Geometric and Dynamical Aspects of Measure Theory

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Doktori Értekezés

Budapest, 2006

A Doktori Értekezés a Nemzeti Kutatási és Technológia Hivatal támogatásával
jött létre amely támogatás forrása a Kutatási és Technológiai Innovációs Alap.

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Introduction

In Hungarian Mathematics it is difficult to avoid the influence of Paul Erdős the "uncrowned king of problem solvers", even if someone works in Analysis. In my Approximation Theory course Vera T. Sós quite often sent him to class as a "substitute". My first "student research" paper was related to a simple problem connected to the Lebesgue density theorem which he recommended to me for investigation after I had asked some questions in class. Later in my career I worked most of the time as a problem solver. As I learned more and more Mathematics the problems I solved got deeper and deeper, more and more difficult. I have found it most interesting to work on questions which were related to Geometric Measure Theory and later to Dynamical Systems and to Ergodic Theory.

In this dissertation I have collected some of my favorite problems, which belong to a more or less straight line of developing ideas.

In 1987/88 I learned two problems which are important for this dissertation. The first one was asked by W. F. Pfeffer and was related to Henstock–Kurzweil integration. To be more specific, this problem was about the existence of Regular Special Partitions. The other problem was the Gradient Problem of C. E. Weil.

I was able to solve Pfeffer's problem in a few months and this became the starting point of a long and fruitful cooperation. Based on the solution of this problem I was invited for a year and a half to teach and do research with him at the Davis campus of the University of California and later we continued working via email. Pfeffer was the "main ideologist" as he worked on his higher dimensional generalization of the Henstock–Kurzweil integral. When he encountered some problems related to Geometric Measure Theory he contacted me. I was quite often able to help him by a solution. The main theme of this dissertation is not generalized integrals. Hence I do not discuss most of my results in this field. The interested reader can find its account of in [11]. I discuss in Section 1.1 of Chapter 1 the result about the existence of Regular Special Partitions. It is interesting that, although special partitions did not play an important role during the later development of Pfeffer's integrals, other people, like M. Ash, J. Cohen, H. Fejzić, C. Freiling, and D. Rinne got interested in them. Some of them were motivated by questions coming from multidimensional Fourier Analysis. They improved the constants appearing in my proof (see [33]), and in [1] they applied special partitions in the study of generalizations of the wave equation. In Section 1.2 I discuss their observation. I have selected to discuss only one other result which was an answer to a question of Pfeffer originating in generalized integration. This result, presented in Section 1.3, concerns Lebesgue density and dispersion points and gives for lipeomorphic images of measurable sets in $\mathbb{R}^m$ a density topology version of Brouwer's Theorem on the Invariance of Domain.

Later I learned more about Dynamical Systems and Ergodic Theory. In Chapter 2 I discuss two areas. Section 2.1 is the bridge between generalized integrals and Ergodic Theory. Motivated by ideas of generalized integrals I got interested in the convergence behavior of Birkhoff sums of non-L^1 functions. Motivated by a result of P. Major and by a question of M. Laczkovich I have showed strange convergence properties of the same function with respect to Birkhoff sums along two independent rotations, or in a more general setting, along generators of a free $\mathbb{Z}^2$ action. In Section 2.2 the one-parameter family of tent maps, T_a is studied. K. M. Brucks and M. Misiurewicz showed that for Lebesgue almost every parameter value the trajectory of the turning point is dense in the dynamical core of the tent map T_a. By an intricate argument it is showed in a joint paper with K. M. Brucks that the exceptional parameter values are not only of zero Lebesgue measure but they are σ-porous, which is a significant sharpening of the Brucks–Misiurewicz result. Some of the techniques, I figured out while I was working on this argument, provided a good heuristic background for the solution of the Gradient Problem, discussed in Chapter 4.

Chapter 3 is about the solution of a problem of J. A. Haight and H. v. Weizsäcker. I discussed my results from Chapter 2 during a trainride with R. D. Mauldin and he immediately suggested that I should start working on the Haight–Weizsäcker problem, which is the following: suppose $f : (0, \infty) \to \mathbb{R}$ is measurable, is it true that $\sum_{n=1}^{\infty} f(nx)$ converges, or diverges almost everywhere. To put it in a simple way: this question asks about the existence of a zero-one law for $\sum f(nx)$. In [23] we showed that there are examples when there is no such zero-one law. This problem was among the favorite problems of J-P. Kahane, so suddenly I started to receive emails and faxes from him and a joint project started with him and R. D. Mauldin. The results, which are related to generalizations of the Haight–Weizsäcker problem can be found in Section 3.2. Instead of working with the integers $\mathbb{N}$, other "discrete" subsets Λ of $[0, \infty)$ are considered and convergence properties of $\sum_{\lambda \in \Lambda} f(\lambda x)$, or by a logarithmic "change of variables", of $\sum_{\lambda \in \Lambda} f(x + \lambda)$ are investigated, see also [21] and [22]. It turned out that a certain class of Λ's: the Λ^{α^k} sets were of particular interest. R. D. Mauldin had suggested studying them and in Section 3.3 the results of the joint paper [24] are presented.

In Chapter 4 finally an answer to the gradient problem is given. It took 15 years of work until in 2002 I managed to solve it. In Section 4.1 the numerous partial results and "by-products" of these 15 years are presented. While in Section 4.2 the ideas behind the final solution are outlined.

In this dissertation apart from the original Mathematical Reviews refereed research papers I also used material from several of the written versions of my invited addresses presented at different conferences. (These are: [11] [20], [18] and [19].)

Acknowledgements. I would like to thank Á. Császár, and M. Laczkovich for teaching me the techniques of Real Analysis and Measure Theory. I would also like to thank K. M. Brucks, J-P. Kahane, R. D. Mauldin, W. F. Pfeffer and C. E. Weil for suggesting to me problems to work on and for their cooperation. I am grateful to the Eötvös University, to the University of California at Davis, to the University of Wisconsin at Milwaukee, to the Michigan State University and to the University of North Texas for providing me the opportunity work and do research. I should also acknowledge the financial support I received during the past 18 years from OTKA, FKFP, the Soros Foundation and the US National Science Foundation.

Finally, special thanks to my Mother, to my late Father and to Emőke Greschik.

Chapter 1

Problems related to generalized integration

The main topic of this dissertation is not the theory of the Henstock-Kurzweil integral. However we would like to remind the reader to the definition of this integral in dimension one.

Definition 1.0.1. If $A \subset \mathbb{R}$ is an interval, $f : A \to \mathbb{R}$ is HK-integrable (Henstock-Kurzweil integrable) on A and its integral, denoted by $(HK)\int_A f$, equals I if for all $\epsilon > 0$, there exists $\delta : A \to (0, +\infty)$ such that

$$\left| \sum_{i=1}^{p} f(x_i)|A_i| - I \right| < \epsilon$$

for all δ-fine partitions $P = \{(A_i, x_i)\}_{i=1}^{p}$ of A.

Recall that in the above definition the collection $P = \{(A_i, x_i)\}_{i=1}^{p}$ is a *partition of* A if the intervals A_i are non-overlapping, their union is A, and $x_i \in A_i$ for all i. Denote by $B(x, r)$ the open ball of radius r centered at x. The partition P is δ-*fine* when $A_i \subset B(x_i, \delta(x_i))$ for all i.

1.1 Special partitions

To obtain a defintion of the Henstock-Kurzweil integral of functions defined on rectangles $A \subset \mathbb{R}^2$ (or $A \subset \mathbb{R}^m$), one can do the obvious modifications of the above one-dimensional definition. However, as it often turns out in higher dimensional integration, instead of allowing arbitrary partitions, integrals based on regular partitions are more useful.

The *regularity* of an m-dimensional interval is the ratio of its shortest and longest side. An interval is r-regular when its regularity is greater or equal than r. Given $r > 0$, allowing only r-regular intervals in the definition of the HK-integral one can obtain a "regular" generalization for higher dimensions. However, there are examples when A and B are non-overlapping intervals, the generalized "regular" integrals $\int_A f$, and $\int_B f$ exist, but $\int_{A\cup B} f$ does not. W. Pfeffer in [50] suggested to solve this problem by using *special* partitions. The partition $P = \{(A_i, x_i)\}$ is special when it is a partition and x_i is a *vertex* of A_i for each i. Then the problem arises whether for any $\delta : A \to (0, \infty)$, $r > 0$, there exists a δ-fine r-regular special partition of A. In [7] we answered this problem for planar partitions. We showed that

Theorem 1.1.1. *If the rectangle $A \subset \mathbb{R}^2$, and $r \geq 1/1000 > 0$ then for all $\delta : A \to (0, \infty)$ there exists a δ-fine r-regular special partition of A.*

In dimensions higher than two, the problem about the existence of special partitions is still unsolved.

In the later development of higher dimensional integration special partitions did not turn out to be as useful as they were expected to be. However, motivated by possible applications to harmonic analysis H. Fejzić, C. Freiling and D. Rinne in [33] did work on improving the regularity constant of the previous theorem. They showed that in the above result *the constant* $1/1000$ *can be replaced by* $1/\sqrt{2}$. It would be interesting to see whether $1/\sqrt{2}$ is the best constant possible? We believe that probably not.

1.2 Application of special partitions to the (generalized) wave equation

It is well-known that *if* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *is smooth* (C^∞) *then* $\partial^2 f/\partial x \partial y = 0$ *on* $\mathbb{R}^2$ *implies* $f(x, y) = a(x) + b(y)$.

In [1] the authors studied different generalizations of the wave equation. They used generalized concepts of differentiation and studied whether the above theorem remained valid. They also studied the connection of these results with the theory of the uniqueness for multiple trigonometric series.

Let f be a function of two variables and set $\Delta f(x, y; h, k) = f(x + h, y + k) - f(x+h, y) - f(x, y+k) + f(x, y)$. Let $Df(x, y) = \lim_{h,k\to 0} \Delta f(x, y; h, k)/hk$. In [1] there is a short and neat proof for the following result of K. Bogel, [3]: *if* $Df(x, y) = 0$ *on* $\mathbb{R}^2$ *then* $f(x, y) = a(x) + b(y)$.

Definition. If there is a number s such that for arbitrarily large M's $\sup_{1/M \le |h/k| \le M} |\frac{\Delta f(x,y;h,k)}{hk} - s| \to 0$ as $\max\{|h|, |k|\} \to 0$ then s is the generalized (x, y)-derivative of f in the restricted sense, and is denoted by $D^r f(x, y)$.

In [1] it was observed that it is a corollary of Theorem 1.1.1 that *if* $D^r f(x, y) = 0$ *on* $\mathbb{R}^2$ *then* $f(x, y) = a(x) + b(y)$.

1.3 Lipschitz Transformations

Working with sets of bounded variation the best choice for transformations is the class of *lipeomorphisms* (bi-lipschitz mappings).

By $\mathrm{int}(A)$ we denote the interior of the set $A \subset \mathbb{R}^m$. In this section the m-dimensional Lebesgue measure of the measurable set A is denoted by $\lambda_m(A)$. If $A \subset \mathbb{R}^m$ is Lebesgue measurable, $p \in \mathbb{R}^m$ is called a density point of A whenever

$$\lim_{r\to 0} \frac{\lambda_m((B(p, r) \cap A))}{\lambda_m(B(p, r))} = 1.$$

Dispersion points of a measurable set A are the density points of the complement of A. The density points form the *essential interior*, the dispersion points form the *essential exterior* and the points which are in the complement of the union of the essential interior and exterior belong to the *essential boundary* when we use the terminology coming from the density topology. Recall that if $A \subset \mathbb{R}^m$ is measurable then by Lebesgue's density theorem almost every point of $\mathbb{R}^m$ is either a density, or a dispersion point of A.

If $A, B \subset \mathbb{R}^m$ the function $f : A \to B$ is Lipschitz with constant L if $||f(x) - f(y)|| \leq L||x - y||$ for $x, y \in A$, where the norm is the Euclidean norm. The function mapping $A \subset \mathbb{R}^m$ onto $B \subset \mathbb{R}^m$ is a lipeomorphism with constants L and L' if both f and f^{-1} are Lipschitz with constants L and L' respectively.

