A NEW FRACTAL DIMENSION: THE TOPOLOGICAL HAUSDORFF DIMENSION

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Abstract. We introduce a new concept of dimension for metric spaces, the so called *topological Hausdorff dimension*. It is defined by a very natural combination of the definitions of the topological dimension and the Hausdorff dimension. The value of the topological Hausdorff dimension is always between the topological dimension and the Hausdorff dimension, in particular, this new dimension is a non-trivial lower estimate for the Hausdorff dimension.

We examine the basic properties of this new notion of dimension, compare it to other well-known notions, determine its value for some classical fractals such as the Sierpiński carpet, the von Koch snowflake curve, Kakeya sets, the trail of the Brownian motion, etc.

As our first application, we generalize the celebrated result of Chayes, Chayes and Durrett about the phase transition of the connectedness of the limit set of Mandelbrot's fractal percolation process. They proved that certain curves show up in the limit set when passing a critical probability, and we prove that actually 'thick' families of curves show up, where roughly speaking the word thick means that the curves can be parametrized in a natural way by a set of large Hausdorff dimension. The proof of this is basically a lower estimate of the topological Hausdorff dimension of the limit set. For the sake of completeness, we also give an upper estimate and conclude that in the non-trivial cases the topological Hausdorff dimension is almost surely strictly below the Hausdorff dimension.

Finally, as our second application, we show that the topological Hausdorff dimension is precisely the right notion to describe the Hausdorff dimension of the level sets of the generic continuous function (in the sense of Baire category) defined on a compact metric space.

1. Introduction

The term 'fractal' was introduced by Mandelbrot in his celebrated book [13]. He formally defined a subset of a Euclidean space to be a fractal if its topological dimension is strictly smaller than its Hausdorff dimension. This is just one example to illustrate the fundamental role these two notions of dimension play in the study of fractal sets. To mention another such example, let us recall that the topological dimension of a metric space X is the infimum of the Hausdorff dimensions of the metric spaces homeomorphic to X, see [8].

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The main goal of this paper is to introduce a new concept of dimension, the so called *topological Hausdorff dimension*, that interpolates the two above mentioned dimensions in a very natural way. Let us recall the definition of the (small inductive) topological dimension.

Definition 1.1. Set $\dim_t \emptyset = -1$. The *topological dimension* of a non-empty metric space X is defined by induction as

$$\dim_t X = \inf\{d : X \text{ has a basis } \mathcal{U} \text{ such that } \dim_t \partial U \leq d-1 \text{ for every } U \in \mathcal{U}\}.$$

Our new dimension will be defined analogously, however, note that this second definition will not be inductive, and also that it can attain non-integer values as well. The Hausdorff dimension of a metric space X is denoted by $\dim_H X$, see e.g. [4] or [14]. In this paper we adopt the convention that $\dim_H \emptyset = -1$.

Definition 1.2. Set $\dim_{tH} \emptyset = -1$. The *topological Hausdorff dimension* of a non-empty metric space X is defined as

$$\dim_{tH} X = \inf\{d : X \text{ has a basis } \mathcal{U} \text{ such that } \dim_H \partial U \leq d-1 \text{ for every } U \in \mathcal{U}\}.$$

(Both notions of dimension can attain the value ∞ as well.)

It was not this analogy that initiated the study of this new concept. Our original motivation was that this notion grew out naturally from our investigations of the following topic. B. Kirchheim proved in [10] that for the generic continuous function (in the sense of Baire category) defined on $[0,1]^d$, for every $y \in \operatorname{int} f([0,1]^d)$ we have $\dim_H f^{-1}(y) = d-1$, that is, as one would expect, 'most' level sets are of Hausdorff dimension $d-1$. The next problem is about generalizations of this result to fractal sets in place of $[0,1]^d$.

Problem 1.3. Describe the Hausdorff dimension of the level sets of the generic continuous function (in the sense of Baire category) defined on a compact metric space.

It has turned out that the topological Hausdorff dimension is the right concept to deal with this problem. We will essentially prove that the value $d-1$ in Kirchheim's result has to be replaced by $\dim_{tH} K - 1$, see the end of this introduction or Section 6 for the details.

