the number $V^{\mathcal{A}}F(B)$ has still meaning for each $B \in \mathcal{A}_A$, and $(V^{\mathcal{A}}F)\lfloor A = V^{\mathcal{A}}(F\lfloor A)$. Thus the standard $\mathcal{A}$-variation requires no *relativization* (see below).

Remark 4.3 If F is an additive continuous function in $\mathbf{R}^m$ and A is a figure, it is easy to deduce from Proposition 1.1 that $V^{\mathcal{BV}}F(A) = V^{\mathcal{F}}F(A)$. Thus for additive continuous functions in $\mathbf{R}^m$, we say "standard variation" instead of "standard $\mathcal{A}$-variation," and write VF instead of $V^{\mathcal{A}}F$.

Lemma 4.4 *Let F be an additive continuous function in $\mathbf{R}^m$. If $A \in \mathcal{BV}$ then $VF(A) \leq V_*^{\mathcal{BV}}F(A)$, and if $A \in \mathcal{F}$ then*

$$V_*^{\mathcal{F}}F(A) = V_*^{\mathcal{BV}}F(A) = VF(A)\,.$$

In particular, $V_*^{\mathcal{F}}F(E) = V_*^{\mathcal{BV}}F(E)$ whenever $E \subset \mathbf{R}^m$ is open, or either E or ∂E is thin.

PROOF. Proceeding towards a contradiction, assume $V_*^{\mathcal{BV}}F(A) < VF(A)$ for an $A \in \mathcal{BV}$. There is a nonoverlapping collection $\{A_1, \dots, A_n\} \subset \mathcal{BV}_A$ such that $V_*^{\mathcal{BV}}F(A) < \sum_{k=1}^{n} |F(A_k)|$. Choose a positive $\eta < 1/(2m)$ and find a gage δ on A so that

$$V_{\eta,\delta}^{\mathcal{BV}}F(A) < \sum_{k=1}^{n} |F(A_k)|\,.$$

Given $\varepsilon > 0$, it follows from [9, Lemma 7.2] that in each A_k there is an η-regular δ-fine $\mathcal{BV}$-partition $P_k = \{(B_1^k, x_1^k), , \dots, (B_{p_k}^k, x_{p_k}^k)\}$ such that

$$\sum_{i=1}^{p_k} |F(B_i^k)| \geq \left| F\left(\bigcup_{i=1}^{p_k} B_i^k\right)\right| > |F(A_k)| - \frac{\varepsilon}{n}\,.$$

Since $P = \bigcup_{k=1}^{n} P_k$ is an η-regular δ-fine $\mathcal{BV}$-partition in A,

$$V_{\eta,\delta}^{\mathcal{BV}}F(A) \geq \sum_{k=1}^{n}\sum_{i=1}^{p_k} |F(B_i^k)| > \sum_{k=1}^{n} |F(A_k)| - \varepsilon$$

and a contradiction follows from the arbitrariness of ε.

Now let A be a figure. Using Remark 4.3 and [11, Lemma 11.3.4 and Proposition 11.3.7], we can modify the previous argument to show $VF(A) \le V_*^{\mathcal{F}}F(A)$. Observe that $\delta : x \mapsto \operatorname{dist}(x, \partial A)$ is a gage in A, and that every δ-fine $\mathcal{BV}$-partition is a $\mathcal{BV}$-partition in A. Thus $V_{\eta,\delta}^{\mathcal{BV}}F(A) \le VF(A)$ for each positive $\eta < 1/(2m)$, and we conclude

$$VF(A) \le V_*^{\mathcal{F}}F(A) \le V_*^{\mathcal{BV}}F(A) \le VF(A) .$$

Clearly $V_*^{\mathcal{F}}F(E) = V_*^{\mathcal{BV}}F(E) = 0$ if E is thin. As each open set is the union of countably many nonoverlapping cubes, the lemma is proved.

Example 4.5 In dimension one $\mathcal{BV} = \mathcal{F}$, and so we write only V_* and V_{e*} instead of $V_*^{\mathcal{A}}$ and $V_{e*}^{\mathcal{A}}$, respectively. Let C be the Cantor ternary set in $A = [0,1]$, and let F be an additive continuous function in $\mathbf{R}$ whose distribution function extends the Cantor function in A [11, Example 5.3.11]. Since the function $\delta : x \mapsto \operatorname{dist}(x, C)$ is an e-gage on A and a gage on $A - C$, we have $V_{e*}F(A) = V_*F(A - C) = 0$. On the other hand, it follows from Lemma 4.4 that $V_*F(A) = VF(A) = F(A) = 1$, and so $V_*(C) = 1$.

Lemma 4.6 *If F is a real-valued function on $\mathcal{A}$, then the measures $V_*^{\mathcal{A}}F$ and $V_{e*}^{\mathcal{A}}F$ are Borel regular.*

Proof. We prove the lemma for $V_*^{\mathcal{A}}F$ using the technique of [14, Theorem 3.15]. The proof for $V_{e*}^{\mathcal{A}}F$ is analogous. Assume $V_*^{\mathcal{A}}F(E) < +\infty$, choose an $\varepsilon > 0$, and fix a positive $\eta < 1/(2m)$. Find a gage δ on E so that

$$V_{\eta,\delta}^{\mathcal{A}}F(E) < V_*^{\mathcal{A}}F(E) + \varepsilon ,$$

and let $E_n = \{x \in E : \delta(x) > 1/n\}$ for $n = 1, 2, \ldots$.