W.F.Pfeffer asked the author about how density points of Lebesgue measurable sets are transformed by lipeomorphisms. We proved in [8] that Lebesgue density and dispersion points are preserved by lipeomorphisms, that is, the following theorem holds:

Theorem 1.3.1. *Suppose that A and B are measurable subsets of $\mathbb{R}^m$, the function f maps A onto B, and f is a lipeomorphism with constants L and L'. Then f maps density or dispersion points of A into density or dispersion points of B respectively.*

If f and f^{-1} are defined on $\mathbb{R}^m$ then this statement is an easy exercise in measure theory. Assume that a lipeomorphism f is defined on the set $A \subset \mathbb{R}^m$ and maps A onto $B \subset \mathbb{R}^m$. Then it is still easy to prove that dispersion points of A are mapped into dispersion points of B. The statement about density points is more involved. Unfortunately, in general one cannot find a lipeomorphic extension $\tilde{f} : \mathbb{R}^m \to \mathbb{R}^m$ of $f : A \to B$ and reduce the problem to the easy case. One can only obtain by Kirszbraun's theorem a Lipschitz function $\tilde{f} : \mathbb{R}^m \to \mathbb{R}^m$ such that $\tilde{f}|_A = f$ and $\tilde{f}$ is Lipschitz but not necessarily a lipeomorphism. The most difficult part of our proof is related to the range of $\tilde{f}$ and is of topological nature. If the range of $\tilde{f}$ contains large holes then it might happen that a density point is transformed into a non-density point. To be more precise assume that $p \in A$ is a density point of A. One has to show that $f(p) = \tilde{f}(p) \in B$ is a density point of B. To prove this we have to show that the range of $\tilde{f}$ cannot contain big holes near $\tilde{f}(p)$. In fact, we prove that there exists a $c > 0$, such that c depends only on the Lipschitz properties of f, and $\tilde{f}(B(p,r))$ contains a relatively big ball, namely $B(f(p), cr) \subset \tilde{f}(B(p,r))$, whenever $r > 0$ is sufficiently small.

We also remark that the almost every version of our theorem, that is, almost every density points are mapped into density points, follows easily from the result about dispersion points and Lebesgue's density theorem. The point in our theorem is that we prove our result about *all* density points.

We point out that Theorem 1.3.1 implies that each lipeomorphism function maps the *essential boundary* of its domain onto the essential boundary of its image. This corollary of Theorem 1.3.1 motivated W.F.Pfeffer's original question. In conjunction with [32, Theorem 4.5.11] the above corollary implies that the lipeomorphic image of a *Caccioppoli set* is again a Caccioppoli set. The usual proof of this fact rests on interpreting Caccioppoli sets as *integral currents* - a technique substantially more involved than our result (see [32, Chapter 4]).

Theorem 1.3.1 can be viewed as the density topology version of the famous Brouwer's Theorem on the Invariance of Domain [39, Chapter VI, §6, Theorem VI 9, p.95]:

Theorem 1.3.2. *Let X be an arbitrary subset of $\mathbb{R}^m$ and h a homeomorphism of X on another subset $h(X)$ of $\mathbb{R}^m$. Then if x is an interior point of X, $h(x)$ is an interior point of $h(X)$.*

Our result about the invariance of domain theorem for lipeomorphisms, is also related to quasiconformal/quasisymmetric maps, for the details see [62].

Chapter 2

Non-L^1 functions, Birkhoff sums and exceptional sets in the tent family

2.1 The Ergodic Theorem and functions not in L^1

By the ergodic theorem if f is periodic by 1 and is Lebesgue integrable on $[0,1]$ then

$$M_n^\alpha f(x) \stackrel{\text{def}}{=} \frac{1}{n+1}\sum_{k=0}^{n} f(x+k\alpha) \to \int_0^1 f$$

for any irrational number α and for almost every $x \in \mathbb{R}$. Given an irrational number α there are non-integrable functions f such that $M_n^\alpha f(x)$ converges for almost every x as $n \to \infty$ (this will be a special case of Theorem 2.1.2). (In this section unless otherwise specified we always work with the one-dimensional Lebesgue measure and $|A|$ denotes the Lebesgue measure of the set A.) What happens if we have this property for many α's? In Theorem 2.1.2 (see [10]) we showed that if $f : \mathbb{R} \to \mathbb{R}$ is an arbitrary measurable function and there exists a set of positive measure of α's for which $M_n^\alpha f(x)$ converges for almost every x, then f should be integrable on $[0,1]$, that is, the following theorem holds:

Theorem 2.1.1. *Let $f:\mathbb{R}\to\mathbb{R}$ be a given measurable function, periodic by 1. For an $\alpha\in\mathbb{R}$ put*

$$M_n^\alpha f(x)=\frac{1}{n+1}\sum_{k=0}^{n} f(x+k\alpha).$$

Let Γ_f denote the set of those α's in $(0,1)$ for which $M_n^\alpha f(x)$ converges for almost every $x\in\mathbb{R}$. Then from $|\Gamma_f|>0$ it follows that f is integrable on $[0,1]$.

Hence, when $|\Gamma_f|>0$ by the ergodic theorem all these limits $M_n^\alpha f(x)$ should be of the same value, namely, $\int_0^1 f$. On the other hand, in Theorem 2.1.2 (see [10]) for any given sequence of irrationals $\{\alpha_1,\alpha_2,...\}$, independent over $\mathbb{Q}$, we construct functions f which are *non-integrable* on $[0,1]$, and $M_n^{\alpha_j} f(x)\to 0$ for almost every $x\in\mathbb{R}$ and any $j=1,2,\dots$. The precise statement of this result is the following theorem:

Theorem 2.1.2. *For any sequence of independent irrationals $\{\alpha_j\}_{j=1}^\infty$ there exists a function $f:\mathbb{R}\to\mathbb{R}$, periodic by 1 such that $f\notin L^1[0,1]$ and $M_n^{\alpha_j} f(x)\to 0$ for almost every $x\in[0,1]$.*

This result implies that Γ_f can be dense for non-integrable functions. In [60] R. Svetic improves this result by showing that *there exists a non-integrable $f:\mathbb{T}\to\mathbb{R}$ such that Γ_f is c-dense in $\mathbb{T}$.* (A set $S\subset\mathbb{T}$ is *c-dense* if the cardinality of $S\cap I$ equals continuum for every open interval $I\subset\mathbb{T}$.)

It is still not known whether Γ_f can be of Hausdorff dimension 1.

Answering our question P. Major in [43] gave an example showing that with non-Lebesgue integrable functions one should be very cautious to obtain a "non-absolute ergodic theorem". He showed that *there exists a function*

$f : X \to \mathbb{R}$, and $S_1, S_2 : X \to X$ ergodic transformations on a probability space (X, μ) such that

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} f(S_1^k x) = 0 \ \textit{a.e. and}$$

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} f(S_2^k x) = a \neq 0 \ \textit{a.e.}$$

By the ergodic theorem clearly $f \notin L^1(X, \mu)$. M. Laczkovich asked whether there exist α, β irrational rotations of the unit circle $\mathbb{T}$ and a measurable function $f : \mathbb{T} \to \mathbb{R}$ for which

$$\frac{1}{N+1} \sum_{j=0}^{N} f(x + j\alpha) \to 1 \text{ and } \frac{1}{N+1} \sum_{j=0}^{N} f(x + j\beta) \to 0?$$

I could solve the problem of M. Laczkovich about two rotations. Suggested by a referee a more general version of this theorem was proved in [9].

Assume that a $\mathbb{Z}^2$ action is generated by S and T on a finite nonatomic Lebesgue measure space $(X, \mathcal{S}, \mu)$ and $T^j S^k$ for all $(j, k) \in \mathbb{Z}^2$ is a measure preserving transformation on X. We say that the group action generated by T and S is free if $T^j S^k x \neq x$ for $(j, k) \neq (0, 0)$ and μ a.e. x.

Theorem 2.1.3. *Assume that $(X, \mathcal{S}, \mu)$ is a finite non-atomic Lebesgue measure space and $S, T : X \to X$ are two μ-ergodic transformations which generate a free $\mathbb{Z}^2$ action on X. Then for any $c_1, c_2 \in \mathbb{R}$ there exists a μ-measurable function $f : X \to \mathbb{R}$ such that*

$$M_N^S f(x) = \frac{1}{N+1} \sum_{j=0}^{N} f(S^j x) \to c_1, \ \textit{and}$$

$$M_N^T f(x) = \frac{1}{N+1} \sum_{j=0}^{N} f(T^j x) \to c_2, \text{ for } \mu\text{-almost every } x \text{ as } N \to \infty.$$

Free $\mathbb{Z}^2$ actions on Lebesgue spaces are natural generalizations of independent rotations of the circle. This implies that the above theorem provides a positive solution to M. Laczkovich's question.

The proof of this theorem was based on the following rather technical Lemma.

Lemma 2.1.4. *Suppose that the transformations S, T satisfy the assumptions of the Theorem 2.1.3. Assume that K and N are arbitrary positive integers and g_0 is a bounded measurable function with support E^0. Set $\epsilon = 1/K$. Then there exists another bounded measurable function g_1 such that*

- *a)* $\int_X g_1 d\mu = \int_X g_0 d\mu$,
- *b) If E^1 denotes the support of g_1 then $\mu(\cup_{k=-N}^{N} S^{-k} E^1) < 2\epsilon$,*
- *c)* $\sup_{x\in X} |g_1(x)| \leq \frac{1}{\epsilon} \sup_{x\in X} |g_0(x)|$,
- *d)* $|\sum_{k=0}^{M} (g_1 - g_0)(T^k x)| \leq \frac{2}{\epsilon} \sup_{x\in X} |g_0(x)|$ *for* $M = 0, 1, ...,$ *and all* $x \in X$, *and*
- *e) If $E_{1,0}$ denotes the support of $g_1 - g_0$ we have $E_{1,0} \subset \cup_{k=0}^{K} T^k E^0$.*

The crucial step of the proof of this lemma is the application of a generalization of Rohlin's lemma.

Given a number N denote by R_N the rectangle $\{(j, k) : 1 \leq j \leq N, 1 \leq k \leq 2N\}$. Observe that translated copies of R_N form a partition of $\mathbb{Z}^2$, that is, R_N is a *tiling* set in the sense of [48]. By Theorem 2 of [48] Rohlin's lemma is valid for the above free $\mathbb{Z}^2$ actions and R_N. This means the following:
For any $\epsilon > 0$ there is a set $B \in \mathcal{S}$ such that

- i) $\{T^j S^k B : (j, k) \in R_N\}$ are disjoint sets, and

- ii) $\mu(\cup_{(j,k)\in R_N} T^j S^k B) > 1 - \epsilon$.

For further results of different nature, but related to non-Lebesgue integrable functions and the ergodic theorem, we refer to [67] and [68].

2.2 Exceptional sets in one dimensional dynamics

Let (X, ρ) be a compact metric space, $E \subset X$, $x \in X$, and $\delta > 0$. We define $\gamma(E, x, \delta)$ to be the minimum of 1 and the number defined by

$$\sup\{2\eta \mid \eta > 0 \text{ and there exists } y \in X \text{ such that } B(y, \eta) \subset B(x, \delta) \setminus E\}.$$

If no such y exists, we set $\gamma(E, x, \delta) = 0$. We can now define the *porosity of E in X*. For more detailed discussions on porosity see [54, 61, 69].

Definition 2.2.1. If $x \in E$, then we define the *porosity of E in X at x* to be

$$p(E, x) = \limsup_{\delta \to 0^+} \frac{\gamma(E, x, \delta)}{\delta}.$$

If $p(E, x) > 0$, then E is said to be *porous in X at x*. We say that E is *porous in X* if $p(E, x) > 0$ for all $x \in E$. Any subset of X which can be written as a countable union of sets, each porous in X, is said to be *σ-porous in X*. If $A \subset X$ is σ-porous, then we say $X \setminus A$ is co-σ-porous.

Notice that if X contains no isolated points, then any countable subset of X is σ-porous.

For $a \in (1, 2]$ set $T_a(x) = ax$ for $0 \le x \le \frac{1}{2}$ and $T_a(x) = a(1 - x)$ for $\frac{1}{2} \le x \le 1$. We refer to this family of maps as the family of *tent maps*.

We restrict our attention to the parameters a from $[\sqrt{2}, 2]$. If $\sqrt{2} < a^m \le 2$ for some $m \in \{1, 2, 2^2, 2^3, \dots\}$, then the nonwandering set of T_a consists of

m disjoint closed intervals and a finite number of periodic points [59, p.78]. Moreover, for such a the map T_a^m restricted to any one of those intervals is a tent map with slope a^m, so is affinely conjugate to T_{a^m}. Thus, getting corresponding results for smaller parameter values is easy. We work with T_a restricted to its core, $[T_a^2(\frac{1}{2}), T_a(\frac{1}{2})]$; the core is the smallest forward invariant interval containing the turning point $\frac{1}{2}$. The term trajectory will always refer to the forward trajectory.