We would also like to mention another potentially very interesting motivation of this new concept. Unlike most well-known notions of dimension, such as packing or box-counting dimensions, the topological Hausdorff dimension is smaller than the Hausdorff dimension. As it is often an important and difficult task to estimate the Hausdorff dimension from below, this gives another reason why to study the topological Hausdorff dimension.

It is also worth mentioning that there is another recent approach by M. Urbański [16] to combine the topological dimension and the Hausdorff dimension. However, his new concept, called the transfinite Hausdorff dimension is quite different in nature from ours, e.g. it takes ordinal numbers as values.

Next we say a few words about the main results and the organization of the paper.

In Section 3 we investigate the basic properties of the topological Hausdorff dimension. Among others, we prove the following.

Theorem 3.4. $\dim_t(X) \leq \dim_{tH}(X) \leq \dim_H(X)$.

We also verify that $\dim_{tH} X$ satisfies some standard properties of a dimension, such as monotonicity, bi-Lipschitz invariance and countable stability for closed sets. Moreover, we check that this concept is genuinely new, since we show that $\dim_{tH} X$ cannot be expressed as a function of $\dim_t X$ and $\dim_H X$.

In Section 4 we compute $\dim_{tH} X$ for some classical fractals, like the Sierpiński triangle and carpet, the von Koch curve, etc. For example

Theorem 4.4. Let T be the Sierpiński carpet. Then $\dim_{tH}(T) = \frac{\log 6}{\log 3} = \frac{\log 2}{\log 3} + 1$.

(Note that $\dim_t T = 1$ and $\dim_H T = \frac{\log 8}{\log 3}$ while the Hausdorff dimension of the triadic Cantor set equals $\frac{\log 2}{\log 3}$.)

We also consider Kakeya sets (see [4] or [14]). Unfortunately, our methods do not give any useful information concerning the Kakeya Conjecture.

Theorem 4.6. For every $d \in \mathbb{N}^+$ there exist a compact Kakeya set of topological Hausdorff dimension 1 in $\mathbb{R}^d$.

Following [11] by T. W. Körner we prove somewhat more, since we essentially show that the generic element of a carefully chosen space is a Kakeya set of topological Hausdorff dimension 1.

We show that the trail of the Brownian motion almost surely (i.e. with probability 1) has topological Hausdorff dimension 1 in every dimension except perhaps 2 and 3. These two cases remain the most intriguing open problems of the paper.

Problem 4.8. Determine the almost sure topological Hausdorff dimension of the trail of the d-dimensional Brownian motion for $d = 2$ or 3.

As our first application in Section 5 we generalize a result of Chayes, Chayes and Durrett about the phase transition of the connectedness of the limit set of Mandelbrot's fractal percolation process. This limit set $M = M^{(p,n)}$ is a random Cantor set, which is constructed by dividing the unit square into $n \times n$ equal subsquares and keeping each of them independently with probability p, and then repeating the same procedure recursively for every subsquare. (See Section 5 for more details.)

Theorem 5.1 (Chayes-Chayes-Durrett, [2])**.** There exists a critical probability $p_c = p_c^{(n)} \in (0,1)$ such that if $p < p_c$ then M is totally disconnected almost surely, and if $p > p_c$ then M contains a nontrivial connected component with positive probability.

It will be easy to see that this theorem is a special case of our next result.

Theorem 5.2. For every $d \in [0,2)$ there exists a critical probability $p_c^{(d)} = p_c^{(d,n)} \in (0,1)$ such that if $p < p_c^{(d)}$ then $\dim_{tH} M \leq d$ almost surely, and if $p > p_c^{(d)}$ then $\dim_{tH} M > d$ almost surely (provided $M \neq \emptyset$).

Theorem 5.1 essentially says that certain curves show up at the critical probability, and our proof will show that even 'thick' families of curves show up, which roughly speaking means a 'Lipschitz copy' of $C \times [0,1]$ with $\dim_H C > d-1$.

We also give a numerical upper bound for $\dim_{tH} M$ which implies the following.

Corollary 5.18. Almost surely

$$\dim_{tH} M < \dim_H M \text{ or } M = \emptyset.$$

In Section 6 we answer Problem 1.3 as follows.

Corollary 6.21. If K is a compact metric space with $\dim_t K > 0$ then $\sup\{\dim_H f^{-1}(y) : y \in \mathbb{R}\} = \dim_{tH} K - 1$ for the generic $f \in C(K)$.