We claim $V_{\eta,1/n}^{\mathcal{A}}F(E_n) = V_{\eta,1/n}^{\mathcal{A}}F(\operatorname{cl} E_n)$. As $V_{\eta,1/n}^{\mathcal{A}}F(E_n) \le V_{\eta,1/n}^{\mathcal{A}}F(\operatorname{cl} E_n)$, it suffices to obtain a contradiction by supposing this inequality is sharp. Then there is an η-regular $(1/n)$-fine $\mathcal{A}$-partition $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ anchored in $\operatorname{cl} E_n$ for which

$$V_{\eta,1/n}^{\mathcal{A}}F(E_n) < \sum_{i=1}^{p} |F(A_i)| .$$

Employing the additivity and continuity of F, it is easy to modify P so that it becomes anchored in E_n and still satisfies the other conditions, a contradiction.

From the claim, we infer

$$\begin{aligned} \inf_{\sigma} V_{\eta,\sigma}^{\mathcal{A}}F(\operatorname{cl} E_n) &\le V_{\eta,1/n}^{\mathcal{A}}F(\operatorname{cl} E_n) = V_{\eta,1/n}^{\mathcal{A}}F(E_n) \le V_{\eta,\delta}^{\mathcal{A}}F(E_n) \\ &\le V_{\eta,\delta}^{\mathcal{A}}F(E) < V_*^{\mathcal{A}}F(E) + \varepsilon , \end{aligned}$$

where the infimum is taken over all gages σ on E. The arbitrariness of η yields $V_*^{\mathcal{A}}F(\mathrm{cl}\,E_n) \le V_*^{\mathcal{A}}F(E) + \varepsilon$, and thus

$$V_*^{\mathcal{A}}F\left(\bigcup_{n=1}^{\infty} \mathrm{cl}\,E_n\right) = \lim V_*^{\mathcal{A}}F(\mathrm{cl}\,E_n) \le V_*^{\mathcal{A}}F(E) + \varepsilon\,;$$

for $\{\mathrm{cl}\,E_n\}$ is an increasing sequence of closed sets.

Since $E - N_\delta \subset \bigcup_{n=1}^{\infty} \mathrm{cl}\,E_n$, it follows from the arbitrariness of ε that there is a Borel set B such that $E - N_\delta \subset B$ and $V_*^{\mathcal{A}}F(E) = V_*^{\mathcal{A}}F(B)$. Now the thin set N_δ is contained in a thin Borel set C [6, Section 2.1, Theorem 1]. As $V_*^{\mathcal{A}}F(C) = 0$, the lemma is proved.

Our next result improves on [3, Theorem 3.3]. Its proof is similar to that given in [1, Theorem 1] for an abstract measure space with a derivation base.

Theorem 4.7 *If F is a function defined on the family $\mathcal{BV}$, then*

$$V_{e*}^{\mathcal{F}}F(E) = \int_E \overline{D}^{\mathcal{F}}|F|\,d\lambda \qquad \text{and} \qquad V_{e*}^{\mathcal{BV}}F(E) \le \int_E \overline{D}^{\mathcal{BV}}|F|\,d\lambda$$

for each measurable set $E \subset \mathbf{R}^m$.

Proof. As $V_{e*}^{\mathcal{A}}F = V_{e*}^{\mathcal{A}}|F|$, we suppose $F \ge 0$. Select a measurable set $E \subset \mathbf{R}^m$ and note that the integral $I = \int_E \overline{D}^{\mathcal{A}}F\,d\lambda$ exists (possibly equal to $+\infty$), since $\overline{D}^{\mathcal{A}}F \ge 0$ is a measurable function.

First we prove the inequality $V_{e*}^{\mathcal{A}}F(E) \le I$. If the set $E_\infty = \{x \in E : \overline{D}^{\mathcal{A}}F(x) = +\infty\}$ has positive measure, then $I = +\infty$ and the inequality holds. If E_∞ is negligible, then $V_{e*}^{\mathcal{A}}F(E_\infty) = \int_{E_\infty} \overline{D}^{\mathcal{A}}F\,d\lambda = 0$, and no generality is lost by assuming $E_\infty = \emptyset$. Under this assumption, the measurable sets

$$E_n = \{x \in E \cap U(\mathbf{0}, n) : \overline{D}^{\mathcal{A}}F(x) < n\},\ n = 1, 2, \dots\,,$$

form an increasing sequence whose union is E, and so it suffices to prove the inequality for each E_n.

Consequently, we may assume from the onset $I < +\infty$ and there is an open set $U \subset \mathbf{R}^m$ such that $E \subset U$ and $|U| < +\infty$. Let

$$G(A) = F(A) - \int_{A \cap E} \overline{D}^{\mathcal{A}}F\,d\lambda$$

for each $A \in \mathcal{A}$, and observe that the set $N = \{x \in E : \overline{D}^{\mathcal{A}}G(x) \ne 0\}$ is negligible [12, Chapter IV, Theorem 6.3].

Choose an $\varepsilon > 0$ and a positive $\eta < 1/(2m)$, and define an e-gage δ on E as follows: if $x \in N$ let $\delta(x) = 0$, and if $x \in E - N$ select $\delta(x) > 0$ so that $U(x, \delta(x)) \subset U$ and $G(A) < \varepsilon|A|$ for each η-regular $A \in \mathcal{A}$ with $x \in A$

and $d(A) < \delta(x)$. Now given an η-regular δ-fine $\mathcal{A}$-partition $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ anchored in E, we obtain

$$\sum_{i=1}^{p} F(A_i) < \sum_{i=1}^{p} \left(\varepsilon |A_i| + \int_{A_i \cap E} \overline{D}^{\mathcal{A}} F \, d\lambda \right) < \varepsilon |U| + I$$

and so $V_{\eta,\delta}^{\mathcal{A}} F(E) \leq \varepsilon |U| + I$. The desired inequality follows from the arbitrarines of ε and η.