In [4] K. M. Brucks and M. Misiurewicz proved that for almost every (with respect to Lebesgue measure) $a \in [\sqrt{2}, 2]$, the T_a trajectory of the turning point $\frac{1}{2}$ is dense in $[T_a^2(\frac{1}{2}), T_a(\frac{1}{2})]$. Letting $\mathcal{D}$ denote those parameters $a \in [\sqrt{2}, 2]$ such that the closure of the trajectory of the turning point under T_a is $[T_a^2(\frac{1}{2}), T_a(\frac{1}{2})]$. In a joint paper with K. M. Brucks we proved (see [5]):

Theorem 2.2.2. *The set* $[\sqrt{2}, 2] \setminus \mathcal{D}$ *is* σ*-porous.*

For a detailed survey of these concepts we refer to [69] and the appendix of [61]. Each σ-porous set in $\mathbb{R}$ is of the first category and of zero Lebesgue measure. These sets arise quite often as exceptional sets. For example, Preiss and Zajíček verified that the set of points of Fréchet-nondifferentiability of any continuous convex function on a Banach space with a separable dual is σ-porous [51]. However, Konjagin showed that the set $E = \{x \in \mathbb{R} \mid \sum_{n=1}^{\infty} |\frac{\sin(n!\pi x)}{n}| \leq 1\}$ is a closed non-σ-porous set of zero Lebesgue measure [69, Chapter 5]. This shows that the σ-ideal of σ-porous sets is a proper subset of the σ-ideal of measure zero first category sets. Therefore Theorem 2.2.2 strengthens the result in [4]. To obtain this stronger result, a more delicate (refined) study of the kneading properties of tent maps was necessary. Some of these techniques might be of independent interest.

Chapter 3

The Haight-Weizsäcker problem and related topics

3.1 Zero-one law for $\sum f(nx)$

In the Summer of 1998 at the conference held in Miskolc I was talking about the results of Section 2.1. On the trainride to Miskolc I discussed these results with Dan Mauldin and he suggested to look at an unsolved problem from 1970, originating from the Diplomarbeit of Heinrich von Weizsäcker [64].

Basic problem: Assume $f : (0, +\infty) \to \mathbb{R}$ is a measurable function. Is it true that $\sum_{n=1}^{\infty} f(nx)$ either converges almost everywhere, or diverges almost everywhere, that is, whether there is a zero-one law for $\sum f(nx)$.

Again in this section we use exclusively the Lebesgue measure, by measurable we mean Lebesgue measurable and the Lebesgue measure of the set A will again be denoted by $|A|$.

The history of this problem is also interesting.

The following problem was asked by K. L. Chung in 1957 in the American Mathematical Monthly:

Problem [34]: If $f \in C[0, \infty)$, $f \geq 0$, and $\int_0^{\infty} f(x)dx = \infty$ then show that there exists $x > 0$, for which $\sum_{n=1}^{\infty} f(nx) = \infty$.

It is not difficult to see ([64]), that if $\int_0^{\infty} f(x)dx < \infty$, then $\sum_{n=1}^{\infty} f(nx) < \infty$ almost everywhere.

The basic problem appeared not only in H. v. Weizsäcker's Diplomarbeit but also in a paper of J. A. Haight [35]. The main result of [35] is the verification of the existence of a set $H \subset (0, \infty)$ of infinite measure, for which for all $x, y \in H,\ x \neq y$ the ratio x/y is not an integer, and furthermore $*$ for all $x > 0$, $nx \notin H$ if n is sufficiently large.
From this last property it follows that if $f(x) = \chi_H(x)$, the characteristic function of H, then $\sum_{n=1}^{\infty} f(nx) < \infty$ almost everywhere. Since H is of infinite measure we also have $\int_0^{\infty} f(x)dx = \infty$.

Lekkerkerker [42] started to study sets with property $*$.

Haight stated the following special version of the basic problem:

Question: If f is the characteristic function of a measurable set $A \subset (0, \infty)$, then what is the answer to our basic problem?

Before answering this question we discuss the very interesting history of the similar problem for periodic functions. This is the case when $f : \mathbb{R} \to \mathbb{R}$ is a periodic measurable function and without limiting generality we assume that its period $p = 1$.

If instead of the series $\sum f(nx)$ we consider only the sequence $f(nx)$ then Mazur and Orlicz [46] proved in 1940 that

$$\limsup_{n\to\infty} f(nx) = \operatorname{ess\,sup} f$$

for almost every x. This property holds in the more general case if the sequence (n) is being replaced by an arbitrary sequence (λ_n) converging to ∞.

This implies that if the periodic function f is the characteristic function of a set of positive measure, then for almost every x we have $\sum_n f(nx) = \infty$.

Thus in the periodic case we have a zero-one law.

In this case it is more interesting to consider the Cesàro 1 means of the partial sums of our series.

Khinchin conjecture [40] (1923): Assume that $E \subset (0,1)$ is a measurable set and $f(x) = \chi_E(\{x\})$, where $\{x\}$ denotes the fractional part of x. Is it true that for almost every x

$$\frac{1}{k}\sum_{n=1}^{k} f(nx) \to |E|?$$

It was a known result of H. Weyl [65] from 1916, that there is a positive answer to the above question if f is Riemann integrable.

Raikov [53] (1936), and F. Riesz [55] (1945) showed that if f is periodic by one and it is Lebesgue integrable on $[0,1]$ then for an arbitrary integer $q > 1$

$$\frac{1}{k}\sum_{n=1}^{k} f(q^n x) \to |E|$$

holds for almost every x.

On the other hand, Erdős [31] showed in 1949 that there exists such a sequence $a_n \to \infty$ and a function $f \in L^1(0,1)$ for which

$$\frac{1}{k}\sum_{n=1}^{k} f(a_n x) \to |E|$$

does not hold for almost all x.

Finally, in 1969 Marstrand [45] showed that the Khinchin conjecture is not true.

Now we return to the non-periodic case and to the problem of Haight and Weizsäcker. In a joint paper with Dan Mauldin [23] in 1998 we gave a negative answer to it. Our main result is the following.

Theorem 3.1.1. *There exists a measurable function* $f : (0, +\infty) \to \{0, 1\}$ *and two nonempty intervals* $I_F,\ I_\infty \subset [\frac{1}{2}, 1)$ *such that for every* $x \in I_\infty$ *we have* $\sum_{n=1}^{\infty} f(nx) = +\infty$ *and for almost every* $x \in I_F$ *we have* $\sum_{n=1}^{\infty} f(nx) < +\infty$. *The function* f *is the characteristic function of an open set* E.

The details of the proof can be found in [23], we just make a few remarks. In this chapter we will use the notation log for logarithm in base 2. Clearly, $\sum_{n=1}^{\infty} f(nx) = \sum_{n=1}^{\infty} f(2^{\log x + \log n})$, that is, changing notation it is sufficient to investigate the additive version of our problem. This means that we are interested in the convergence properties of the series $\sum_{n=1}^{\infty} f(x + \log n)$.

The key tool is Kronecker's Theorem on simultaneous inhomogenous approximation [25], p. 53. Here we state a special case of it which was used in [23]. ($||x||$ denotes the distance to the closest integer.)

Theorem 3.1.2. *Assume* $\theta_1, ..., \theta_L \in \mathbb{R}$ *and* $(\alpha_1, ..., \alpha_L)$ *is a real vector. The following two statements are equivalent:*

- *A) For every* $\epsilon > 0$*, there exists* $p \in \mathbb{Z}$ *such that*

$$||\theta_j p - \alpha_j|| < \epsilon, \text{ for } 1 \le j \le L.$$

- *B) If* $(u_1, ..., u_L)$ *is a vector consisting of integers and*

$$u_1\theta_1 + ... + u_L\theta_L \in \mathbb{Z},$$

then

$$u_1\alpha_1 + ... + u_L\alpha_L \in \mathbb{Z}.$$

In the next section we illustrate our method by discussing an easier problem. Actually, this problem was a warmup exercise to [23]. This problem can be handled by the homogeneous version of Kronecker's theorem (the case when all α_k's are zero). In the Fall of 1998 this problem was suggested by the author as one of the research problems for the annual Schweitzer Mathematical contest.

3.1.1 Problem 4 of the Schweitzer Mathematical contest in 1998

Problem:
Assume $H \subset \mathbb{R}$ is an arbitrary measurable set and define the sequence (a_n) by

$$a_n = a_n(H) = \left| [0,1] \setminus \bigcup_{k=n}^{2n} (H + \log_2 k) \right| .$$

Show that there exists a set $H \subset \mathbb{R}$ such that it is of positive measure, it is periodic by one, and the sequence $(a_n(H))$ does not belong to any ℓ_p space $(1 \le p < \infty)$.

Remark: The problem can be generalized many ways, instead of $\log_2 k$ one can use any sequence converging to ∞, and the rate of convergence of $a_n(H)$ to 0 can also be made as slow as we wish. However, the following solution shows how to obtain this result by using diophantine approximation.

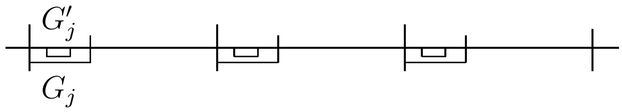
Solution: Assume $1 \le p < \infty$ and $N_1 < N_2 < \dots$ is a monotone subsequence of $\mathbb{N}$ satisfying

$$(N_j - N_{j-1})3^{-j^2} > 1. \tag{3.1}$$

If we deal with the homogeneous case in Kronecker's theorem (Theorem 3.1.2), that is all $\alpha_k = 0$, then condition B is always satisfied. Hence, the corresponding approximation problem can always be solved and one can find $q_j \in \mathbb{N}$ (Krocneker's theorem gives only $q_j \in \mathbb{Z}$ but it is not difficult to find $q_j \in \mathbb{N}$) such that

$$||q_j \log k|| < \frac{3^{-j}}{4}, \qquad \forall k = N_{j-1}, \dots, 2N_j.$$

Set
$$G_j \overset{\text{def}}{=} \bigcup_{m\in\mathbb{Z}} \left(\frac{m}{q_j}, \frac{m+3^{-j}}{q_j}\right) \qquad \text{and} \qquad H \overset{\text{def}}{=} \mathbb{R}\setminus \cup_{j=1}^{\infty} G_j$$



Then one can easily see that

$$G_j + \log k \supset \bigcup_{m\in\mathbb{Z}} \left(\frac{m+\frac{3^{-j}}{4}}{q_j}, \frac{m+\frac{3\cdot 3^{-j}}{4}}{q_j}\right) \overset{\text{def}}{=} G'_j$$

holds for $k = N_{j-1}, \ldots, 2N_j$.

Thus if $N_{j-1} < n \leq N_j$ then

$$G'_j \cap [0,1] \subset [0,1] \setminus \cup_{k=n}^{2n} (H + \log k).$$

Therefore, for the above n's we have $a_n \geq 3^{-j}/2$ and if $j > p$ then $a_n^p > 3^{-jp} > 3^{-j^2}$. By using (3.1) this implies $\sum_{N_{j-1}<n\leq N_j} a_n^p > 1$ and hence $(a_n) \notin \ell_p$. $\square$

3.2 Generalizations related to the zero-one law for $\sum f(nx)$

In 1975 Haight investigated in [36] some generalizations of [35]. He showed that if $\Lambda \subset [0,+\infty)$ is an arbitrary countable set such that its only accumulation point is $+\infty$ then there exists a measurable set $E \subset (0,+\infty)$ such that for all $x, y \in E$, $x \neq y$, $x/y \notin \Lambda$, and for a fixed x there exist only finitely many $\lambda \in \Lambda$ for which $\lambda x \in E$. This implies that choosing $f = \chi_E$ we have $\sum_{\lambda\in\Lambda} f(\lambda x) < \infty$, but $\int_{\mathbb{R}^+} f(x)dx = \infty$.

This problem motivated our joint work with J-P. Kahane and D. Mauldin ([21] and [22]). Below some results concerning the additive version of the generalized problem are listed.

Given Λ an infinite discrete set of nonnegative numbers, and $f : \mathbb{R} \to \mathbb{R}$, a nonnegative function, we consider the sum

$$s(x) = \sum_{\lambda \in \Lambda} f(x + \lambda), \tag{3.2}$$

and the complementary subsets of $\mathbb{R}$:

$$C = C(f, \Lambda) = \{x : s(x) < \infty\}, \qquad D = D(f, \Lambda) = \{x : s(x) = \infty\}.$$

Definition 3.2.1. Λ has type 1 if, for every f, either $C(f, \Lambda) = \mathbb{R}$ a.e. or $C(f, \Lambda) = \emptyset$ a.e. (or equivalently $D(f, \Lambda) = \emptyset$ a.e. or $D(f, \Lambda) = \mathbb{R}$ a.e.). Otherwise, Λ has type 2.