(If $\dim_t K = 0$ then the generic $f \in C(K)$ is one-to-one, thus every non-empty level set is of Hausdorff dimension 0.)

If K is also sufficiently homogeneous, e.g. self-similar then we can actually say more.

Corollary 6.23. If K is a self-similar compact metric space with $\dim_t K > 0$ then $\dim_H f^{-1}(y) = \dim_{tH} K - 1$ for the generic $f \in C(K)$ and the generic $y \in f(K)$.

In the course of the proofs, as a spin-off, we also provide a sequence of equivalent definitions of $\dim_{tH} K$ for compact metric spaces. Perhaps the most interesting one is the following.

Corollary 6.13. If K is a compact metric space then $\dim_{tH} K$ is the smallest number d for which K can be covered by a finite family of compact sets of arbitrarily small diameter such that the set of points that are covered more than once has Hausdorff dimension $d-1$.

It can actually also be shown that in the equation $\sup\{\dim_H f^{-1}(y) : y \in \mathbb{R}\} = \dim_{tH} K - 1$ (for the generic $f \in C(K)$) the supremum is attained. On the other hand, one cannot say more in a sense, since there is a K such that for the generic $f \in C(K)$ there is a *unique* $y \in \mathbb{R}$ for which $\dim_H f^{-1}(y) = \dim_{tH} K - 1$. Moreover, in certain situations we can replace 'the generic $y \in f(K)$' with 'for every $y \in \operatorname{int} f(K)$' as in Kirchheim's theorem. The results of this last paragraph are to appear elsewhere, see [1].

Finally, in Section 7 we list some open problems.

2. Preliminaries

Let (X,d) be a metric space. We denote by $\operatorname{cl} H$, $\operatorname{int} H$ and ∂H the closure, interior and boundary of a set H. For $x \in X$ and $H \subseteq X$ set $d(x,H) = \inf\{d(x,h) : h \in H\}$. Let $B(x,r)$ and $U(x,r)$ stand for the closed and open ball of radius r centered at x, respectively. More generally, for a set $H \subseteq X$ we define $B(H,r) = \{x \in X : d(x,H) \leq r\}$ and $U(H,r) = \{x \in X : d(x,H) < r\}$. The diameter of a set H is denoted by $\operatorname{diam} H$. We use the convention $\operatorname{diam} \emptyset = 0$. For two metric spaces (X, d_X) and (Y, d_Y) a function $f \colon X \to Y$ is *Lipschitz* if there exists a constant $C \in \mathbb{R}$ such that $d_Y(f(x_1), f(x_2)) \leq C \cdot d_X(x_1, x_2)$ for all $x_1, x_2 \in X$. The smallest such constant C is called the Lipschitz constant of f and denoted by $\operatorname{Lip}(f)$. A function $f\colon X \to Y$ is called *bi-Lipschitz* if f is a bijection and both f and f^{-1} are Lipschitz. Let X be a metric space, $s \geq 0$ and $\delta > 0$, then

$$\mathcal{H}^s_\infty(X) = \inf\left\{\sum_{i=1}^\infty (\operatorname{diam} U_i)^s : X \subseteq \bigcup_i U_i\right\},$$

$$\mathcal{H}^s_\delta(X) = \inf\left\{\sum_{i=1}^\infty (\operatorname{diam} U_i)^s : X \subseteq \bigcup_i U_i, \forall i \operatorname{diam} U_i \leq \delta\right\},$$

$$\mathcal{H}^s(X) = \lim_{\delta \to 0+} \mathcal{H}^s_\delta(X).$$

The *Hausdorff dimension of* X is defined as

$$\dim_H X = \inf\{s \geq 0 : \mathcal{H}^s(X) = 0\}.$$

For more information on these concepts see [4] or [14].