Next assume $\mathcal{A} = \mathcal{F}$ and $V_{e*}^{\mathcal{F}} F < I$. Fix an integer $n \geq 1$. For each $x \in E_\infty$ there is a positive $\eta_x < 1/(2m)$ such that given $\theta > 0$, we can find an η_x-regular figure $A \subset U(x, \theta)$ with $x \in A$ and $F(A) > n|A|$. Given an integer $k \geq 1$, let $C_k = \{x \in E_\infty : \eta_x > 1/k\}$, and find an e-gage δ on E_∞ so that

$$V_{1/k,\delta}^{\mathcal{F}} F(E_\infty) < V_{e*}^{\mathcal{F}} F(E_\infty) + 1 \leq V_{e*}^{\mathcal{F}} F(E) + 1 < +\infty \, .$$

The family $\mathcal{C}$ of all $(1/k)$-regular figures A with $d(A) < \delta(x)$ for an $x \in A \cap C_k$ and $F(A) > n|A|$ is a Vitali cover of $C_k - N_\delta$. Using Vitali's covering theorem [12, Chapter IV, Theorem 3.1] and the negligibility of N_δ, find a $(1/k)$-regular δ-fine $\mathcal{F}$-partition $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ anchored in E_∞ such that $F(A_i) > n|A_i|$ for $i = 1, \ldots, p$ and $\sum_{i=1}^{p} |A_i| > |C_k|/2$. It follows

$$|C_k| < 2 \sum_{i=1}^{p} |A_i| < \frac{2}{n} \sum_{i=1}^{p} F(A_i) \leq \frac{2}{n} V_{1/k,\delta}^{\mathcal{F}} F(E_\infty) \leq \frac{2}{n} [V_{e*}^{\mathcal{F}} F(E) + 1]$$

and, as $\{C_k\}$ is an increasing sequence whose union is E_∞, we obtain

$$|E_\infty| \leq \frac{2}{n} [V_{e*}^{\mathcal{F}} F(E) + 1] \, .$$

By the arbitrariness of n, the set E_∞ is negligible. In view of this, we can proceed with the argument assuming the statements made in the third paragraph of this proof.

Choose a positive $\eta < 1/(2m)$ and find an e-gage δ on E with $V_{\eta,\delta}^{\mathcal{F}} F(E) < I$. Making δ smaller, we may assume $N \subset N_\delta$ and $U(x, \delta(x)) \subset U$ for each $x \in E$. Given $\varepsilon > 0$, the family $\mathcal{K}$ of all η-regular figures B with $d(B) < \delta(x)$ for an $x \in B \cap E$ and $G(B) > -\varepsilon |B|$ is a Vitali cover of $E - N_\delta$. Hence there is a disjoint sequence $\{B_i\}$ in $\mathcal{K}$ whose union covers E almost entirely. For $i = 1, 2, \ldots$, select an $x_i \in B_i$ so that $d(B_i) < \delta(x_i)$, and observe that for each integer $p \geq 1$, the collection $\{(B_1, x_1), \ldots, (B_p, x_p)\}$ is an η-regular δ-fine $\mathcal{F}$-partition in U anchored in E. Thus

$$I = \sum_{i=1}^{\infty} \int_{B_i \cap E} \overline{D}^{\mathcal{F}} F \, d\lambda \leq \sum_{i=1}^{\infty} F(B_i) + \varepsilon \sum_{i=1}^{\infty} |B_i| \leq V_{\eta,\delta}^{\mathcal{F}} F(E) + \varepsilon |U|$$

and a contradiction follows from the arbitrariness of ε.

Note that the second part of the previous proof does not apply to $\mathcal{A} = \mathcal{BV}$. Indeed, Vitali's covering theorem cannot be used, since the sets from $\mathcal{BV}$ need not be closed. In view of Lemma 3.1, the difficulty disappears when F is an additive continuous function in $\mathbf{R}^m$. This and Lemma 4.6 yield the following corollary.

Corollary 4.8 *If F is an additive continuous function in $\mathbf{R}^m$, then*

$$V_{e*}^{\mathcal{F}} F(E) = V_{e*}^{\mathcal{BV}} F(E) = \int_E \overline{D}|F| \, d\lambda$$

for each measurable set $E \subset \mathbf{R}^m$. In particular, $V_{e*}^{\mathcal{F}} F = V_{e*}^{\mathcal{BV}} F$.

In view of Corollary 4.8, for additive continuous functions in $\mathbf{R}^m$, we now say "e-variation" instead of "e-$\mathcal{A}$-variation," and write $V_{e*}F$ instead of $V_{e*}^{\mathcal{A}} F$.

Corollary 4.9 *An additive continuous function F in $\mathbf{R}^m$ is derivable almost everywhere in a set $E \subset \mathbf{R}^m$ if and only if E has σ-finite measure $V_{e*}F$.*

Proof. If $E = \bigcup_{n=1}^{\infty} E_n$ and $V_{e*}F(E_n) < +\infty$ for $n = 1, 2, \ldots$, then F is almost derivable almost everywhere in E_n by Theorem 4.7. This and Theorem 3.3 imply F is derivable almost everywhere in E.

