MULTIPLIERS FOR GENERALIZED RIEMANN INTEGRALS

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Abstract. La présente note traite d'une intégrale de Riemann généralisée dans l'espace euclidien $\mathbb{R}^m$. Les résultats suivants sont annoncés. (1) Une fonction g est un multiplicateur (c'est-à-dire que fg est intégrable dès que f l'est) si et seulement si g est bornée et à variation bornée. (2) Le dual de l'espace des fonctions intégrables dans un ensemble A à variation bornée est isomorphe à l'espace des fonctions bornées à variation bornée s'annulant hors de A. En outre, chaque élément du dual est représenté par une formule intégrale usuelle. (3) Une formule d'intégration par parties est valide sur des domaines lipschitziens et pour des champs continus qui sont ponctuellement lipschitziens partout sauf éventuellement sur un ensemble de mesure de Hausdorff $\mathcal{H}^{m-1}$ σ-finie.

In dimension one, the problem of recovering derivative by integration has been solved a long time ago by Denjoy and Perron [8, Chapters 6 and 8]. In higher dimensions, however, a substantial progress was made only in the eighties. Following the ideas of Henstock, Kurzweil, and McShane, several authors suggested various Riemann type extensions of the Lebesgue integral which provide the Gauss-Green theorem for any differentiable (not necessarily continuously) vector field. Among these generalized Riemann integrals, the most viable appear those introduced in [6, Chapters 12 and 13]: they are defined over bounded BV sets, are invariant with respect to lipeomorphic changes of coordinates, and integrate functions with large, topologically unrestricted, sets of singularities. They have a common extension, referred to as the R_*-integral below, and as the continuous integral in [5, Section 9].

Our main result is a complete solution of the multiplier problem for the R_*-integral. Specifically, we show the family of all functions, called *multipliers*, whose product with every locally R_*-integrable function is again locally R_*-integrable is the algebra of all locally bounded locally BV functions. Using this, we obtain an integral representation of continuous linear functionals on the space of all functions that are R_*-integrable in a bounded BV set.

1. Preliminaries

The set of all real numbers is denoted by $\mathbb{R}$, and the ambient space of this note is $\mathbb{R}^m$ where $m \geq 1$ is a fixed integer. By $x \cdot y$ we denote the usual inner product of $x, y \in \mathbb{R}^m$, and let $|x| = \sqrt{x \cdot x}$. The closed ball of radius $r > 0$ about the origin of $\mathbb{R}^m$ is denoted by B_r. The symbols $\mathrm{cl}\, E$, ∂E, and $d(A)$ denote, respectively, the closure, boundary, and diameter of a set $E \subset \mathbb{R}^m$.

The Lebesgue measure in $\mathbb{R}^m$ is denoted by $\mathcal{L}^m$; however, for $E \subset \mathbb{R}^m$ we write $|E|$ instead of $\mathcal{L}^m(E)$. The *essential closure* $\mathrm{cl}_* E$ and *essential boundary* $\partial_* E$ of

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a measurable set $E \subset \mathbb{R}^m$ is, respectively, the set of all nondispersion points of E and all nondispersion points of E that are not density points of E. We say E is *essentially closed* whenever $\mathrm{cl}_* E$ is a closed set. A *thin set* is a set $S \subset \mathbb{R}^m$ whose $(m-1)$-dimensional Hausdorff measure $\mathcal{H}^{m-1}$ is σ-finite.

The linear space of all locally bounded locally BV functions in $\mathbb{R}^m$ is denoted by BV_{loc}^∞, and the linear space of all bounded BV functions in $\mathbb{R}^m$ with compact support is denoted by BV_c^∞. If $g \in BV_c^\infty$, we denote by Dg and $\|g\|$ the distributional gradient and total variation of g, respectively. For $n = 1, 2, \ldots$, topologize the family

$$BV_n = \left\{ g \in BV_c^\infty : \operatorname{supp} g \subset B_n \text{ and } \|g\| + |g|_\infty \leq n+1 \right\}$$

by the L^1 norm $|\cdot|_1$. Each BV_n is a compact set, and the largest topology τ in BV_c^∞ for which all inclusion maps $BV_n \hookrightarrow BV_c^\infty$ are continuous is a nonmetrizable locally convex Hausdorff topology that is sequential and sequentially complete. We view the family $\boldsymbol{\mathcal{BV}}$ of all bounded BV subsets of $\mathbb{R}^m$ as a closed subspace of (BV_c^∞, τ). The *perimeter* and *unit exterior normal* of $E \in \boldsymbol{\mathcal{BV}}$ are denoted by $\|E\|$ and ν_E, respectively. Moreover, we let

$$r(E) = \begin{cases} \frac{|E|}{d(E)\|E\|} & \text{if } d(E)\|E\| > 0, \\ 0 & \text{otherwise.} \end{cases}$$