Example 3.2.2. $\Lambda = \cup_{k \in \mathbb{N}} \Lambda_k$, $\Lambda_k = 2^{-k}\mathbb{N} \cap [k, k+1)$. Theorem 3.2.5 below shows that Λ has type 1.

Example 3.2.3. Let $\Lambda = \{\log n, n = 1, 2, ...\}$. It follows from Theorem 3.1.1 that Λ has type 2. This was the topic of Section 3.1. This result is seen here from a different perspective which, in addition, allows us to see that this is the case for very many sets Λ.

Definition 3.2.4. Λ is asymptotically dense provided when it is ordered in the increasing order, the distance between two consecutive points tends to zero, or equivalently:

$$\forall a > 0, \quad \lim_{x \to \infty} \#(\Lambda \cap [x, x + a]) = \infty.$$

Otherwise, Λ is asymptotically lacunary, or equivalently, Λ has lacunae of the form $(x_j, x_j + l)$, $l > 0$, $x_j \to +\infty$.

In both of the preceding examples Λ is asymptotically dense. We note that our notion of asymptotically lacunary is not the same as the usual one of a "lacunary sequence."

Theorem 3.2.5. *Let (n_k) be an increasing sequence of positive integers and let $\Lambda = \cup_{k\in\mathbb{N}}\Lambda_k$ where $\Lambda_k = 2^{-k}\mathbb{N} \cap [n_k, n_{k+1})$. Then Λ is of type 1.*

In the next theorem $C_0^+(\mathbb{R})$ denotes the set of nonnegative, continuous functions tending to zero at infinity.

Theorem 3.2.6. *The set $\Lambda = \{\log n : n = 1, 2, ...\}$ has type 2. Moreover, for some $f \in C_0^+(\mathbb{R})$, C has full measure on the half-line $(0, \infty)$ and D contains the half line $(-\infty, 0)$. If for each c, $\int_c^{+\infty} e^y g(y)dy < +\infty$, then $C(g, \Lambda) = \mathbb{R}$ a.e. If $g \in C_0^+(\mathbb{R})$ and $C(g, \Lambda)$ is not of the first category (meager), then $C = \mathbb{R}$ a.e. Finally, there is some $g \in C_0^+(\mathbb{R})$ such that $C(g, \Lambda) = \mathbb{R}$ a.e. and $\int_0^{+\infty} e^y g(y)dy = +\infty$.*

Theorem 3.2.7. *Let (n_k) be a given increasing sequence of positive integers. There is an increasing sequence of integers $(m(k))$ such that the set $\Lambda = \cup_{k\in\mathbb{N}}\Lambda_k$ with $\Lambda_k = 2^{-m(k)}\mathbb{N} \cap [n_k, n_{k+1})$ is of type 2.*

Theorem 3.2.8. *If Λ is asymptotically lacunary, then Λ has type 2. Moreover, for some $f \in C_0^+(\mathbb{R})$, there exist intervals I and J, I to the left of J, such that $C(f, \Lambda)$ contains I and $D(f, \Lambda)$ contains J.*

Theorem 3.2.9. *Suppose that there exist three intervals I, J, K such that $J = K + I - I$ (algebraic sum), I is to the left of J, and dist$(I, J) \geq |I|$, and two sequences (y_j) and (N_j) tending to infinity ($y_j \in \mathbb{R}^+$, $N_j \in \mathbb{N}$) such that, for each j, $y_j - I$ contains a set of N_j points of Λ independent from $\Lambda \cap (y_j - I)$ in the sense that the additive groups generated by these sets have only 0 in common. Then Λ has type 2. Moreover, for some $f \in C_0^+(\mathbb{R})$, $D(f, \Lambda)$ contains I and $C(f, \Lambda)$ has full measure on K.*

3.3 The study of the Λ^{α^k} sets

In [24] a specific class of Λ sets was investigated.

Given a sequence of natural numbers $n_1 < n_2 < ...$, it was shown that $\Lambda^{(\frac{1}{2})^k} := \cup_{k=1}^{\infty}\Lambda_k^{(\frac{1}{2})^k}$, where $\Lambda_k^{(\frac{1}{2})^k} := (\frac{1}{2})^k\mathbb{Z} \cap [n_k, n_{k+1})$ is of type 1. See Theorem 3.2.5. But by Theorem 3.2.7 this type 1 set could be adjusted to become a type 2 set by making it become asymptotically dense at a faster rate as follows: there is a sequence $m(k) \to \infty$ such that $\Lambda^{(\frac{1}{2})^{m(k)}} := \cup_{k=1}^{\infty}\Lambda_k^{(\frac{1}{2})^{m(k)}}$, $\Lambda^{(\frac{1}{2})^{m(k)}} := (\frac{1}{2})^{m(k)}\mathbb{Z} \cap [n_k, n_{k+1})$ is of type 2.

A number $t > 0$ is called a translator of Λ if $(\Lambda + t) \setminus \Lambda$ is finite.

Condition (†) is said to be satisfied if $T(\Lambda)$, the countable additive semi-group of translators of Λ, is dense in $\mathbb{R}^+$.

We showed that condition (†) implies that $C(f, \Lambda)$ is either $\emptyset$, $\mathbb{R}$, or a closed left half-line modulo sets of measure zero. The two sets just described above both satisfy condition (†). So condition (†) is not enough to determine whether Λ is of type 1 or type 2.

In this section for a given $\alpha \in (0,1)$, we are interested in the sets $\Lambda^{\alpha^k} := \cup_{k=1}^{\infty}\Lambda_k^{\alpha^k}$, $\Lambda_k^{\alpha^k} = \alpha^k\mathbb{Z} \cap [n_k, n_{k+1})$.

If $\alpha = \frac{1}{q}$ for some $q \in \{2, 3, ...\}$, then a slight modification of the proof of Theorem 3.2.5 shows that $\Lambda^{(\frac{1}{q})^k}$ is of type 1 and condition (†) is satisfied. If $\alpha \notin \mathbb{Q}$, then one can apply Theorem 3.2.9 to show that Λ^{α^k} is of type 2.

Thus, there remains the difficult case when $\alpha = \frac{p}{q}$ with $(p, q) = 1$, $p, q > 1$, $p < q$. In this case we show that $\Lambda^{(\frac{p}{q})^k}$ is of type 2.

Suppose $p, q > 1$ are relative prime integers and $1 < p < q$. For ease of notation we denote $\Lambda^{(\frac{p}{q})^k}$ by Λ.

Fix an integer r such that $r > 8$, $(r, p) = 1$, and $(r, q) = 1$.

The main result of [24] is the following:

Theorem 3.3.1. *The set* Λ *defined above is of type 2. Moreover, there exists a characteristic function* $f : \mathbb{R} \to \mathbb{R}^+$ *such that almost every point in* $[\frac{1}{r}, \frac{2}{r}]$ *belongs to* $C(f, \Lambda)$, $|D(f, \Lambda) \cap [1 + \frac{1}{r}, 1 + \frac{2}{r}]| > \frac{1}{8r}$ *and*

$|D(f,\Lambda) \cap [-1+\frac{1}{r}, -1+\frac{2}{r}]| > \frac{1}{8r}$.

The problem of showing Theorem 3.3.1 is not easy even in the special case when $p = 2$, $q = 3$. When working on this problem, for a while it seemed that we needed some information on the distribution of $\{(\frac{3}{2})^k\}$, (where $\{\cdot\}$ denotes the fractional part). To our surprize, and showing why the case $\Lambda^{(\frac{2}{3})^k}$ was difficult, it turned out, see [27] that in 1980 it was not even known, whether $\{(\frac{3}{2})^k\}$ is uniformly distributed, or even dense in $[0,1]$. These questions were extensively studied previously (see further references in [27]) and according to a recent letter from G. Choquet remain open at this time. Fortunately, we found a way avoiding any information about the distribution of $\{(\frac{3}{2})^k\}$, (or of $\{(\frac{q}{p})^k\}$) to determine that $\Lambda^{(\frac{2}{3})^k}$, (or $\Lambda^{(\frac{p}{q})^k}$) is of type 2. Another novelty is that in the cases $\Lambda^{(\frac{p}{q})^k}$, $(p > 1)$ condition (†) is not satisfied and in Theorem 3.3.1 we also show that there exists a characteristic function f such that $C(f,\Lambda)$ does not equal $\emptyset$, $\mathbb{R}$, or a left half-line modulo sets of measure zero. This structure of $C(f,\Lambda)$ has not been seen before and casts a little more light on the question of what the possible structure could be.

Chapter 4

The gradient problem

It is well-known that if $f : (a, b) \to \mathbb{R}$ is everywhere differentiable on the open interval (a, b) then its derivative, f' is Darboux and Baire one. It is less widely known that derivatives have one more property, the so called Denjoy–Clarkson property see [28], [29]. This property states that if (α, β) is any open interval then its inverse image by the derivative, that is, $(f')^{-1}(\alpha, \beta)$ is either empty, or of positive one-dimensional Lebesgue measure. This means that if inverse images of open intervals are nonempty then they are "heavy". Of course, if the function is continuously differentiable then this inverse image is a nonempty open set and hence heavy. So the Denjoy–Clarkson property is really interesting for functions with discontinuous derivatives. In [63] it was also shown that the Denjoy–Clarkson property is posessed by every approximate derivative and kth Peano derivative. Complete characterization of derivatives is still not available. See on this topic A. Bruckner's survey article [6].

The gradient problem of C. E. Weil is about the multidimensional version of the Denjoy–Clarkson property.

Assume $n \geq 2$, $G \subset \mathbb{R}^n$ is an open set and $f : G \to \mathbb{R}$ is an everywhere differentiable function. Then $\nabla f = (\partial_1 f, ..., \partial_n f)$ maps G into $\mathbb{R}^n$. Assume $\Omega \subset \mathbb{R}^n$ is open. Is it true that $(\nabla f)^{-1}(\Omega) = \{\mathbf{p} \in G : \nabla f(\mathbf{p}) \in \Omega\}$ is either empty, or of positive n-dimensional Lebesgue measure? For continously differentiable functions, like in the one-dimensional case, the answer is obviously positive.

One other, equivalent and interesting way to look at the gradient problem is the following special case. Denote by B_1 the open unit ball in the n-dimensional Euclidean space. Does there exist a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that its gradient equals the null vector at the origin and (Lebesgue) almost everywhere the norm of the gradient is bigger than one. This means that $(\nabla f)^{-1}(B_1)$ is nonempty, but of zero Lebesgue measure. So our differentiable function due to vanishing gradient is not changing much in a neighborhood of the origin, but almost everywhere, "with probability one" it changes rapidly.

The gradient problem was one of the well-known and famous unsolved problems in Real Analysis. It was around since the paper [63] appeared in the 1960s. I have learned it in 1987 from Clifford Weil. This was my first visit to the US and I quite clearly remember as after being introduced to him, standing in front of the Alamo in San Antonio, he suggested me to try to solve this problem. I also received the warning that this was a difficult problem... So I started to work on it. Of course, I was not working on this problem continuously but when I learned something new, I returned to it, worked for a few months and when I felt that I had no breakthrough I gave it up for some time... In case I obtained a partial result which was sufficiently interesting I published it and/or gave talks about it at different conferences. This was going on till the end of the Summer of 2002 when I finally managed to solve it. In 1990 at the Fourteenth Summer Symposium on Real Analysis the gradient problem was advertised and appeared in print in [71]. For most of the time when I worked on the gradient problem I tried to prove a theorem stating an n-dimensional version of the Denjoy-Clarkson property but there was something always missing... A few years ago I even made a bet with L. Zajíček about the way the solution goes. In the end

I had to revise my view of the problem and started to work seriously on a two-dimensional (counter)example. This led to the solution, and I lost the bet. There is the usual moral that at a correct trial, both sides should receive a fair hearing. One should have a flexible thinking to be able to look at and analyse a problem from both ends.

In this chapter I would like to go through the road leading to the final solution, while the details of the proof can be found in the research article [17].

The answer to the gradient problem is a two-dimensional counterexample. In [17] it is shown that there exists a nonempty open set $G \subset \mathbb{R}^2$, a differentiable function $f : G \to \mathbb{R}$ and an open set $\Omega_1 \subset \mathbb{R}^2$ for which there exists a $\mathbf{p} \in G$ such that $\nabla f(\mathbf{p}) \in \Omega_1$ but for almost every (in the sense of two-dimensional Lebesgue measure, λ_2,) $\mathbf{q} \in G$ the gradient $\nabla f(\mathbf{q})$ is not in Ω_1.