Conversely, if F is derivable almost everywhere in E, there is a negligible set $E_0 \subset E$ such that $D|F|(x) = |DF(x)| < +\infty$ for each $x \in E - E_0$. Letting

$$E_n = \{x \in (E - E_0) \cap U(\mathbf{0}, n) : D|F(x)| < n\}$$

for $n = 1, 2, \ldots$, Theorem 4.7 yields

$$V_{e*}F(E_n) = \int_{E_n} D|F|(x) \, d\lambda(x) \leq n|U(\mathbf{0}, n)| < +\infty \, .$$

Since $V_{e*}F(E_0) = 0$ and $E = \bigcup_{n=0}^{\infty} E_n$, the corollary is proved.

As with the derivates, we *relativize* the concept of variations. Let $A \in \mathcal{A}$, $E \subset A$, and let F be a real-valued function on $\mathcal{A}$. Given a positive $\eta < 1/(2m)$ and a nonnegative function δ on E, set

$$V_{\eta,\delta}^{\mathcal{A}} F_A(E) = \sup_P \sum_{i=1}^{p} |F(A_i)|$$

where the supremum is taken over all η-regular $\mathcal{A}$-partitions $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ in A anchored in E that are δ-fine. The *$\mathcal{A}$-variation* of F on E *relative* to A is the number

$$V_*^{\mathcal{A}} F_A(E) = \sup_{0<\eta<\frac{1}{2m}} \inf_{\delta} V_{\eta,\delta}^{\mathcal{A}} F_A(E)$$

where the infimum is taken over all gages δ on E.

The *e-$\mathcal{A}$-variation* $V_{e*}^{\mathcal{A}}F_A(E)$ of F on E *relative* to A, as well as the measures $V_*^{\mathcal{A}}F_A$ and $V_{e*}^{\mathcal{A}}F_A$ in A, are defined in the obvious way. Since the variations of F relative to A do not depend on the values F takes on $\mathcal{A} - \mathcal{A}_A$, they remain meaningful when F is defined only on $\mathcal{A}_A$.

Let $A \in \mathcal{A}$ and let F be a real-valued function on $\mathcal{A}_A$. A direct calculation shows

$$(V_*^{\mathcal{A}}F_A)\lfloor A \leq V_*^{\mathcal{A}}(F\lfloor A) \qquad \text{and} \qquad (V_{e*}^{\mathcal{A}}F_A)\lfloor A \leq V_{e*}^{\mathcal{A}}(F\lfloor A)\,.$$

Since the boundary of a figure is thin and closed, it is easy to see that the inequalities above become *equalities* when $\mathcal{A} = \mathcal{F}$.

Lemma 4.10 *Let $A \in \mathcal{BV}$, and let F be an additive continuous function in A. Then*

$$V_*^{\mathcal{BV}}F_A(E) = V_*^{\mathcal{BV}}(F\lfloor A)(E) \quad \textit{and} \quad V_{e*}^{\mathcal{BV}}F_A(E) = V_{e*}(F\lfloor A)(E)$$

for every set $E \subset A$.

PROOF. Seeking a contradiction, suppose $V_*^{\mathcal{BV}}F_A(E) < V_*^{\mathcal{BV}}(F\lfloor A)(E)$ for a set $E \subset A$. There is a positive $\eta < 1/(2m)$ and a gage δ on E such that $V_{\eta^{m+1},\delta}^{\mathcal{BV}}F_A(E) < V_{\eta,\delta}^{\mathcal{BV}}(F\lfloor A)(E)$. Find an η-regular δ-fine $\mathcal{BV}$-partition $\{(A_1,x_1),\dots,(A_p,x_p)\}$ anchored in E so that

$$V_{\eta^{m+1},\delta}^{\mathcal{BV}}F_A(E) < \sum_{i=1}^{p} |(F\lfloor A)(A_i)| = \sum_{i=1}^{p} |F(A \odot A_i)|\,.$$

With no loss of generality, we may assume $A - \operatorname{int}^c A \subset N_\delta$. Making δ smaller, we may also assume that $\{(A \odot A_1, x_1),\dots,,(A \odot A_p, x_p)\}$ is an η^{m+1}-regular $\mathcal{BV}$-partition [10, Lemma 1.2], a contradiction.

Thus $V_*^{\mathcal{BV}}F_A(E) \geq V_*^{\mathcal{BV}}(F\lfloor A)(E)$ and, as the reverse inequality always holds, the first equality is established. The proof of the second equality is similar.

Remark 4.11 If $A \in BV$ and F is a real-valued function on $\mathcal{BV}_A$, then

$$V_*^{\mathcal{A}}(F\lfloor A)(\mathbf{R}^m - \operatorname{cl} A) = 0\,.$$

However, if A is not a figure, we may have $V_*^{\mathcal{A}}(F\lfloor A)(\operatorname{cl} A - A) > 0$ even when F is an additive continuous function in A (cf. Example 5.8). Whether $V_{e*}(F\lfloor A)(\mathbf{R}^m - \operatorname{cl} A) = 0$ for an additive continuous function F in A is unclear.

In view of Lemma 4.10, for an additive continuous function F in $A \in \mathcal{A}$, we write $V_{e*}F_A$ instead of $V_{e*}^{\mathcal{A}}F_A$.

Proposition 4.12 *Let $A \in \mathcal{BV}$. For an additive continuous function F in A the following conditions hold.*

1. *F is derivable relative to A almost everywhere in A if and only if $V_{e*}F_A$ is σ-finite.*

2. *If F is almost derivable relative to A at every $x \in \operatorname{int}^c A - T$, where T is a thin set, then $V_*^{\mathcal{BV}} F_A$ is σ-finite and absolutely continuous.*

PROOF. By Lemma 4.10, the measure $V_{e*}F_A$ is σ-finite if and only if the set A has σ-finite measure $V_{e*}(F\lfloor A)$. Since $A - \operatorname{int}^c A$ is a negligible set, the first condition follows from Corollary 4.9 and Lemma 3.5.