2. Charges

A *charge* is an additive τ-continuous function defined on $\boldsymbol{\mathcal{BV}}$. If F is a charge and A is a locally BV set, define a charge $F \lfloor A$ by letting

$$(F \lfloor A)(B) = F(A \cap B)$$

for each $B \in \boldsymbol{\mathcal{BV}}$. We say F is a *charge in* A whenever $F = F\lfloor A$. The linear spaces of all charges and of all charges in a locally BV set A are denoted by CH and $CH(A)$, respectively. The following are canonical examples of charges.

(1) If $f \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$ and $F(A) = \int_A f \, d\mathcal{L}^m$ for each bounded BV set A, then F is a charge by the absolute continuity of the Lebesgue integral. It is called the *indefinite Lebesgue integral* of f, denoted by $\int f \, d\mathcal{L}^m$.

(2) If $v \in C(\mathbb{R}^m; \mathbb{R}^m)$ and $F(A) = \int_{\partial_* A} v \cdot \nu_A \, d\mathcal{H}^{m-1}$ for each bounded BV set A, then F is a charge by [6, Proposition 11.2.8]. It is called the *flux* of v, denoted by $\int v \cdot \nu \, d\mathcal{H}^{m-1}$.

For a charge F and $n = 1, 2, \ldots$, let

$$\|F\|_n = \sup\left\{ |F(B)| : B \in \boldsymbol{\mathcal{BV}} \cap BV_n \right\},$$

and observe $\left\{ \|\cdot\|_n \right\}$ is an increasing sequence of seminorms, which defines a Fréchet topology σ in CH. If $A \in \boldsymbol{\mathcal{BV}}$, then $\|F\|_n$ are equivalent *Banach norms* on $CH(A)$ for all sufficiently large n.

Proposition 2.1. *Let F be a charge, and let E be a bounded BV subset of $\mathbb{R}^{m+1}$. For $\mathcal{L}^1$-almost all $t \in \mathbb{R}$, the section $E^t = \left\{ x \in \mathbb{R}^m : (x,t) \in E \right\}$ belongs to $\boldsymbol{\mathcal{BV}}$, and the function $t \mapsto F(E^t)$ belongs to $L^1(\mathbb{R})$. Letting*

$$(F \times \mathcal{L}^1)(E) = \int_{\mathbb{R}} F(E^t) \, dt$$

for each bounded BV set $E \subset \mathbb{R}^{m+1}$ defines a charge $F \times \mathcal{L}^1$ in $\mathbb{R}^{m+1}$. The map $F \mapsto F \times \mathcal{L}^1$ is a continuous linear map from $\big(CH(\mathbb{R}^m), \sigma\big)$ to $\big(CH(\mathbb{R}^{m+1}), \sigma\big)$.

Denote by CH^* and $(BV_c^\infty)^*$ the dual spaces of (CH, σ) and (BV_c^∞, τ), respectively. We endow $(BV_c^\infty)^*$ with a Fréchet topology ν induced by the uniform convergence on each BV_n. For a nonnegative function $g \in BV_c^\infty$,

$$\Sigma_g = \big\{(x,t) \in \mathbb{R}^m \times \mathbb{R} : 0 < t < g(x)\big\}$$

is a bounded BV subset of $\mathbb{R}^{m+1}$. In view of Proposition 2.1, the number

$$\langle F, g\rangle = (F \times \mathcal{L}^1)(\Sigma_{g^+}) - (F \times \mathcal{L}^1)(\Sigma_{g^-})$$

is defined for each $F \in CH$ and each $g \in BV_c^\infty$.

Proposition 2.2. *The pairing $(F,g) \mapsto \langle F, g\rangle$ is a nondegenerate separately continuous bilinear functional on $(CH \times BV_c^\infty, \sigma \times \tau)$.*

Theorem 2.3. *For $T \in (BV_c^\infty)^*$ and $E \in \boldsymbol{\mathcal{BV}}$, let $\Lambda(T)(E) = T(\chi_E)$. Then $\Lambda(T)$ is a charge, and the map*

$$\Lambda : \big((BV_c^\infty)^*, \nu\big) \to (CH, \sigma) : T \mapsto \Lambda(T)$$

is a linear homeomorphism with $\Lambda^{-1} : F \mapsto \langle F, \cdot\rangle$.