The above result was a breakthrough in this area. Ideas of our method later led to an alternate construction, based on infinite games. See the recent paper of J. Malý and M. Zelený, [44]. It is interesting to see that heuristics from different areas of Mathematics, namely our method from Dynamical Systems, and the Malý–Zelený method from Descriptive Set Theory can motivate alternate proofs of the same difficult problem in Real Analysis.

4.1 The results on the way to the solution of the gradient problem

4.1.1 The $\mathcal{H}_1$ Denjoy-Clarkson property

It is not too difficult to prove (see [12]) that if one replaces the n-dimensional Lebesgue measure by the one-dimensional Hausdorff measure, $\mathcal{H}_1$, then the Denjoy–Clarkson property holds in the multidimensional case as well:

Theorem 4.1.1. *Assume $G \subset \mathbb{R}^n$ is open, $f : G \to \mathbb{R}$ is a differentiable function. Then for any $\Omega \subset \mathbb{R}^n$ open set $(\nabla f)^{-1}(\Omega)$ is either empty, or of positive $\mathcal{H}_1$ measure.*

So inverse images of open sets by the gradient map at least should be heavy in the $\mathcal{H}_1$ sense.

In another research paper, [38], Holický, Malý, Weil, and Zajíček verified the following theorem (for the definition of porosity see Section 2.2):

Theorem 4.1.2. *Let $G \subset \mathbb{R}^n$ be an open set and let f be a (Fréchet) differentiable function on G. Suppose $\Omega \subset \mathbb{R}^n$ is an open set such that $(\nabla f)^{-1}(\Omega) \neq \emptyset$. Then the following assertions hold.*

- *(i) $(\nabla f)^{-1}(\Omega)$ is porous at none of its points.*
- *(ii) If $H \subset \mathbb{R}^n$ is open and $H \cap (\nabla f)^{-1}(\Omega) \neq \emptyset$, then $L(H \cap (\nabla f)^{-1}(\Omega))$ is of one-dimensional Lebesgue mesure zero for no nonzero linear function $L : \mathbb{R}^n \to \mathbb{R}$. In particular, $H \cap (\nabla f)^{-1}(\Omega)$ is of positive one-dimensional Hausdorff measure.*
- *(iii) If $H \subset \mathbb{R}^n$ is open and $H \cap (\nabla f)^{-1}(\Omega) \neq \emptyset$, then $H \cap (\nabla f)^{-1}(\Omega)$ is not σ-porous.*

Observe that in (ii) of the above theorem one can take any one-dimensional projection as the linear function showing that any one-dimensional projection of $(\nabla f)^{-1}(\Omega)$ is of positive $\mathcal{H}_1$ measure, when $(\nabla f)^{-1}(\Omega)$ is nonempty.

4.1.2 The "paradoxical convexity" property

The above results point in the direction that the counterexample function of [17] should be quite complicated. In the next result we show that this function is even more complicated. For a while I believed that such functions do not exist, but they do. I think that this "paradoxical convexity" property of the counterxample functions to the gradient problem was widely ignored by even those people who worked on the gradient problem, though this was the property which finally led to the convexity construction in [17]. So assume that $G_0 \subset \mathbb{R}^2$ and $f : G_0 \to \mathbb{R}$ is a differentiable counterexample function to the gradient problem. This means that there exists an $\mathbf{x}_0 \in G_0$, a $\mathbf{p}_0 \in \mathbb{R}^2$ and an $\eta_0 > 0$ such that $\nabla f(\mathbf{x}_0) = \mathbf{p}_0$ and $\lambda_2((\nabla f)^{-1}(B(\mathbf{p}_0, \eta_0))) = 0$, where λ_2 denotes the Lebesgue measure in the plane and $B(\mathbf{p}_0, \eta_0)$ is the open ball of radius η_0 centered at $\mathbf{p}_0$. Set $F_0 = cl((\nabla f)^{-1}(B(\mathbf{p}_0, \eta_0/2)))$. Since ∇f is a Baire one function on F_0 it has a point of continuity, $\mathbf{x}_1 \in F_0$. From the definition of F_0 it follows that $\nabla f(\mathbf{x}_1) \in cl(B(\mathbf{p}_0, \eta_0/2))$. Hence, $B(\nabla f(\mathbf{x}_1), \eta_0/2) \subset B(\mathbf{p}_0, \eta_0)$. Choose $\delta > 0$ such that for any $\mathbf{y} \in B(\mathbf{x}_1, \delta) \cap F_0$ we have $||\nabla f(\mathbf{x}_1) - \nabla f(\mathbf{y})|| < \eta_0/4$. Then $B(\mathbf{x}_1, \delta) \cap F_0 \subset (\nabla f)^{-1}(B(\mathbf{p}_0, \eta_0))$ and hence $\lambda_2(B(\mathbf{x}_1, \delta) \cap F_0) = 0$. On the other hand, if we set $\mathcal{R} = \nabla f(B(\mathbf{x}_1, \delta)) \cap B(\mathbf{p}_0, \eta_0/2)$ then $\mathcal{R} \subset B(\nabla f(\mathbf{x}_1), \eta_0/4)$. Hence, $\mathcal{G} = B(\mathbf{p}_0, \eta_0/2) \setminus cl(\mathcal{R})$ is a nonempty open set, and the density of $(\nabla f)^{-1}(B(\mathbf{p}_0, \eta_0/2))$ in F_0 implies that $\mathcal{R}$ is nonempty either. By a suitable change of variable we can assume that $\mathbf{p}_0 = \mathbf{0}$ and $\eta_0/2 = 1$. We can also take $G = B(\mathbf{x}_1, \delta)$ and work with the restriction of f onto this set. Hence, the conditions of the following theorem are satisfied (see [15]):

Theorem 4.1.3. *Assume that f is a differentiable function on $G \subset \mathbb{R}^2$. Set $\overline{\Delta}_1 = cl\{\mathbf{x} \in G : \nabla f(\mathbf{x}) \in B(\mathbf{0}, 1)\}$. Suppose $\overline{\Delta}_1$ is nonempty and $\lambda_2(\overline{\Delta}_1) = 0$. Put $\mathcal{R} = B(\mathbf{0}, 1) \cap \nabla f(G)$ and $\mathcal{G} = B(\mathbf{0}, 1) \setminus cl(\mathcal{R})$. Then $\mathcal{G}$ is a convex open subset of the plane and $\mathcal{G} \neq \emptyset$ implies that for any $\mathbf{p} \in int(cl(\mathcal{R}))$ we have $\mathcal{H}_1(\{\mathbf{y} : \nabla f(\mathbf{y}) = \mathbf{p}\}) > 0$.*

By a suitable linear change of variable and rescaling we can achieve that $\mathcal{G} \neq \emptyset$ and $\mathbf{0} \in \mathcal{R}$. Then $B(\mathbf{0}, 1)$ contains a half disk such that for any point, $\mathbf{p}$ belonging to this half disk we have $\mathcal{H}_1(\{\mathbf{x} : \nabla f(\mathbf{x}) = \mathbf{p}\}) > 0$. This shows that our differentiable function is very different from the smooth surfaces we got used to. For example, if one takes the open upper half sphere $x_3 = f(x_1, x_2) = \sqrt{x_1^2 + x_2^2}$ then for any $\mathbf{p} \in \mathbb{R}^2$ we have that either

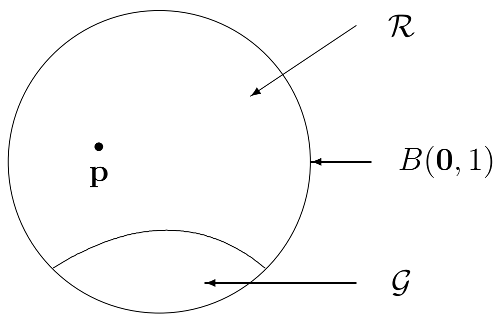


Figure 4.1: "Paradoxical convexity" in the range

$(\nabla f)^{-1}(\mathbf{p})$ is empty, or it consists of a single point only. So the price we need to pay for nonempty but λ_2-measure zero ∇f preimages of open sets is that we have many points with large preimages in the sense of $\mathcal{H}_1$ measure.

Since $int(cl(\mathcal{R}))$ is uncountable Theorem 4.1.3 also implies that $(\nabla f)^{-1}(B(\mathbf{0},1))$ is of non-sigma finite $\mathcal{H}_1$ measure.

Therefore, it is possible that $(\nabla f)^{-1}(B(\mathbf{0},1)))$ is nonempty and of zero λ_2 measure, but it is at least of non-sigma finite $\mathcal{H}_1$ measure. So there is a gap which needs to be explored.

Question 4.1.4. Assume that $G \subset \mathbb{R}^n$ is an open set, $f : G \to \mathbb{R}$ is a differentiable function and $\Omega \subset \mathbb{R}^n$ is such an open set that $(\nabla f)^{-1}(\Omega) \neq \emptyset$. Then, what can we say about the Hausdorff-dimension of $(\nabla f)^{-1}(\Omega)$?

Recently M. Zelený made progress in this direction and in a preprint [70] he constructed a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, $(n \geq 2)$ such that the set $(\nabla f)^{-1}(B(0,1))$ is a nonempty set of Hausdorff dimension 1.

4.1.3 Functions with many tangent planes

Now we return to Theorem 4.1.3. Assume that the assumptions of this theorem are satisfied and $\mathbf{p} \in int(cl(\mathcal{R}))$. Put $f^{\mathbf{p}}(\mathbf{x}) = f(\mathbf{x}) - \mathbf{p} \cdot \mathbf{x}$, where

$\cdot$ denotes the usual dot/scalar product in $\mathbb{R}^2$. We use the following notation for the level sets $f_c^{\mathbf{p}} \stackrel{\text{def}}{=} \{\mathbf{x} : f^{\mathbf{p}}(\mathbf{x}) = c\}$. By scrutinizing the proof of Theorem 4.1.3 one can see that for a set of positive λ_1 measure of c's, there exists $\mathbf{x}_c$ on the level set $f_c^{\mathbf{p}}$ such that $\nabla f^{\mathbf{p}}(\mathbf{x}_c) = \mathbf{0}$, that is, $\nabla f(\mathbf{x}_c) = \mathbf{p}$. This means that we can find many points where the tangent plane to the surface given by the graph of f has gradient $\mathbf{p}$.

This turned my attention to differentiable functions with many tangent planes. Results in this direction were published in [16]. The main result of this paper is the following.

Theorem 4.1.5. *There exists a C^1 function $f : \mathbb{R}^2 \to \mathbb{R}$, a nowhere dense closed set $E \subset [0,1] \times [0,1]$ of zero λ_2 measure, and a nonempty open set $H \subset \mathbb{R}^3$ such that for any $(a,b,c) \in H$ we can find an $(x_0, y_0) \in E$ for which the equation of the tangent plane to the surface $z = f(x,y)$ at the point $(x_0, y_0, f(x_0, y_0))$ is $z = ax + by + c$.*

So we can expect more than differentiability. There are even C^1 functions for which we have the above construction. Observe that H in the parameter space is a three-dimensional open set and E is a measure zero two-dimensional one. So we gain not only measure, but dimension as well, we have a "Peano" map into the parameter space. In dimension one we got used to the fact that C^1 functions are sufficiently smooth. However in higher dimensions it is not true any more. In the theory of partial differential equations there are several results like the Sobolev Lemma ([56] Th. 7.25) where smoothness assumptions and consequences depend on the dimension.

Recall the Morse–Sard theorem [47], [57].

Theorem 4.1.6. *Let M, N be manifolds of dimension m, n and $f : M \to N$ a C^r map. If $r > \max\{0, m-n\}$ then $f(\Sigma_f)$ has measure zero in N, where Σ_f denotes the critical points of f. The set of regular values of f is residual and therefore dense.*

Hence, if $f : \mathbb{R}^2 \to \mathbb{R}$ is a C^2 function then for any $(a, b) \in \mathbb{R}^2$ the set of critical values $c_{(a,b)}$ of $f(x, y) - ax - by$ is of λ_1 measure zero. Thus, it cannot contain an interval, moreover by Fubini's theorem the set

$$\{(a, b, c) \in \mathbb{R}^3 : \text{ the plane } z = ax + by + c \text{ is tangent to } z = f(x, y)\}$$

is of λ_3 measure zero. Therefore, it has empty interior.