If the assumption of the second condition is satisfied, then F is derivable relative to A almost everywhere in A by Theorem 3.3 and Lemma 3.5. In particular, $V_{e*}F_A$ is σ-finite according to the previous paragraph. Making T larger, we may assume it contains $A - \operatorname{int}^c A$. Now choose a negligible set $E \subset A$ and a positive $\eta < 1/(2m)$. For $n = 1, 2, \ldots$, let

$$E_n = \{x \in E - T : n - 1 \le \overline{D}_\eta |F|_A(x) < n\}$$

and find open sets U_n so that $E_n \subset U_n$ and $|U_n| < \eta 2^{-n}/n$. Given $x \in E_n$ there is a $\delta_n(x) > 0$ such that $U(x, \delta_n(x)) \subset U_n$ and $|F(B)| < n|B|$ for every η-regular $B \in \mathcal{BV}$ with $x \in B$ and $B \subset A \cap U(x, \delta_n(x))$. Since $E - T$ is the disjoint union of the sets E_n, the formula

$$\delta(x) = \begin{cases} \delta_n(x) & \text{if } x \in E_n, \\ 0 & \text{if } x \in E \cap T, \end{cases}$$

defines a gage on E. For an η-regular $\mathcal{BV}$-partition $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ in A anchored in E that is δ-fine, we obtain

$$\begin{aligned} \sum_{i=1}^{p} |F(A_i)| &= \sum_{n=1}^{\infty} \sum_{x_i \in E_n} |F(A_i)| < \sum_{n=1}^{\infty} \sum_{x_i \in E_n} n|A_i| \\ &\le \sum_{n=1}^{\infty} n|U_n| < \sum_{n=1}^{\infty} \eta 2^{-n} = \eta\,. \end{aligned}$$

Thus $V_{\eta,\delta}^{\mathcal{BV}} F_A(E) \le \eta$, and so $V_*^{\mathcal{BV}} F_A(E) = 0$ by the arbitrariness of η. An application of Proposition 4.2 completes the proof.

Example 4.13 Let F be the flux of a uniformly continuous vector field v on $A \in \mathcal{BV}$. If v is almost differentiable relative to A at every $x \in A - T$, where T is a thin set, then $V_*^{\mathcal{BV}} F_A$ is σ-finite and absolutely continuous (Example 3.4 and Proposition 4.12).

Proposition 4.14 *Let $A \in \mathcal{BV}$, let $\Phi : A \to \mathbf{R}^m$ be a lipeomorphism, and let F be a real-valued function defined on $\mathcal{BV}_{\Phi(A)}$. Then*

$$V_*^{\mathcal{BV}}(F \circ \Phi)_A = (V_*^{\mathcal{BV}} F_{\Phi(A)}) \circ \Phi \quad \textit{and} \quad V_{e*}^{\mathcal{BV}}(F \circ \Phi)_A = (V_{e*}^{\mathcal{BV}} F_{\Phi(A)}) \circ \Phi \,.$$

If both A and $\Phi(A)$ are figures and F is an additive continuous function in $\Phi(A)$, then

$$V_*^{\mathcal{F}}(F \circ \Phi)_A = (V_*^{\mathcal{F}} F_{\Phi(A)}) \circ \Phi \quad \textit{and} \quad V_{e*}^{\mathcal{F}}(F \circ \Phi)_A = (V_{e*}^{\mathcal{F}} F_{\Phi(A)}) \circ \Phi \,.$$

PROOF. If $x \in A$ and $E \subset A$, let $x^\bullet = \Phi(x)$ and $E^\bullet = \Phi(E)$. There are positive constants a, b such that

$$a|x-y| \le |x^\bullet - y^\bullet| \le b|x-y|$$

for all $x, y \in A$. If $B \subset A$ is a figure, then there is a sequence $\{K_n\}$ of subfigures of $B^\bullet$ such that $\lim |K_n| = |B^\bullet|$ and $\sup \|K_n\| \le \gamma \|B^\bullet\|$ where $\gamma \ge 1$ is a constant depending only on the dimension m [11, Lemma 12.6.4]. Since $B^\bullet$ is essentially closed, we also have $d(B^\bullet) = \lim d(K_n)$.

Suppose $V_*^{\mathcal{BV}} F_{A^\bullet}(E^\bullet) < V_*^{\mathcal{BV}}(F \circ \Phi)_A(E)$ for an $E \subset A$. There is a positive $\eta < 1/(2m)$ such that given a positive $\eta^\bullet < 1/(2m)$, we can find a gage $\delta^\bullet$ on $E^\bullet$ with

$$V_{\eta^\bullet,\delta^\bullet}^{\mathcal{BV}} F_{A^\bullet}(E^\bullet) < V_{\eta,\delta}^{\mathcal{BV}}(F \circ \Phi)_A(E)$$

for any gage δ on E. We make the appropriate choices: let $\eta^\bullet = (a/b)^m(\eta/\gamma)$, and define a gage δ on E by setting $\delta(x) = \delta^\bullet(x^\bullet)/b$ for each $x \in E$. Now find an η-regular δ-fine $\mathcal{BV}$-partition $\{(A_1, x_1), \dots, (A_p, x_p)\}$ in A anchored in E so that

$$V_{\eta^\bullet,\delta^\bullet}^{\mathcal{BV}} F_{A^\bullet}(E^\bullet) < \sum_{i=1}^{p} |F \circ \Phi(A_i)| = \sum_{i=1}^{p} |F(A_i^\bullet)| \,.$$