Let X be a *complete* metric space. A set is *somewhere dense* if it is dense in a non-empty open set, and otherwise it is called *nowhere dense*. We say that $M \subseteq X$ is *meager* if it is a countable union of nowhere dense sets, and a set is called *co-meager* if its complement is meager. By Baire's Category Theorem co-meager sets are dense. It is not difficult to show that a set is co-meager iff it contains a dense G_δ set. We say that the generic element $x \in X$ has property $\mathcal{P}$, if $\{x \in X : x \text{ has property } \mathcal{P}\}$ is co-meager. The term 'typical' is also used instead of 'generic'. Our two main examples will be $X = C(K)$ endowed with the supremum metric (for some compact metric space K) and $X = \mathcal{K}$, that is, a certain subspace of the non-empty compact subsets of $\mathbb{R}^d$ endowed with the Hausdorff metric (i.e. $d_H(K_1, K_2) = \min\{r : K_1 \subseteq B(K_2, r) \text{ and } K_2 \subseteq B(K_1, r)\}$). See e.g. [9] for more on these concepts.

3. Basic properties of the topological Hausdorff dimension

Let X be a metric space. Since $\dim_t X = -1 \iff X = \emptyset \iff \dim_H X = -1$, we easily obtain

Fact 3.1. $\dim_{tH} X = 0 \iff \dim_t X = 0$.

As $\dim_H X$ is either -1 or at least 0, we obtain

Fact 3.2. *The topological Hausdorff dimension of a non-empty space is either* 0 *or at least* 1.

These two facts easily yield

Corollary 3.3. *Every metric space with a non-trivial connected component has topological Hausdorff dimension at least one.*

The next theorem states that the topological Hausdorff dimension is between the topological and the Hausdorff dimension.

Theorem 3.4. *For every metric space* X

$$\dim_t X \leq \dim_{tH} X \leq \dim_H X.$$

Proof. We can clearly assume that X is non-empty. It is well-known that $\dim_t X \leq \dim_H X$ (see e.g. [8]), which easily implies $\dim_t X \leq \dim_{tH} X$ using the definitions. The second inequality is obvious if $\dim_H X = \infty$. If $\dim_H X < 1$ then $\dim_t X = 0$ (since $\dim_t X \leq \dim_H X$ and $\dim_t X$ only takes integer values) and by Fact 3.1 we obtain $\dim_{tH} X = 0$, hence the second inequality holds. Therefore we may assume that $1 \leq \dim_H X < \infty$. The following lemma is basically [14, Thm. 7.7.]. It is only stated there in the special case $X = A \subseteq \mathbb{R}^n$, but the proof works verbatim for all metric spaces X.

Lemma 3.5. *Let X be a metric space and $f\colon X \to \mathbb{R}^m$ be Lipschitz. If $s > m$ then*

$$\int^{\star} \mathcal{H}^{s-m}\left(f^{-1}(y)\right) \, d\lambda_m(y) \leq c(m) \operatorname{Lip}(f)^m \mathcal{H}^s(X), \tag{3.1}$$

where $\int^{\star}$ denotes the upper Lebesgue integral, λ_m the m-dimensional Lebesgue measure and $c(m)$ is a finite constant depending only on m.

Now we return to the proof of Theorem 3.4. We fix $x_0 \in X$ and define $f\colon X \to \mathbb{R}$ by $f(x) = d_X(x, x_0)$. Using the triangle inequality it is easy to see that f is Lipschitz with $\operatorname{Lip}(f) \leq 1$. We fix $n \in \mathbb{N}^+$ and apply Lemma 3.5 for f and $s = \dim_H X + \frac{1}{n} > 1 = m$. Hence

$$\int^{\star} \mathcal{H}^{s-1}(f^{-1}(y)) \, d\lambda_1(y) \leq c(1)\mathcal{H}^s(X) = 0.$$

Thus $\mathcal{H}^{s-1}(f^{-1}(y)) = \mathcal{H}^{\dim_H X + \frac{1}{n} - 1}(f^{-1}(y)) = 0$ holds for a.e. $y \in \mathbb{R}$. Since this is true for all $n \in \mathbb{N}^+$, we obtain that $\dim_H f^{-1}(y) \leq \dim_H(X) - 1$ for a.e. $y \in \mathbb{R}$. From the definition of f it follows that $\partial U(x_0, y) \subseteq f^{-1}(y)$. Hence there is a neighborhood basis of x_0 with boundaries of Hausdorff dimension at most $\dim_H(X) - 1$, and this is true for all $x_0 \in X$, so there is a basis with boundaries of Hausdorff dimension at most $\dim_H(X) - 1$. By the definition of the topological Hausdorff dimension this implies $\dim_{tH} X \leq \dim_H X$. $\square$

There are some elementary properties one expects from a notion of dimension. Now we verify some of these for the topological Hausdorff dimension.