3. The R-integral

A charge F is *derivable* at $x \in \mathbb{R}^m$ if $\lim\big[F(B_n)/|B_n|\big] \neq \pm\infty$ exists for each $\{B_n\}$ in $\boldsymbol{\mathcal{BV}}$ with $\lim d\big(B_n \cup \{x\}\big) = 0$ and $\inf r\big(B_n \cup \{x\}\big) > 0$. If all these limits exist, they have the same value called the *derivate* of F at x, denoted by $F'(x)$.

A charge F is called AC if for each $\varepsilon > 0$ there is a $\delta > 0$ such that $\big|F(E)\big| < \varepsilon$ whenever $E \subset B_{1/\varepsilon}$ is a BV set with $|E| < \delta$. It is well-known each AC charge F is derivable almost everywhere with $F' \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$ and $\int F'\, d\mathcal{L}^m = F$.

A nonnegative function δ defined on a set $E \subset \mathbb{R}^m$ is called a *gage* on E whenever its null set is thin. A *partition* is a collection (possibly empty) $P = \big\{(A_1, x_1), \dots, (A_p, x_p)\big\}$ where $A_1, \dots, A_p$ are disjoint bounded BV sets, and $x_1, \dots, x_p$ are points of $\mathbb{R}^m$. Given $\varepsilon > 0$ and a gage δ on $E \subset \mathbb{R}^m$, the partition P is called

- *ε-regular* if $r\big(A_i \cup \{x_i\}\big) > \varepsilon$ for $i = 1, \dots, p$,
- *δ-fine* if $x_i \in E$ and $d\big(A_i \cup \{x_i\}\big) < \delta(x_i)$ for $i = 1, \dots, p$.

The *critical variation* of a charge F on a set $E \subset \mathbb{R}^m$ is the number

$$V_*F(E) = \sup_{\varepsilon > 0} \inf_{\delta} \sup_{P} \sum_{i=1}^{p} \big|F(A_i)\big|$$

where δ is a gage on E, and $P = \big\{(A_1, x_1), \dots, (A_p, x_p)\big\}$ is an ε-regular δ-fine partition. The extended real-valued function $V_*F : E \mapsto V_*F(E)$ is a Borel regular measure in $\mathbb{R}^m$, which coincides with the classical variation of F on every compact interval. The following proposition was proved in [2, Proposition 3.3].

Proposition 3.1. *A charge F is AC if and only if V_*F is absolutely continuous and locally finite.*

A charge F is called AC_* whenever the measure V_*F is absolutely continuous. The linear space of all AC_* charges or all AC_* charges in a bounded BV set A is denoted by CH_{AC_*} or $CH_{AC_*}(A)$, respectively. While neither of the spaces (CH_{AC_*}, σ) and $\big(CH_{AC_*}(A), \sigma\big)$ is complete, both are barrelled [3, Theorem 4.1]. It turns out many important properties of AC charges remain valid for AC_* charges.

Definition 3.2. A function f defined on $\mathbb{R}^m$ is *locally R-integrable* whenever there is an AC_* charge F such that $F'(x) = f(x)$ for almost all $x \in \mathbb{R}^m$. The charge F, uniquely determined by f, is called the *indefinite R-integral* of f. If A is a bounded BV set, the number $F(A)$ is called the *R-integral* of f over A.

The linear space of all locally R-integrable functions is denoted by $R_{\mathrm{loc}}(\mathbb{R}^m)$. Clearly $L^1_{\mathrm{loc}}(\mathbb{R}^m) \subset R_{\mathrm{loc}}(\mathbb{R}^m)$, and the indefinite R-integral F of $f \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$ coincides with the indefinite Lebesgue integral of f. Thus without danger of confusion, we denote F by $\int f \, d\mathcal{L}^m$, and $F(A)$ by $\int_A f \, d\mathcal{L}^m$ for each bounded BV set A. The next theorem, proved in [2, Theorem 4.6], shows the R-integral is akin to the generalized Riemann integrals of [1, Definition 5.1].

Theorem 3.3. *A function f defined on $\mathbb{R}^m$ is locally R-integrable if and only if there is a charge F satisfying the following condition: given $\varepsilon > 0$, we can find a gage δ on $B_{1/\varepsilon}$ so that*

$$\sum_{i=1}^{p} \Big| f(x_i)|A_i| - F(A_i) \Big| < \varepsilon$$

for each ε-regular δ-fine partition $\big\{(A_1, x_1), \dots, (A_p, x_p)\big\}$. In this case F is the indefinite R-integral of f.