A simple calculation shows that $\{(A_1^\bullet, x_1^\bullet), \dots, (A_p^\bullet, x_p^\bullet)\}$ is an $\eta^\bullet$-regular $\delta^\bullet$-fine $\mathcal{BV}$-partition in $A^\bullet$ anchored in $E^\bullet$, a contradiction. Thus

$$V_{\eta,\delta}^{\mathcal{BV}}(F \circ \Phi)_A(E) \le V_{\eta^\bullet,\delta^\bullet}^{\mathcal{BV}} F_{A^\bullet}(E^\bullet) \,. \tag{8}$$

An identical argument shows that

$$V_{\eta,\delta}^{\mathcal{F}}(F \circ \Phi)_A(E) \le V_{\eta^\bullet,\delta^\bullet}^{\mathcal{F}} F_{A^\bullet}(E^\bullet) \tag{9}$$

when the dimension $m = 1$. If $m \ge 2$ and both A and $A^\bullet$ are figures, the assumption

$$V_{\eta^\bullet,\delta^\bullet}^{\mathcal{F}} F_{A^\bullet}(E^\bullet) < V_{\eta,\delta}^{\mathcal{F}}(F \circ \Phi)_A(E)$$

for a set $E \subset A$ leads to the existence of an η-regular δ-fine $\mathcal{F}$-partition $\{(A_1, x_1), \dots, (A_p, x_p)\}$ in A anchored in E so that

$$V_{\eta^\bullet,\delta^\bullet}^{\mathcal{F}} F_{A^\bullet}(E^\bullet) < \sum_{i=1}^{p} |F \circ \Phi(A_i)| = \sum_{i=1}^{p} |F(A_i^\bullet)| \,.$$

As before, $\{(A_1^\bullet, x_1^\bullet), \ldots, (A_p^\bullet, x_p^\bullet)\}$ is an $\eta^\bullet$-regular $\delta^\bullet$-fine $\mathcal{BV}$-partition in $A^\bullet$ anchored in $E^\bullet$, however, this is not yet a contradiction. On the other hand, if F is an additive continuous function in $A^\bullet$, then for $i = 1, \ldots, p$, we can find $\eta^\bullet$-regular figures $B_i \subset A_i^\bullet$ with $d(B_i)$ arbitrarily close to $d(A_i^\bullet)$ and such that

$$V_{\eta^\bullet,\delta^\bullet}^{\mathcal{F}} F_{A^\bullet}(E^\bullet) < \sum_{i=1}^{p} |F(B_i)| \,.$$

The collection $Q = \{(B_1, x_1^\bullet), \ldots, (B_p, x_p^\bullet)\}$ is still not an $\mathcal{F}$-partition, since $x_i^\bullet$ may be outside B_i. However, no more than 2^m of the points $x_1^\bullet, \ldots, x_p^\bullet$ can coalesce; for $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ is an $\mathcal{F}$-partition (Remark 4.1) and Φ is bijective. Thus Q can be easily modified to an $\eta^\bullet$-regular $\delta^\bullet$-fine $\mathcal{F}$-partition $\{C_1, x_1^\bullet), \ldots, (C_p, x_p^\bullet)\}$ in $A^\bullet$ anchored in $E^\bullet$ for which

$$V_{\eta^\bullet,\delta^\bullet}^{\mathcal{F}} F_{A^\bullet}(E^\bullet) < \sum_{i=1}^{p} |F(C_i)| \tag{10}$$

(cf. [11, Theorem 12.7.6]). As (10) is contradictory, (9) holds again.

The reverse inequalities to (8) and (9) are obtained by applying (8) and (9) to the inverse map Φ^{-1} and the function $F \circ \Phi$. The proofs for the essential variations are identical.

5 Riemann type integrals

Definition 5.1 A real-valued function f defined on $A \in \mathcal{A}$ is called $\mathcal{A}$-*integrable* if there is an additive continuous function F in A satisfying the following condition: given $\varepsilon > 0$, we can find a gage δ on A so that

$$\sum_{i=1}^{p} \Big| f(x_i)|A_i| - F(A_i) \Big| < \varepsilon$$

for each ε-regular δ-fine $\mathcal{A}$-partition $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ in A.

It follows from [9, Propositions 7.7 and 7.8] and [11, Theorem12.2.2] that F, called the *indefinite $\mathcal{A}$-integral* of f in A, is uniquely determined by f. If f is $\mathcal{A}$-integrable in A, it is $\mathcal{A}$-integrable in each $B \in \mathcal{A}_A$, and $F\lceil\mathcal{A}_B$ is the indefinite $\mathcal{A}$-integral of f in B. The real number $F(A)$ is called the $\mathcal{A}$-*integral* of f over A, denoted by $\int_A f\,d\lambda$. Since the $\mathcal{F}$-integral, $\mathcal{BV}$-integral, and Lebesgue integral coincide on the intersections of their domains ([9, Proposition 5.8] and [11, Theorem 11.4.5]), this notation leads to no confusion.

Let $A \in \mathcal{A}$. Since the $\mathcal{A}$-integral of f over A does not depend on the values of f in a negligible set ([9, Corollary 5.9] and [11, Corollary 11.4.7]), the concepts of $\mathcal{A}$-integrability and $\mathcal{A}$-integral can be readily extended to functions defined almost everywhere in A. We shall assume such an extension has been made, and denote by $\mathcal{A}(A)$ the family of all functions defined almost everywhere in A that are $\mathcal{A}$-integrable. If A is a figure, then $\mathcal{BV}(A) \subset \mathcal{F}(A)$ but we do not know whether the inclusion is proper.