Extension of the classical dimension. Theorem 3.4 implies that the topological Hausdorff dimension of a countable set equals zero, moreover, for open subspaces of $\mathbb{R}^d$ and for smooth d-dimensional manifolds the topological Hausdorff dimension equals d.

Monotonicity. Let $X \subseteq Y$. If $\mathcal{U}$ is a basis in Y then $\mathcal{U}_X = \{U \cap X : U \in \mathcal{U}\}$ is a basis in X, and $\partial_X(U \cap X) \subseteq \partial_Y U$ holds for all $U \in \mathcal{U}$. This yields

Fact 3.6 (Monotonicity). *If $X \subseteq Y$ are metric spaces then $\dim_{tH} X \leq \dim_{tH} Y$.*

Bi-Lipschitz invariance. First we prove that the topological Hausdorff dimension does not increase under Lipschitz homeomorphisms. An easy consequence of this that our dimension is bi-Lipschitz invariant, and does not increase under an injective Lipschitz map on a compact space. After obtaining corollaries of Theorem 3.7 we give some examples illustrating the necessity of certain conditions in this theorem and its corollaries.

Theorem 3.7. *Let X, Y be metric spaces. If $f\colon X \to Y$ is a Lipschitz homeomorphism then $\dim_{tH} Y \leq \dim_{tH} X$.*

Proof. Since f is a homeomorphism, if $\mathcal{U}$ is a basis in X then $\mathcal{V} = \{f(U) : U \in \mathcal{U}\}$ is a basis in Y, and $\partial f(U) = f(\partial U)$ for all $U \in \mathcal{U}$. The Lipschitz property of f implies that $\dim_H \partial V = \dim_H \partial f(U) = \dim_H f(\partial U) \leq \dim_H \partial U$ for all $V = f(U) \in \mathcal{V}$. Thus $\dim_{tH} Y \leq \dim_{tH} X$. $\square$

This immediately implies the following two statements.

Corollary 3.8 (Bi-Lipschitz invariance)**.** *Let X, Y be metric spaces. If $f: X \to Y$ is bi-Lipschitz then* $\dim_{tH} X = \dim_{tH} Y$.

Corollary 3.9. *If K is a compact metric space, and $f: K \to Y$ is* one-to-one *Lipschitz then* $\dim_{tH} f(K) \leq \dim_{tH} K$.

The following example shows that we cannot drop injectivity here. First we need a well-known lemma.

Lemma 3.10. *Let $M \subseteq \mathbb{R}$ be measurable with positive Lebesgue measure. Then there exists a Lipschitz onto map $f: M \to [0,1]$.*

Proof. Let us choose a compact set $C \subseteq M$ of positive Lebesgue measure. Define $f: M \to [0,1]$ by

$$f(x) = \frac{\lambda\left((-\infty, x) \cap C\right)}{\lambda(C)},$$

where λ denotes the one-dimensional Lebesgue measure. Then it is not difficult to see that f is Lipschitz (with $\operatorname{Lip}(f) \leq \frac{1}{\lambda(C)}$) and $f(C) = [0,1]$. ☐

Example 3.11. Let $K \subseteq \mathbb{R}$ be a Cantor set (that is, a set homeomorphic to the middle-thirds Cantor set) of positive Lebesgue measure. By Fact 3.1, $\dim_{tH} K = \dim_t K = 0$. Using Lemma 3.10 there is a Lipschitz map $f: K \to [0,1]$ such that $f(K) = [0,1]$. By Theorem 3.4, $\dim_{tH}[0,1] = 1$, hence $\dim_{tH} K = 0 < 1 = \dim_{tH}[0,1] = \dim_{tH} f(K)$.

The next example shows that Corollary 3.9 does not hold without the assumption of compactness. We even have a separable metric counterexample.

Example 3.12. Let C be the middle-thirds Cantor set, and $f: C \times C \to [0,2]$ be defined by $f(x,y) = x + y$. It is well-known and easy to see that f is Lipschitz and $f(C \times C) = [0,2]$. Therefore one can select a subset $X \subseteq C \times C$ such that $f|_X$ is a bijection from X onto $[0,2]$. Then X is separable metric. Monotonicity and $\dim_t(C \times C) = 0$ imply $\dim_{tH} X \leq \dim_{tH}(C \times C) = 0$. Therefore, f is one-to-one and Lipschitz on X but $\dim_{tH} X = 0 < 1 = \dim_{tH}[0,2] = \dim_{tH} f(X)$.