C^1 functions in the plane can exhibit many other pathological properties. One is the famous example of Whitney [66] which gives a C^1 function defined on $\mathbb{R}^2$ and a continuous nondegenerate arc γ such that $\nabla f = \mathbf{0}$ on γ but f is not constant along γ. If we relax the assumption to differentiability Körner [41] constructed a non-constant differentiable function defined on the plane for which any two points in $\mathbb{R}^2$ can be connected by a continuous curve, γ such that $\nabla f = \mathbf{0}$ everywhere on γ but at endpoints.

It is also worth to say a few words about the one-dimensional version of Theorem 4.1.5. So assume that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function. If $y = ax + b$ is the tangent of f at $(x_0, f(x_0))$ then we set $S_1(x_0) = (a, b)$. Is it possible that $S_1(\mathbb{R})$ has nonempty interior?

In dimension one C^1 functions are sufficiently smooth for the Morse–Sard theorem. Thus, the answer is no. For any $a \in \mathbb{R}$ the critical values of $f(x) - ax$ are of measure zero. Therefore, for a fixed a we have only λ_1-measure zero b's in the range of S_1.

One might expect that for differentiable fuctions maybe the answer is positive. But it is still not. This time another one of my favorite theorems in Real Analysis can be used. Part of the Denjoy-Young-Saks theorem ([58], Chap. IX, (3.7) Theorem, p. 267) implies the following. Assume $y = a'x + b'$ denotes a given line in $\mathbb{R}^2$ and π' denotes the orthogonal projection onto this line. Assume $f : \mathbb{R} \to \mathbb{R}$ is an arbitrary function and E' denotes the π' image of those points of the graph of f where the tangent to f is perpendicular to $y = a'x + b'$, see Figure 4.2. Then $\lambda_1(E') = 0$. This again implies that $\lambda_2(S_1(\mathbb{R})) = 0$ even for differentiable functions.

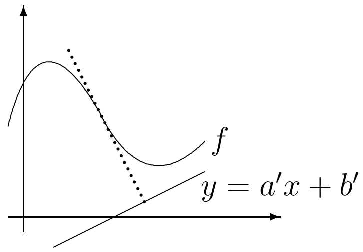


Figure 4.2: Projection π' onto $y = a'x + b'$

4.1.4 The level set structure

Trying to answer the gradient problem in dimension two I also got interested in the level set structure of functions defined on the plane. (Anyone who likes hiking encounters similar type level sets on hiking and other geographic maps.) So assume $f : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function. At critical points the level sets can look very strange. So it seems to be natural to assume that the gradient is nonvanishing.

Assuming this, is it true that the level sets, $\{(x, y) : f(x, y) = c\}$ consist of differentiable arcs? The answer is no, see [13]. If one takes

$$f(x,y) = \begin{cases} \frac{y^3 - yx^4}{y^2 + x^4}, & \text{if} \quad (x,y) \neq (0,0); \\ 0, & \text{if} \quad (x,y) = (0,0). \end{cases}$$

then the following theorem shows that the level set can contain bifurcation points. On the left side of Figure 4.3 this function is plotted. For better viewing on the 3D image the y axis is pointing from left to right. On the right side of this picture the level set structure of this function is plotted in the plane, the axes are pointing in the usual directions. Properties of this function are given in the following theorem.

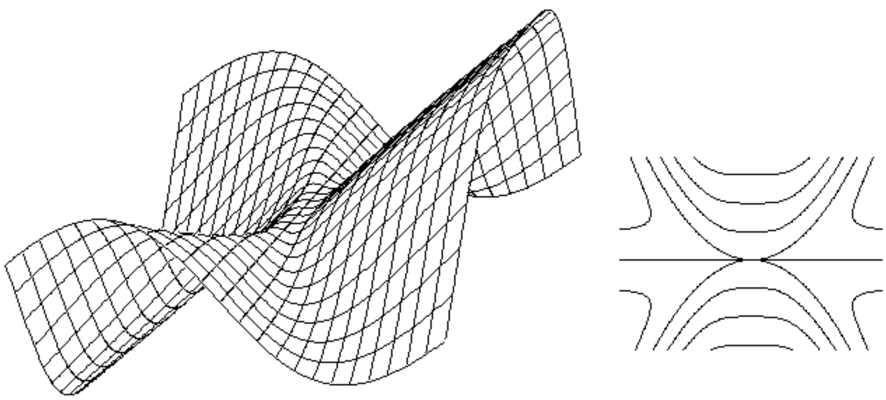


Figure 4.3: $f(x,y)$ with bifurcation

Theorem 4.1.7. *There exists a differentiable function* $f:\mathbb{R}^2\to\mathbb{R}^2$ *such that* $\nabla f(x,y)\neq\mathbf{0}$ *for all* $(x,y)\in\mathbb{R}^2$ *and* $S_0=\{(x,y): f(x,y)=0\}=\{(x,y): |y|=x^2,\ \text{or } y=0\}$*, and hence one cannot find a neighborhood* G *of* $(0,0)$ *such that* $U\cap S_0$ *is homeomorphic to an open interval.*

However, when I was thinking about the gradient problem Theorem 4.1.7 was not a serious obstruction since by using the Baire one property of the gradient and adding a suitable linear function, like in the argument preceding Theorem 4.1.3, one can work with differentiable functions such that the gradient is bounded away from zero. This extra assumption is sufficient to get rid of bifurcation points and the following theorem holds.

Theorem 4.1.8. *Suppose that* $G\subset\mathbb{R}^2$ *is open,* $f:G\to\mathbb{R}$ *is differentiable and* $||\nabla f(x,y)||>r>0$ *for all* $(x,y)\in G$*. If* $c=f(x_0,y_0)$ *for an* $(x_0,y_0)\in G$ *then there exists a neighborhood* G_0 *of* (x_0,y_0) *such that* $S_c=G_0\cap\{(x,y): f(x,y)=c\}$ *is homeomorphic to an open interval and* S_c *has a tangent at each of its points.*

Assuming sufficient smoothness one might obtain a similar result by using the implicit function theorem. However, even for a generalized version of the two-dimensional implicit function theorem (see [26]), an assumption is used that the partial derivative of $f(x, y)$ with respect to y is non-vanishing plus some other assumptions are also necessary about the local boundedness of the (generalized) partial derivatives. It is intersting to observe that for the inverse function theorem differentiability is sufficient. In [52] it is proved that if the mapping $f : G \to \mathbb{R}^n$ is differentiable on an open set $G \subset \mathbb{R}^n$ and $\det f'(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in G$ then f is a local diffeomorphism. This theorem is standard if f is sufficiently smooth, but when only differentianility is assumed then there is nontrivial topology, namely, Degree Theory in the background.

The direction of research coming from Theorems 4.1.7 and 4.1.8 was continued by M. Elekes. The results in [30] show that if a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ has nonvanishing gradient then locally the level set $\{\mathbf{x} \in \mathbb{R}^2 : f(\mathbf{x}) = c\}$ is homeomorphic either to an open interval or (at bifurcation points) to the union of finitely many line segments passing through a point. The bifurcation points form a discrete set.

4.1.5 Partials and approximate continuity of derivatives

Definition 4.1.9. The function $f : \mathbb{R}^n \to \mathbb{R}^m$ is approximate continuous at $\mathbf{x}$ if there exists an $E \subset \mathbb{R}^n$ such that $\mathbf{x}$ is a Lebesgue density point of E and $\lim_{\mathbf{y}\to\mathbf{x};\mathbf{y}\in E} f(\mathbf{y}) = f(\mathbf{x})$. The set of approximate continuity points of f will be denoted by A_f.

In [49] G. Petruska verified that if $f : \mathbb{R} \to \mathbb{R}$ is the derivative of a function F, then f takes each of its values on its approximate continuity points, that is, $f(\mathbb{R}) = f(A_f)$. This implies the one-dimensional Denjoy–Clarkson property. Hence I was interested in higher-dimensional generalizations of this result.

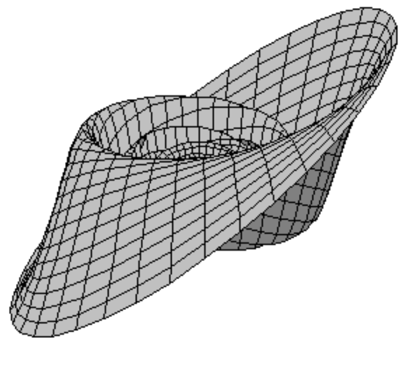


Figure 4.4: $f(r,\phi)$

In [14] I proved the following results. If one considers partial derivatives then the following theorem holds.

Theorem 4.1.10. *Assume $F:\mathbb{R}^n \to \mathbb{R}$ is differentiable and $i \in \{1,...,m\}$. Then $f = \partial_i F$ takes each of its values on A_f.*

This implies that partials of differentiable functions have the Denjoy–Clarkson property. The assumption that F is differentiable is needed in the above theorem since the existence of partial derivatives is not sufficient as the following theorem shows.

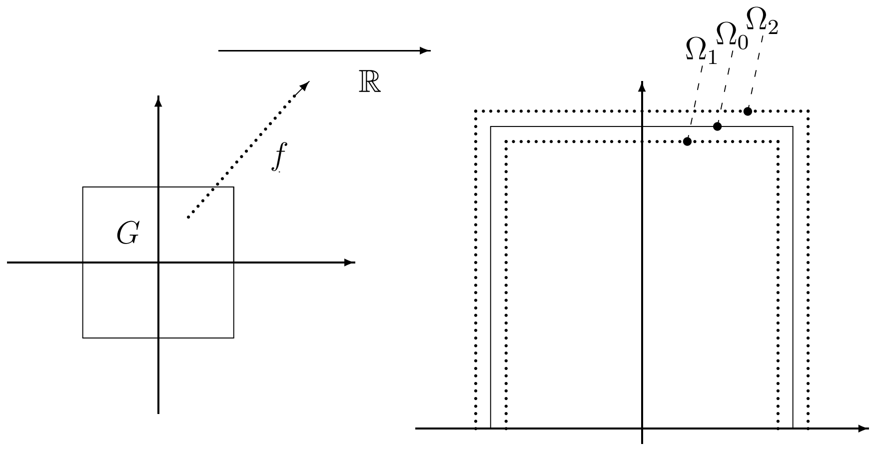
Theorem 4.1.11. *There exists a continuous function $F:\mathbb{R}^2 \to \mathbb{R}$ such that $f = \partial F/\partial x_1$ exists everywhere and f does not take each of its values on A_f.*

For the gradient of differentiable functions we have a similar theorem.

Theorem 4.1.12. *There exists a differentiable function $f:\mathbb{R}^2 \to \mathbb{R}$ such that ∇f does not take each of its values on $A_{\nabla f}$.*

The function of the above theorem can be defined in polar coordinates as

$$f(r,\phi) = \begin{cases} r^2 \sin\left(\phi + \frac{1}{r^2}\right), & \text{if } r \neq (0,0); \\ 0, & \text{if } r = 0. \end{cases}$$



Domain and range of f Range of ∇f, the gradient space

Figure 4.5: G and the sets Ω in the gradient space

This function can be seen on Figure 4.4. The gradient of this function vanishes only at $\mathbf{0}$ and this value is not in $A_{\nabla f}$.

Question 4.1.13. Assume $f : \mathbb{R}^n \to \mathbb{R}$ is a differentiable function. We call a point $\mathbf{y} \in \mathbb{R}^n$ a regular value of ∇f if there is an $\mathbf{x} \in A_{\nabla f}$ such that $\nabla F(\mathbf{x}) = \mathbf{y}$. Denote the set of regular values by $REG(\nabla F)$. The question in [14] about the density of $REG(\nabla F)$ in the range of ∇F receives a negative answer by the negative answer to the gradient problem. On the other hand our question about the characterization of $REG(\nabla F)$ remains open.

4.2 Outline of the construction of the counterexample to the gradient problem

We put $G = (-1, 1) \times (-1, 1)$, $\Omega_0 = [-\frac{1}{2}, \frac{1}{2}] \times [0, 2]$, $\Omega_1 = (-0.49, 0.49) \times (0, 1.99)$, and $\Omega_2 = [-0.51, 0.51] \times [0, 2.01]$. On the right side of Figure 4.5

the boundary of Ω_0 is marked by solid line, while the boundaries of Ω_1 and Ω_2 are marked by dotted lines.

Our answer to the gradient question is the following theorem. The details of the proof can be found in the research article [17], here we discuss only the main ideas of the construction.