Proposition 5.2 *Let $A \in \mathcal{A}$ and $f \in \mathcal{A}(A)$. If F is the indefinite integral of f, then $V_*^{\mathcal{A}} F_A$ is σ-finite and absolutely continuous.*

Proof. Let $E_n = \{x \in A : |f(x)| \leq n\}$ for $n = 1, 2, \ldots$, and let E be a negligible subset of A. With no loss of generality, we may assume f is a real-valued function defined on A such that $f(x) = 0$ for each $x \in E$. In particular $A = \bigcup_{n=1}^{\infty} E_n$. Choose a positive $\eta < 1/(2m)$, and find a gage δ on A so that

$$\sum_{i=1}^{p} \Big| f(x_i)|A_i| - F(A_i) \Big| < \eta$$

for each η-regular δ-fine $\mathcal{A}$-partition $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ in A. If P is anchored in E_n, then

$$\sum_{i=1}^{p} |F(A_i)| \leq \sum_{i=1}^{p} |f(x_i)| \cdot |A_i| + \eta < n|A| + \eta\,,$$

and hence $V_{\eta,\delta}^{\mathcal{A}} F_A(E_n) \leq n|A| + \eta$. If P is anchored in E, then $\sum_{i=1}^{p} |F(A_i)| < \eta$, and so $V_{\eta,\delta}^{\mathcal{A}} F_A(E) \leq \eta$. From the arbitrariness of η, we conclude

$$V_*^{\mathcal{A}} F_A(E_n) \leq n|A| \qquad \text{and} \qquad V_*^{\mathcal{A}} F_A(E) = 0\,,$$

which proves the proposition.

Theorem 5.3 *If F is an additive continuous function in $A \in \mathcal{A}$, then the following conditions are equivalent.*

1. *$V_*^{\mathcal{A}} F_A$ is σ-finite and absolutely continuous.*
2. *DF_A belongs to $\mathcal{A}(A)$, and F is its indefinite $\mathcal{A}$-integral.*

Proof. As $(2 \Rightarrow 1)$ follows immediately from Proposition 5.2, it suffices to prove $(1 \Rightarrow 2)$. By Proposition 4.12, the set E of all $x \in A$ at which $DF_A(x)$ does not exists is negligible. We let

$$f(x) = \begin{cases} DF_A(x) & \text{if } x \in A - E, \\ 0 & \text{if } x \in E, \end{cases}$$

and show that F is the indefinite $\mathcal{A}$-integral of f. To this end, choose a positive $\varepsilon < 1/(2m)$, and find a gage δ_E on E so that $\sum_{j=1}^{q} |F(B_j)| < \varepsilon$ for each ε-regular δ_E-fine $\mathcal{A}$-partition $\{(B_1, y_1), \ldots, (B_q, y_q)\}$ anchored in E; such a gage exists, since $V_*^{\mathcal{A}} F_A(E) = 0$ by our assumptions. On $A - E$ there is a positive function Δ such that

$$\left| f(x)|B| - F(B) \right| < \varepsilon |B|$$

for each $x \in A - E$ and each ε-regular $B \in \mathcal{A}_A$ with $x \in B$ and $d(B) < \Delta(x)$. Now define a gage δ on A by setting

$$\delta(x) = \begin{cases} \Delta(x) & \text{if } x \in A - E, \\ \delta_E(x) & \text{if } x \in E, \end{cases}$$

and select an ε-regular δ-fine $\mathcal{A}$-partition $\{(A_1, x_1), \dots, (A_p, x_p)\}$ in A. Then

$$\sum_{i=1}^{p} \left| f(x_i)|A_i| - F(A_i) \right| < \sum_{x_i \in E} |F(A_i)| + \varepsilon \sum_{x_i \notin E} |A_i| < \varepsilon (1 + |A|),$$

and the desired conclusion follows.

Theorem 5.3 gives the full descriptive definition of the $\mathcal{A}$-integral (cf. [11, Remark 5.3.6]). It facilitates simple proofs of several important results.

Corollary 5.4 *Let $A \in \mathcal{A}$, and let F be the indefinite integral of $f \in \mathcal{A}(A)$. Then $DF_A(x) = f(x)$ for almost all $x \in A$.*

PROOF. By Proposition 5.2 and Theorem 5.3, the derivate $DF_A(x)$ exists for almost all $x \in A$, and F is the indefinite $\mathcal{A}$-integral of DF_A. The corollary follows from [9, Corollary 5.14] and [11, Proposition 6.3.7], which assert that two $\mathcal{A}$-integrable functions with the same indefinite $\mathcal{A}$-integral are equal almost everywhere.

Corollary 5.5 *Let T be a thin set, and let F be an additive continuous function in $A \in \mathcal{A}$ that is almost derivable relative to A at each $x \in \operatorname{int}^c A - T$. Then DF_A belongs to $\mathcal{A}(A)$ and F is its indefinite integral.*

This corollary follows immediately from Proposition 4.12 and Theorem 5.3. Its immediate consequence is the *divergence theorem* for the $\mathcal{A}$-integral.

Theorem 5.6 *Let T be a thin set and let v be a uniformly continuous vector field on $A \in \mathcal{BV}$. If v is almost differentiable relative to A at every $x \in \operatorname{int}^c A - T$, then* $\operatorname{div} v$ *belongs to $\mathcal{BV}(A)$ and*

$$\int_A \operatorname{div} v \, d\lambda = \int_{\partial^* A} v \cdot \nu_A \, d\mathcal{H}.$$

PROOF. By Example 3.4 and Corollary 5.5, the flux of v is the indefinite $\mathcal{BV}$-integral of $\operatorname{div} v$.