Our last example shows that the topological Hausdorff dimension is not invariant under homeomorphisms. Not even for compact metric spaces.

Example 3.13. Let $C_1, C_2 \subseteq \mathbb{R}$ be Cantor sets such that $\dim_H C_1 \neq \dim_H C_2$. We will see in Theorem 3.19 that $\dim_{tH}(C_i \times [0,1]) = \dim_H C_i + 1$ for $i = 1, 2$. Hence $C_1 \times [0,1]$ and $C_2 \times [0,1]$ are homeomorphic compact metric spaces whose topological Hausdorff dimensions disagree.

Stability and countable stability. As the following example shows, similarly to the case of topological dimension, stability does not hold for non-closed sets. That is, $X = \bigcup_{n=1}^{k} X_n$ does not imply $\dim_{tH} X = \max_{1 \leq n \leq k} \dim_{tH} X_n$.

Example 3.14. Theorem 3.4 implies $\dim_{tH}(\mathbb{R}) = 1$, and Fact 3.1 yields $\dim_{tH}(\mathbb{Q}) = \dim_t(\mathbb{Q}) = 0$ and $\dim_{tH}(\mathbb{R} \setminus \mathbb{Q}) = \dim_t(\mathbb{R} \setminus \mathbb{Q}) = 0$. Thus $\dim_{tH} \mathbb{R} = 1 > 0 = \max\{\dim_{tH}(\mathbb{Q}), \dim_{tH}(\mathbb{R} \setminus \mathbb{Q})\}$, and therefore stability fails.

As a corollary, we now show that as opposed to the case of Hausdorff (and packing) dimension, there is no reasonable family of measures inducing the topological Hausdorff dimension. Let us say that a 1-parameter family of measures $\{\mu^s\}_{s\geq 0}$ is *monotone* if $\mu^s(A) = 0$, $s < t$ implies $\mu^t(A) = 0$. The family of Hausdorff (or packing) measures certainly satisfies this criterion. It is not difficult to see that monotonicity implies that the induced notion of dimension, that is, $\dim A = \inf\{s : \mu^s(A) = 0\}$ is countably stable. Hence we obtain

Corollary 3.15. *There is no monotone 1-parameter family of measures* $\{\mu^s\}_{s\geq 0}$ *such that* $\dim_{tH} A = \inf\{s : \mu^s(A) = 0\}$.

However, just like in the case of topological dimension, even countable stability holds for *closed* sets.

Theorem 3.16 (Countable stability for closed sets)**.** *Let X be a separable metric space and $X = \bigcup_{n\in\mathbb{N}} X_n$, where X_n ($n \in \mathbb{N}$) are closed subsets of X. Then* $\dim_{tH} X = \sup_{n\in\mathbb{N}} \dim_{tH} X_n$.

Proof. Monotonicity clearly implies $\dim_{tH} X \geq \sup_{n\in\mathbb{N}} \dim_{tH} X_n$. For the other direction we may assume $\sup_{n\in\mathbb{N}} \dim_{tH} X_n < \infty$. Let $d > \sup_{n\in\mathbb{N}} \dim_{tH} X_n$ be arbitrary. Assume $\mathcal{U}_n$, $n \in \mathbb{N}$ is a countable basis of X_n such that $\dim_H \partial_{X_n} U \leq d-1$ for all $n \in \mathbb{N}$ and $U \in \mathcal{U}_n$. Let $Y = \bigcup\{\partial_{X_n} U : n \in \mathbb{N},\, U \in \mathcal{U}_n\}$. By countable stability of the Hausdorff dimension, $\dim_H Y \leq d - 1$. Using the definition of the topological dimension we obtain $\dim_t(X_n \setminus Y) = 0$ for all $n \in \mathbb{N}$. The set $X_n \setminus Y$ is closed in the separable metric space $X \setminus Y$, and $X \setminus Y = \bigcup_{n\in\mathbb{N}}(X_n \setminus Y)$. By the sum theorem for topological dimension 0, see [3, 1.3.1], $\dim_t(X \setminus Y) = 0$.