A *multiplier* for a family X of functions defined on a set E is a function g defined on E such that $fg \in X$ for each $f \in X$. The collection of all multipliers for the family X is denoted by MX. It follows from [7, Proposition 3.1] that $MR_{\mathrm{loc}}(\mathbb{R}^m) \subset BV^\infty_{\mathrm{loc}}$.

Theorem 3.4. *Let F be the indefinite R-integral of an $f \in R_{\mathrm{loc}}(\mathbb{R}^m)$, and let $g \in BV^\infty_{\mathrm{loc}}$. Then $fg \in R_{\mathrm{loc}}(\mathbb{R}^m)$ and*

$$\int_B fg \, d\mathcal{L}^m = \langle F, g\chi_B \rangle$$

for each bounded BV set B. In particular, $MR_{\mathrm{loc}}(\mathbb{R}^m) = BV^\infty_{\mathrm{loc}}$.

Theorem 3.4 is proved by showing $G : B \mapsto \langle F, g\chi_B \rangle$ is an AC_* charge such that $G' = F'g$ almost everywhere.

For a function f defined on a set $E \subset \mathbb{R}^m$, let $\overline{f}$ be the extension of f to $\mathbb{R}^m$ by zero. If $E \in \boldsymbol{\mathcal{B}V}$, we denote by $BV^\infty(E)$ the family of all functions g defined on E for which $\overline{g}$ belongs to BV^∞_c.

Definition 3.5. A function f defined on a bounded BV set A is called *R-integrable* in A whenever $\overline{f}$ is locally R-integrable. The indefinite R-integral F of $\overline{f}$ is called the *indefinite R-integral* of f, and the number $F(A)$ is called the *R-integral* of f over A.

The linear space of all R-integrable functions in a bounded BV set A is denoted by $R(A)$. The indefinite integral F of a function $f \in R(A)$ is a charge in A, and the map

$$F \mapsto F' \restriction A : CH_{AC_*}(A) \to R(A)$$

is a linear bijection, which induces in $R(A)$ a normed barrelled topology. If $f \in R(A)$, we denote by $\int_A f \, d\mathcal{L}^m$ the R-integral of f over A. The R-integral has the usual properties associated with the word "integral," including the common convergence theorems. Moreover, the R-integral is invariant with respect to lipeomorphisms (i.e., bi-Lipschitzian maps). The term "locally R-integrable," introduced in

Definition 3.2, is now justified: a function f defined on $\mathbb{R}^m$ is locally R-integrable if and only if $f \restriction A$ is R-integrable in A for each bounded BV set A.

Theorem 3.6. *Let A be a bounded BV set. Then $MR(A) = BV^\infty(A)$, and for each $g \in BV^\infty(A)$, the map*

$$T_g : f \mapsto \int_A fg \, d\mathcal{L}^m$$

is a continuous linear functional on $R(A)$. The map $g \mapsto T_g$ is a continuous linear bijection from $\big(BV^\infty(A), \tau\big)$ onto the dual space of $R(A)$ equipped with the w^*-topology.

One of the principal reasons for introducing the R-integral is the divergence theorem for discontinuously differentiable vector fields. If $E \subset \mathbb{R}^m$, we say a vector field $v : E \to \mathbb{R}^m$ is *Lipschitz at* $x \in E$ whenever there are positive numbers c and r such that

$$\big|v(y) - v(x)\big| \le c|y - x|$$

for each $y \in E$ with $|y - x| < r$. Let $E \subset \mathbb{R}^m$ be a closed set, and let $v \in C(E; \mathbb{R}^m)$ be Lipschitz at almost all $x \in E$. In view of Whitney's and Stepanoff's theorems, $\operatorname{div} v$ is uniquely defined almost everywhere in E.

Theorem 3.7. *Let A be a bounded BV set that is essentially closed, let $v \in C(\mathrm{cl}_* A; \mathbb{R}^m)$, and let $S \subset \mathrm{cl}_* A$ be a thin set such that v is Lipschitz at each $x \in \mathrm{cl}_* A - S$. Then $\operatorname{div} v$ is R-integrable in A and*

$$\int_A \operatorname{div} v \, d\mathcal{L}^m = \int_{\partial_* A} v \cdot \nu_A \, d\mathcal{H}^{m-1} .$$

If Ω is a *Lipschitz domain* [4, Section 4.2.1] and f is a BV function in Ω, we can define the *trace* of f, which is a function in $L^1(\partial\Omega, \mathcal{H}^{m-1})$, denoted by $\operatorname{Tr} f$.