Theorem 4.2.1. *There exists a differentiable function* $f : G \to \mathbb{R}$ *such that* $\nabla f(0,0) = (0,1)$ *and* $\nabla f(\mathbf{p}) \notin \Omega_1$ *for almost every* $\mathbf{p} \in G$.

To illustrate this theorem on the left side of Figure 4.5 the domain of f is shown and an arrow from the domain into a copy of $\mathbb{R}$ shows that f maps into $\mathbb{R}$, on the other hand, on the right side of this figure one can see the range of the gradient map, the "gradient space".

Scrutinizing the proof of Theorem 4.2.1 one can also see that f is a Lipschitz function and ∇f stays in the upper half-plane.

If one relaxes the assumption in the above theorem to λ_2 almost everywhere differentiability of f it is very easy to construct some Lipschitz functions. On Figure 4.6 one of the favorite "constructions" of people thinking about the gradient problem can be seen, a sheet of folded paper. This corresponds to a surface which is not changing much globally, although locally with a sufficiently large number of zig-zags one can obtain almost everywhere large values of the gradient. These large gradient values are illustrated by the arrows on the figure. The surface is not differentiable on the folding edges, and elsewhere the gradient takes only two values. If we want to obtain a differentiable function we need to sand down the folding edges which yields small absolute value gradient values but we can still keep the gradient values within a line segment in the plane. We have some freedom of choosing the number of zig-zags and choosing the direction of the folding, that is, choosing the direction of the segment containing the values of the gradient. Slightly modified versions of these folded paper sheet surfaces will be converted into perturbation functions denoted by $\phi_{B_{n,k}}$. The function of Theorem 4.1.12

shown on Figure 4.4 corresponds to a paper folding along a "spiral" and is not far from this construction.

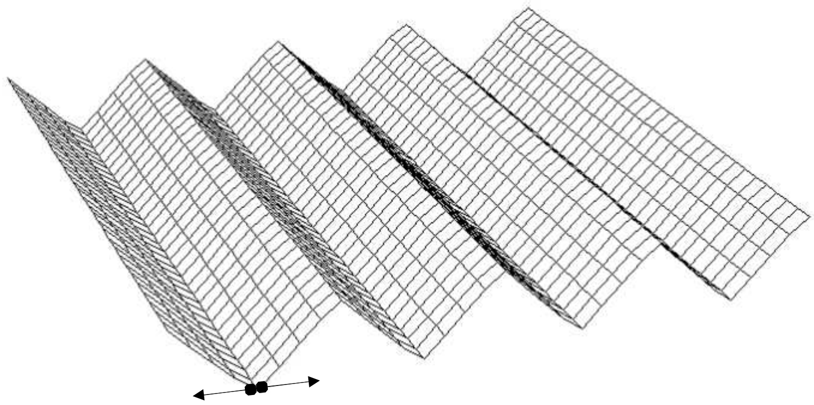


Figure 4.6: Folded paper

Next we outline the proof of Theorem 4.2.1. We start with a function $h_{-1}(x, y) = y$. Then $\nabla h_{-1} = (0, 1)$ everywhere on G. We choose a sequence of functions h_n so that $f(x, y) \stackrel{\text{def}}{=} \sum_{n=-1}^{\infty} h_n(x, y)$ satisfies Theorem 4.2.1. Each function h_{n+1} can be regarded as a perturbation of the previous partial sum $f_n = \sum_{k=-1}^{n} h_k$.

We want to push $\nabla f(\mathbf{p})$ outside of Ω_1 for almost every $\mathbf{p} \in G$.

In the construction we will have a nested sequence of open sets G_n such that for almost every $\mathbf{p}$ outside of G_n we will have $\nabla f(\mathbf{p}) \notin \Omega_1$. We use a "stopping time argument" by not perturbing the function at almost every point, $\mathbf{p}$ any more, once $\nabla f(\mathbf{p}) \notin \Omega_0$. For these points $\sum_{n=-1}^{\infty} h_n(x, y) = \sum_{n=-1}^{n_0} h_n(x, y)$ for a suitable n_0. We show that $\lambda_2(G_n) \to 0$. The main difficulty is to show that f is differentiable at those points $\mathbf{p} \in \cap_{n=0}^{\infty} G_n$ which are subject to infinitely many perturbations.

At step n, $(n = 0, 1, ...)$ we are choosing some disjoint open squares $B_{n,k}$, called *perturbation blocks*. The sides of these squares are not necessarily parallel to the coordinate axes and the choice of the proper direction of these squares is very delicate.

Here is some notation, the open square $B_{n,k}$ is centered at $\mathbf{o}_{B_{n,k}}$. Its sides are parallel to the perpendicular unit vectors $\mathbf{v}_{B_{n,k}}$ and $\mathbf{w}_{B_{n,k}}$. The vector $\mathbf{v}_{B_{n,k}}$ is the direction vector of the block, on the folded paper picture this corresponds to the direction of the folding edges. The angle between this vector and $(0, 1)$ will be in $[-\pi/4, \pi/4]$. The vector $\mathbf{w}_{B_{n,k}}$ is chosen so that its first component is positive and on Figure 4.6 this is parallel to the gradient vectors. We will have

$$B_{n,k} \stackrel{\text{def}}{=} \{\mathbf{o}_{B_{n,k}} + \alpha \mathbf{v}_{B_{n,k}} + \beta \mathbf{w}_{B_{n,k}} \text{ for } |\alpha|, |\beta| < l_{B_{n,k}}\}.$$

For each perturbation block we choose a *perturbation function* $\phi_{B_{n,k}}$ such that this function is continuously differentiable and zero outside $B_{n,k}$, this is a sanded and tamed version of the folded paper but one should essentially think of it as the folded paper surface with smooth/sanded folding edges from Figure 4.6. To be more precise there will be some transient regions close to the boundary of $B_{n,k}$, but the series of the total measures of these transient regions converges and by the Borel–Cantelli lemma λ_2 almost every point is not in a transient region. For a point $\mathbf{p} \in G$ at the nth step there exists at most one perturbation block, $B_n(\mathbf{p})$, containing $\mathbf{p}$. If there is no such block we set $B_n(\mathbf{p}) = \emptyset$ and $\phi_{B_n(\mathbf{p})} = 0$.

Now set $h_n(\mathbf{p}) \stackrel{\text{def}}{=} \phi_{B_n(\mathbf{p})}(\mathbf{p})$ and in [17] we showed that

$$\nabla f(\mathbf{p}) = (0, 1) + \sum_{n=0}^{\infty} \nabla \phi_{B_n(\mathbf{p})}(\mathbf{p}) = \sum_{n=-1}^{\infty} \nabla h_n(\mathbf{p}).$$

It is clear that

$$\nabla f_n(\mathbf{p}) = (0, 1) + \sum_{k=0}^{n} \nabla \phi_{B_k(\mathbf{p})}(\mathbf{p}).$$

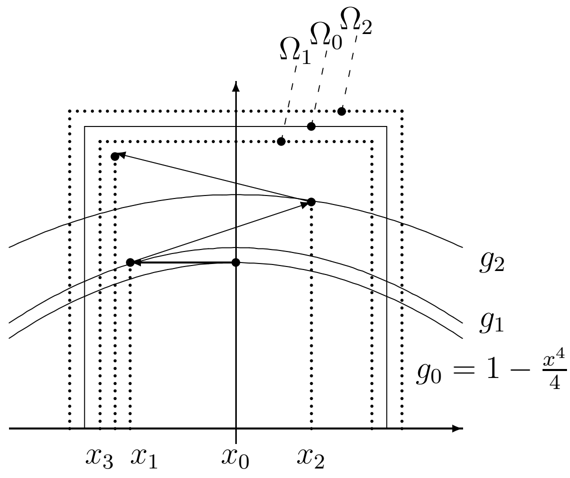


Figure 4.7: trajectory in the gradient space

Next we outline how the perturbation blocks at level n are defined. We proceed by induction.

We choose a sufficiently small $c_0 = 0.004$. We use four perturbation blocks at level 0, these are: $B_{0,1} = (0,1) \times (0,1)$, $B_{0,2} = (-1,0) \times (0,1)$, $B_{0,3} = (-1,0) \times (-1,0)$, and $B_{0,4} = (0,1) \times (-1,0)$. We have $\mathbf{v}_{B_{0,k}} = (0,1)$ and $\mathbf{w}_{B_{0,k}} = (1,0)$ for $k = 1, ..., 4$. We choose the functions $\phi_{B_{0,k}}$, $k = 1, ..., 4$.

Setting $G_0 \approx \cup_k B_{0,k}$ we define an open set and $G \setminus G_0$ is of measure zero. (Due to some technical details G_0 is a little smaller than $\cup_k B_{0,k}$, but for this heuristic argument one can think of it as being equal to it.) For $\mathbf{p} \in G_0$ set $x_0(\mathbf{p}) = 0 = \pi_x(\nabla_{rc,0} f(\mathbf{p})) \approx \pi_x(\nabla f_{-1}(\mathbf{p})) = \pi_x(\nabla h_{-1}(\mathbf{p}))$, where the projection onto the x axis is denoted by π_x, again to avoid technical difficulties one should think of x_0 as the first coordinate of $\nabla h_{-1}(\mathbf{p})$. We also define an auxiliary function $g_{0,\mathbf{p}}(x) = 1 - \frac{x^4}{4}$. These conacve down auxiliary functions can help to determine the direction vectors of the perturbation blocks at the next level.

Assume $n \geq 0$, the constant c_n and the open set G_n are given. Suppose furthermore that for all $\mathbf{p} \in G_n$ the trajectory $\{x_0(\mathbf{p}), ..., x_n(\mathbf{p})\} \subset [-1, 1]$, the perturbation blocks $B_0(\mathbf{p}), ..., B_n(\mathbf{p})$ and the concave down functions $g_{n,\mathbf{p}}$ are defined. Moreover, $|g'_{n,\mathbf{p}}(x)| \le 1$ for all $x \in [-1, 1]$ and ∇f_n is continuous on G_n.

Set $G_n^* = \{\mathbf{p} \in G_n : \nabla f_n(\mathbf{p}) \in \Omega_0\}$. For $\mathbf{p} \in G_n^*$ we define $\mathbf{v}_{\mathbf{p},n+1}$ as the "upward" normal vector of $g_{n,\mathbf{p}}$ at the point $(x_n(\mathbf{p}), g_{n,\mathbf{p}}(x_n(\mathbf{p})))$. We also put $x_{n+1}^*(\mathbf{p}) = \pi_x(\nabla_{mrc,n+1} f(\mathbf{p})) \approx \pi_x(\nabla f_n(\mathbf{p}))$, again to simplify technical difficulties, due to some error terms, one can use this second approximate expression in this heuristic outline, instead of the exact value, $\pi_x(\nabla_{mrc,n+1} f(\mathbf{p}))$ given in [17].

We use Vitali's covering theorem to select the centers of the perturbation blocks at level $n + 1$. In order to continue the perturbation of the gradient of f_n at λ_2 almost every point of G_n^* we cover almost every point of G_n^* by the perturbation blocks $B_{n+1,k}$, $k = 1, 2, ...$ These perturbation blocks are chosen so that if $\mathbf{p}$ is the center of such a block then $\mathbf{v}_{\mathbf{p},n+1}$ is the direction of this block. The values $x_{n+1}^*(\mathbf{p})$ will determine $x_{n+1}(\mathbf{p})$.

Finally, a few words about the functions g_n which play a crucial role in the determination of the direction of the perturbation blocks. The definition of these functions is very delicate. There were two sources of inspiration to define them. The first source was Theorem 4.1.3, it gave the idea of working with concave down functions. The second source was coming from one-dimensional dynamical systems and I learned it while I worked on the proof of Theorem 2.2.2. I was also asked why I used $1 - \frac{x^4}{4}$, instead of the much more natural second order $1 - \frac{x^2}{2}$. The answer is that one needs a varying second derivative, the argument is not working if the second derivative is constant.

Assume $\mathbf{p} \in G$ is fixed and it is subject to infinitely many perturbations. One can think of $(x_{n,\mathbf{p}}, y_{n,\mathbf{p}}) = (x_n, y_n)$, $n = 0, 1, ...,$ as the approximate value $\nabla f_{n-1}(\mathbf{p})$, again we ignore some small error terms. In order to verify that f is differentiable at $\mathbf{p}$ one has to show that (x_n, y_n) converges. (In the language of dynamical systems we need to show that if the "trajectory in the gradient space", (x_n, y_n), is not escaping from Ω_0 then it converges.) For ease of notation we write g_n, instead of $g_{n,\mathbf{p}}$. See Figure 4.7.