Theorem 5.7 *Let $A \in \mathcal{BV}$, and let $\Phi : A \to \mathbf{R}^m$ be a lipeomorphism. If $f \in \mathcal{BV}(\Phi(A))$, then $(f \circ \Phi)J_\Phi$ belongs to $\mathcal{BV}(A)$. If both A and $\Phi(A)$ are figures and $f \in \mathcal{F}(\Phi(A))$, then $(f \circ \Phi)J_\Phi$ belongs to $\mathcal{F}(A)$. In either case,*

$$\int_{\Phi(A)} f \, d\lambda = \int_A (f \circ \Phi) J_\Phi \, d\lambda .$$

PROOF. Let F be the indefinite $\mathcal{A}$-integral of $f \in \mathcal{A}(\Phi(A))$. It follows from Corollary 5.4 and Proposition 3.7 that

$$D(F \circ \Phi)_A(x) = DF_{\Phi(A)}(\Phi(x))J_\Phi(x) = (f \circ \Phi)(x)J_\Phi(x)$$

for almost all $x \in A$. Thus $F \circ \Phi$ is the indefinite $\mathcal{A}$-integral of $(f \circ \Phi)J_\Phi$ by Proposition 4.14 and Theorem 5.3.

Example 5.8 Using [9, Example 5.21], we can construct an open BV set U, and a uniformly continuous vector field v on $A = \mathrm{cl}^* U$ which is differentiable in U and satisfies the following condition: if C is a cell containing U and

$$f(x) = \begin{cases} \operatorname{div} v(x) & \text{if } x \in U, \\ 0 & \text{if } x \in C - U, \end{cases}$$

then f is not $\mathcal{BV}$-integrable in C. Note that $A \in \mathcal{BV}$ and $\operatorname{div} v$ is $\mathcal{BV}$-integrable in A by Theorem 5.6.

Let F be the flux of v in A, let $G = F\lfloor A$, and let C be a cell containing A. Then $V_*^{\mathcal{BV}} G(\mathrm{cl}\, A - A) > 0$, and there is an $S \subset \mathrm{cl}\, A - A$ such that S is not thin and for each $x \in S$, either $\underline{D}G(x) \neq 0$ or $\overline{D}G(x) \neq 0$ (cf. Remarks 3.6 and 4.11).

Indeed, if $V_*^{\mathcal{BV}} G(\mathrm{cl}\, A - A) = 0$, then it follows from Lemmas 4.10 and 3.5, Remark 4.11, and Theorem 5.3 that $H = G\lceil \mathcal{BV}_C$ is the indefinite $\mathcal{BV}$-integral of DH_C in C. The same conclusion follows from Example 3.2, Remark 3.6, and Corollary 5.5 if the set $\{x \in \mathrm{cl}\, A - A : \overline{D}|G|(x) > 0\}$ is thin. Moreover, in both cases we have $DH_C(x) = DG(x) = 0$ for almost all $x \in C - A$ (Theorem 4.7). Since $DH_C(x) = \operatorname{div} v(x)$ for each $x \in U$ (Example 3.2), we have a contradiction.

The next proposition indicates the contrast between the Lebesgue integral and $\mathcal{A}$-integral (cf. Theorem 5.3).

Proposition 5.9 *If F is an additive continuous function in $A \in \mathcal{A}$, then the following conditions are equivalent.*

1. *$V_*^{\mathcal{BV}} F_A$ is finite and absolutely continuous.*
2. *$V_*^{\mathcal{A}} F_A$ is finite and absolutely continuous.*
3. *DF_A belongs to $L^1(A)$, and F is its indefinite Lebesgue integral.*

PROOF. $(2 \Rightarrow 3)$ By Theorem 5.3, the derivate DF_A belongs to $\mathcal{A}(A)$, and F is its indefinite $\mathcal{A}$-integral. Since $\mu = V_*^{\mathcal{A}} F_A$ is an absolutely continuous metric measure in A, all measurable subsets of A are μ-measurable. As μ is also finite and Borel regular (Lemma 4.6),

$$\mu(E) = \inf\{\mu(A \cap U) : U \subset \mathbf{R}^m \text{ is open and } E \subset U\} \tag{11}$$

for every set $E \subset A$. In particular, $G = \mu \lceil \mathcal{A}_A$ is an additive continuous function in A with $V_*^{\mathcal{A}} G_A \leq G$. Indeed, this inequality follows from (11); for it is easy to show $V_*^{\mathcal{A}} G(B) \leq \mu(A \cap U)$ whenever $B \in \mathcal{A}_A$ and $U \subset \mathbf{R}^m$ is an open set containing B.

Applying Theorem 5.3 and [9, Corollary 5.14], we infer $DG_A \in L^1(A)$. Since $|F| \leq VF \leq G$ (Lemma 4.4), the inequality

$$|DF_A| = D|F|_A \leq DG_A$$

holds almost everywhere in A. This and the measurability of DF_A imply $DF_A \in L^1(A)$. Moreover, F is the indefinite Lebesgue integral of DF_A by [9, Proposition 5.8].

$(3 \Rightarrow 1)$ If $DF_A \in L^1(A)$ and F is its indefinite Lebesgue integral, then it is easy to see that $V_*^{\mathcal{BV}} F_A(E) \leq \int_{A \cap U} |DF_A| \, d\lambda$ for each set $E \subset A$ and each open set $U \subset \mathbf{R}^m$ containing E. We conclude $V_*^{\mathcal{BV}} F_A(E) \leq \int_E |DF_A| \, d\lambda$ for every measurable set $E \subset A$, which implies condition 1.

As the implication $(1 \Rightarrow 2)$ is obvious, the proposition is proved.

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