Theorem 3.8. *Let Ω be a Lipschitz domain, $v \in C(\mathrm{cl}\,\Omega; \mathbb{R}^m)$, and let $S \subset \Omega$ be a thin set such that v is Lipschitz at each $x \in \Omega - S$. Then*

$$\int_\Omega g \operatorname{div} v \, d\mathcal{L}^m = \int_{\partial\Omega} (\operatorname{Tr} g)\, v \cdot \nu_\Omega \, d\mathcal{H}^{m-1} - \int_\Omega v \cdot Dg$$

for each $g \in BV^\infty(\Omega)$.

4. The R_*-integral

While the R-integral introduced in Sections 3 is an averaging process of appreciable appeal, it is not adequate for integration over arbitrary bounded BV sets: Theorem 3.7 is false when A is not essentially closed [5, Example 5.21]. Fortunately, a simple extension of the R-integral eliminates this shortcoming and retains all desirable properties of the R-integral.

If f is a function defined on a bounded BV set A and F is a charge, denote by $\boldsymbol{\mathcal{R}}(f, F; A)$ the family of all $B \in \boldsymbol{\mathcal{BV}}(A)$ for which $f \restriction B$ belongs to $R(B)$ and $F \lfloor B$ is the indefinite R-integral of $f \restriction B$. The closure operator in the space $(\boldsymbol{\mathcal{BV}}, \tau)$ is denoted by cl_τ.

Definition 4.1. A function f defined on a BV set A is called *R_*-integrable* in A if there is a charge F in A such that $A \in \mathrm{cl}_\tau \boldsymbol{\mathcal{R}}(f, F; A)$. The charge F, uniquely determined by f, is called the *indefinite R_*-integral* of f, and the number $F(A)$ is called the *R_*-integral* of f over A.

The linear space of all R_*-integrable functions in a bounded BV set A is denoted by $R_*(A)$. Clearly $R(A) \subset R_*(A)$, and arguments similar to those of [5, Sections 8 and 9] show the R_*-integral has the usual "integral properties," including the invariance with respect to lipeomorphic changes of coordinates. The validity of Theorems 3.6 for the R_*-integral is also easy to verify. With no danger of confusion, for $f \in R_*(A)$ we denote by $\int_A f \, d\mathcal{L}^m$ the R_*-integral of f over A. As τ is a sequential topology, it is easy to see no new integral is produced by a further iteration of Definition 4.1.

Lemma 4.2. *Let $A \in \boldsymbol{\mathcal{BV}}$ and $v \in C(\mathrm{cl}\, A; \mathbb{R}^m)$. Suppose there is a family $\boldsymbol{\mathcal{E}} \subset \boldsymbol{\mathcal{BV}}(A)$ such that for each $E \in \boldsymbol{\mathcal{E}}$, the restriction $v \restriction \mathrm{cl}\, E$ is Lipschitz at almost all $x \in E$. If $A \in \mathrm{cl}_\tau \boldsymbol{\mathcal{E}}$, then there is a function f, defined on A and unique up to equivalence, such that given $E \in \boldsymbol{\mathcal{E}}$,*

$$f(x) = \mathrm{div}\,(v \restriction \mathrm{cl}\, E)(x)$$

for almost all $x \in E$. In particular, $f(x) = \mathrm{div}\, v(x)$ for almost all $x \in A$ whenever $A \in \boldsymbol{\mathcal{E}}$.

Under the assumptions of Lemma 4.2, we call the function f the *divergence* of v, still denoted by $\mathrm{div}\, v$. The last claim of Lemma 4.2 shows this is consistent with our previous definition of $\mathrm{div}\, v$ introduced prior to Theorem 3.7.

Theorem 4.3. *Let A be any bounded BV set, and let $v \in C\big(\mathrm{cl}\, A; \mathbb{R}^m\big)$. Suppose there is a family $\boldsymbol{\mathcal{E}} \subset \boldsymbol{\mathcal{BV}}(A)$ with the following property: given $E \in \boldsymbol{\mathcal{E}}$, we can find a thin set $S_E \subset \mathrm{cl}_* E$ so that $v \restriction \mathrm{cl}\, E$ is Lipschitz at each $x \in \mathrm{cl}_* E - S_E$. If $A \in \mathrm{cl}_\tau \boldsymbol{\mathcal{E}}$, then $\mathrm{div}\, v$ is R_*-integrable in A and*

$$\int_A \mathrm{div}\, v \, d\mathcal{L}^m = \int_{\partial_* A} v \cdot \nu_A \, d\mathcal{H}^{m-1} \,.$$

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