The functions g_n, for all n are concave (down) on $\mathbb{R}$ and they are all Lipschitz 1 functions on $[-1, 1]$.

They satisfy the following lemmas:

Lemma 4.2.2. *For all $x \in \mathbb{R}$ and $n = 0, 1, ...$ we have $g_{n+1}(x) \geq g_n(x)$.*

Lemma 4.2.3. *If $x_n \in [-1, 1]$ and $x_n \to x^*$ then there exists $y^* \in \mathbb{R} \cup \{+\infty\}$ such that $y_n \to y^*$.*

Lemma 4.2.4. *If $x_n \in [-1, 1]$, $\liminf x_n = x_* < \limsup x_n = x^*$ and $c = (x_* + x^*)/2$ then $g_n(c) \to \infty$. This, by the uniform Lipschitz property of g_n, implies $y_n \to \infty$ as well.*

These lemmas imply convergence of (x_n, y_n) and differentiability of the function constructed. Lemma 4.2.2 says that the higher n the higher the graph of g_n. This and the non-uniform concavity of the g_n in Lemma 4.2.3 show that convergence of the first coordinates implies that second coordinates have a finite or infinite limit. Finally Lemma 4.2.4 can be used to show that if (x_n, y_n) is a bounded sequence then the first coordinates converge and Lemma 4.2.3 is applicable.

Bibliography

[1] J. M. ASH, J. COHEN, C. FREILING AND D. RINNE, Generalizations of the wave equation, *Trans. Amer. Math.* Soc. **338**, (1993) 57-75.

[2] L. BLOCK, W.A. COPPEL, *Dynamics on one dimension,* Lecture Notes in Math. **1513**, Springer, New York (1992).

[3] K. BOGEL, Über die mehrdimensionale Differentiation, *Jber. Deutsch. Math.-Verein.* **65** (1962), 45-71.

[4] K. BRUCKS, M. MISIUREWICZ, Trajectory of the turning point is dense for almost all tent maps, *Ergodic Theory Dynam. Systems* **16** (1996), no. 6, 1173–1183.

[5] K. M. BRUCKS AND Z. BUCZOLICH, Trajectory of the turning point is dense for a co-σ-porous set of tent maps, *Fund. Math.* **165** (2000) no. 2, 95-123. **Supplement 4.**

[6] A. BRUCKNER, The problem of characterizing derivatives revisited, *Real. Anal. Exch.* **21** (1995/96), no. 1, 112-133.

[7] Z. BUCZOLICH, A general Riemann complete integral in the plane, *Acta Math. Hung.* **57** (1991), 315-323. **Supplement 1.**

[8] Z. BUCZOLICH, Density points and bi-Lipschitz functions in $\mathbf{R}^m$, *Proc. Amer. Math. Soc.* **116** (1992), 53-59. **Supplement 2.**

[9] Z. Buczolich, Ergodic averages and free $\mathbb{Z}^2$ actions, *Fund. Math.*, **160**, (1999), 247-254. **Supplement 5.**

[10] Z. Buczolich, Arithmetic averages of rotations of measurable functions, *Ergod. Th. and Dynam. Sys.* **16** (1996), 1185-1196. **Supplement 3.**

[11] Z. Buczolich, Generalized integrals and related topics, *Real Anal. Exchange* Vol. 23 No. 1, 1997-98, 27-36.

[12] Z. Buczolich, The n-dimensional gradient has the 1-dimensional Denjoy-Clarkson property, *Real Anal. Exch.*, **18** (1992/93), no. 1, 221-224. **Supplement 9.**

[13] Z. Buczolich, Level sets of functions $f(x, y)$ with nonvanishing gradient, *J. Math. Anal. Appl.*, **185** (1994), no. 1, 27-35. **Supplement 12.**

[14] Z. Buczolich, Approximate continuity points of derivatives of functions of several variables, *Acta Math. Hungar.* **67** (1995) no. 3, 229-235. **Supplement 13.**

[15] Z. Buczolich, Another note on the gradient problem of C. E. Weil, *Real Anal. Exch.*, **22** (1996/97), no. 2, 775-784. **Supplement 10.**

[16] Z. Buczolich, Functions of two variables with large tangent plane sets, *J. Math. Anal. Appl.*, **220** (1998), no. 2, 562-570. **Supplement 11.**

[17] Z. Buczolich, Solution to the gradient problem of C. E. Weil, *Rev. Mat. Iberoamericana,* **21** (2005) No. 3., 889-910. **Supplement 14.**

[18] Buczolich Z., A Gradiensprobléma, Matematikai Lapok, (In Hungarian. Talk given at the Big 5 conference) Új sorozat 11. évf. (2002-03) **1**, (2006).

[19] Z. Buczolich, On the gradient problem of C. E. Weil, *Real Anal. Exchange Summer Symposium report 2003,* 161-178.

[20] Z. Buczolich, Birkhoff sums and zero-one law, *Real Anal. Exchange Summer Symposium report 2000,* 13-22.

[21] Z. Buczolich, J-P. Kahane and R.D. Mauldin, On series of translates of positive functions, *Acta Math. Hungar.,* **93(3)** (2001), 171-188. **Supplement 7.**

[22] Z. Buczolich, J-P. Kahane and R.D. Mauldin, Sur les séries de fonctions positives, *C. R. Acad. Sci. Paris Sér. I. Math.* **329** (1999), no. 4, 261-264.

[23] Z. Buczolich and D. Mauldin, On the convergence of $\sum_{n=1}^{\infty} f(nx)$ for measurable functions, *Mathematika,* **131** (2001), no. 4, 785-798. **Supplement 6.**

[24] Z. Buczolich and D. Mauldin, On series of translates of positive functions II., *Indag. Mathem.,* N. S., **12** (3), (2001), 317-327. **Supplement 8.**

[25] J.W.S. Cassels, *An Introduction to Diophantine Approximation,* Cambridge University Press, 1957.

[26] F. Cater, A global implicit function theorem, *Real. Anal. Exchange* **16**, No. 1, (1990/91), 268-273.

[27] Choquet, G., Construction effective de suites $(k(3/2)^n)$. Études des mesures (3/2)-stables, *C. R. Acad Sci. Paris Sér. A-B* **291** (1980), no. 2, A69-A74.

[28] J. A. Clarkson, A property of derivatives, *Bull. Amer. Math. Soc.* **53** (1947) 124-125.

[29] A. Denjoy, Sur une proprieté des fonctions dérivées, *Enseignement. Math.* **18** (1916) 320-328.

[30] M. Elekes, Level sets of differentiable functions of two variables with non-vanishing gradient, *J. Math. Anal. Appl.*, **270** (2002) 369-382.

[31] P. Erdős, On the strong law of large numbers, *Trans. Amer. Math. Soc.,* **67** (1949) 51-56.

[32] H. Federer, *Geometric measure theory*, Die Grundlehren der matematischen Wissenschaften, Band 153, Springer-Verlag New York Inc., New York (1969).

[33] H. Fejzić, C. Freiling and D. Rinne, Two-dimensional partitions, *Real Analysis Exch.* **19** (1993-94), 540- 546.

[34] N.J. Fine and A.R. Hyde, Solution of a problem proposed by K.L. Chung, *Amer. Math. Monthly* **64** (1957), 119-120.

[35] J.A. Haight, A linear set of infinite measure with no two points having integral ratio, *Mathematika* **17** (1970), 133-138.

[36] J.A. Haight, A set of infinite measure whose ratio set does not contain a given sequence, *Mathematika* **22** (1975), 195-201.

[37] M. W. Hirsch, *Differential Topology*, Graduate Texts in Mathematics, **33**, Springer-Verlag, Berlin/New York, 1976.

[38] P. Holický, J. Malý, L. Zajíček, C. E. Weil, A note on the gradient problem, *Real Anal. Exch.*, **22** (1996/97), no. 1, 225-235.

[39] W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton University Press, 1948.

[40] A. Khinchin, Ein Satz über Kettenbruche mit arithmetischen Anwendungen, *Math. Zeit.,* **18** (1923) 289-306.

[41] T. W. Körner, A dense arcwise connected set of critical points–Molehills out of mountains, *J. London Math. Soc.* (2) **38** (1988), 442-452.

[42] C. G. Lekkerkerker, Lattice points in unbounded point sets, I. *Indag. Math.,* **20** (1958) 197-205.

[43] P. Major, A counterexample in ergodic theory, *Acta Sci. Math.* (Szeged) **62** (1996), 247-258.

[44] J. Malý and M. Zelený, A note on Buczolich's solution of the Weil gradient problem: a construction based on an infinite game, to appear in *Acta Mathematica Hungarica.*

[45] J. M. Marstrand, On Khinchin's conjecture about strong uniform distribution, *Proc. London Math. Soc.* **3** 21 (1970) 540-56.

[46] S. Mazur and W. Orlicz, Sur quelques propriétés de fonctions périodiques et presque-périodiques, *Studia Math.* **9** (1940), 1-16.

[47] A. P. Morse, The behavior of a function on its critical set, *Ann. of Math.* (2) **40** (1939), 62-70.

[48] D. O. Ornstein and B. Weiss, Ergodic Theory of Amenable Group Actions. I: The Rohlin Lemma, *Bulletin of the Amer. Math. Soc.* **2** No. 1, (1980), 161-164.

[49] G. Petruska, Derivatives take every value on the set of approximate continuity points, *Acta Math. Hungar.* **42** (1983) no. 3-4, 355-360.

[50] W. F. Pfeffer, The divergence theorem, *Trans. Amer. Math. Soc.* **295** (1986) 665-685.

[51] D. Preiss, L. Zajíček, Fréchet differentiation of convex functions in a Banach space with a separable dual, *Proc. Amer. Math. Soc.* **91**, 202-204 (1984).

[52] M. Radulescu and S. Radulescu, Local inversion theorems without assuming continuous differentiability, *J. Math. Anal. Appl.* **138** (1989), 581-590.

[53] D. A. Raikov, On some arithmetical properties of summable functions, *Mat. Sb.* **1** (43) (1936) 377-84.

[54] D.L. Renfro, *On some various porosity notions,* Preprint, (1995).

[55] F. Riesz, Sur la théorie ergodique, *Comment. Math. Helv.* **17** (1945) 221-39.

[56] W. Rudin, *Functional Analysis*, McGraw-Hill Series in Higher Mathematics, *McGraw-Hill Book Co.,* (1973).

[57] A. Sard, The measure of the critical values of differentiable maps, *Bull. Amer. Math. Soc.* **48** (1942), 883-890.

[58] S. Saks, *Theory of the integral*, Dover, New York, 1964.

[59] S. van Strien, *Smooth dynamics on the interval,* New Directions in Dynamical Systems, London Math. Soc. Lecture Note Ser. **127**, Cambridge Univ. Press, Cambridge - New York, 57-119 (1988).

[60] R. Svetic, A function with locally uncountable rotation set, *Acta. Math. Hungar.* **81 (4)**, (1998), 305-314.

[61] B.S. Thomson, *Real Functions,* Lecture Notes in Math. **1170**, Springer, New York (1985).

[62] J. Väisälä, Porous sets and quasisymmetric maps, *Trans. Amer. Math. Soc.* **299** (1987), 525- 533.

[63] C. E. Weil, On properties of derivatives, *Trans. Amer. Math. Soc.* **114** (1965), 363-376.

[64] H. v. Weizsäcker, Zum Konvergenzverhalten der Reihe $\Sigma_{n=1}^{\infty} f(nt)$ für λ-messbare Funktionen $f : \mathbb{R}^+ \to \mathbb{R}^+$, *Diplomarbeit,* Universität München, 1970.

[65] H. Weyl, Über die Gleichverteilung von Zahlen modulo Eins, *Math. Ann.* **77** (1916) 313-52.

[66] H. Whitney, A function not constant on a connected set of critical points, *Duke Math. J.*, **1**, (1935), 514-517.

[67] J. Woś, The filling scheme and the ergodic theorems of Kesten and Tanny, *Colloquium Mathematicum* **52** (1987), 263-276.

[68] J. Woś, A remark on the existence of the ergodic Hilbert transform, *Colloquium Mathematicum* **53** (1987), 97- 101.

[69] L. Zajíček, Porosity and σ-porosity, *Real Anal. Ex.* **13**, 314-347 (1987-88).

[70] M. Zelený, The Denjoy-Clarkson property with respect to Hausdorff measures for the gradient mapping of functions of several variables, (preprint).

[71] Query 1, *Real Anal. Exch.*, **16** (1990/91), no. 1, 373.

Supplements