Let us fix an open set $V \subseteq X$ and a point $x \in V$. Using that $X \setminus Y$ is a separable subspace of X with topological dimension 0, by the separation theorem for topological dimension zero [3, 1.2.11.] there is a so-called partition between x and $X \setminus V$ disjoint from $X \setminus Y$. This means that there exist disjoint open sets $U, U' \subseteq X$ such that $x \in U$, $X \setminus V \subseteq U'$ and $(X \setminus (U \cup U')) \cap (X \setminus Y) = \emptyset$. In particular, $x \in U \subseteq V$. Moreover, $\partial_X U \cap (X \setminus Y) = \emptyset$, so $\partial_X U \subseteq Y$, thus $\dim_H \partial_X U \leq \dim_H Y \leq d-1$. By the definition of topological Hausdorff dimension we obtain $\dim_{tH} X \leq d$. As $d > \sup_{n\in\mathbb{N}} \dim_{tH} X_n$ was arbitrary, the proof is complete. □

Corollary 3.17. *The same holds for F_σ sets, as well.*

Products. Now we investigate products from the point of view of topological Hausdorff dimension. By product of two metric spaces we will always mean the l^2-product, that is,

$$d_{X\times Y}((x_1, y_1), (x_2, y_2)) = \sqrt{d_X^2(x_1, x_2) + d_Y^2(y_1, y_2)}.$$

First we recall a well-known statement, see [4, Chapters 3 and 7] for the definitions and the proof.

Lemma 3.18. *Let X, Y be non-empty metric spaces such that $\dim_H Y = \overline{\dim}_B Y$, where $\overline{\dim}_B$ is the upper box-counting dimension. Then*

$$\dim_H(X \times Y) = \dim_H X + \dim_H Y.$$

Now we prove our next theorem which provides a large class of sets for which the topological Hausdorff dimension and the Hausdorff dimension coincide.

Theorem 3.19. *Let X be a non-empty separable metric space. Then*

$$\dim_{tH}(X \times [0,1]) = \dim_H(X \times [0,1]) = \dim_H X + 1.$$

Proof. Applying Lemma 3.18 for $Y = [0,1]$ we deduce that $\dim_H(X \times [0,1]) = \dim_H X + \dim_H[0,1] = \dim_H X + 1$. From Theorem 3.4 it follows that $\dim_{tH}(X \times [0,1]) \le \dim_H(X \times [0,1])$. For the opposite inequality we need the following lemma.

Lemma 3.20. *Let X be a non-empty separable metric space and let $d < \dim_H X$ be fixed. Then there exists $x_d \in X$ such that $\dim_H U(x_d, r) \ge d$ holds for every $r > 0$.*

Proof of the Lemma. Assume, on the contrary, that for all $x \in X$ there is an $r_x > 0$ such that $\dim_H U(x, r_x) < d$. Since X is separable, by the Lindelöf property we can select a countable subcover $\{U(x_n, r_n)\}_{n\in\mathbb{N}}$ of the cover $\{U(x, r_x)\}_{x\in X}$. By countable stability of the Hausdorff dimension $\dim_H X = \sup_{n\in\mathbb{N}} \dim_H U(x_n, r_n) \le d$, which is a contradiction. □

We now return to the proof of Theorem 3.19. For a fixed $d < \dim_H X$ assume that $x_d \in X$ is given as in the lemma. Let $\mathcal{U}$ be a basis in $X \times [0,1]$ and $\mathrm{pr}_X : X \times [0,1] \to X$, $\mathrm{pr}_X(x,y) = x$. There exists $U_d \in \mathcal{U}$ such that $(x_d, 1) \in U_d$ and $U_d \cap \left(X \times \left[0, \frac{1}{2}\right]\right) = \emptyset$. Then there is an $r_d > 0$ such that $U(x_d, r_d) \times [1-r_d, 1] \subseteq U_d$. For every $x \in U(x_d, r_d)$ we have $(x,0) \notin U_d$ and $(x,1) \in U_d$, hence $\partial U_d \cap (x \times [0,1]) \neq \emptyset$. Thus $U(x_d, r_d) \subseteq \mathrm{pr}_X(\partial U_d)$. Projections do not increase the Hausdorff dimension, therefore $\dim_H \partial U_d \ge \dim_H U(x_d, r_d) \ge d$. This is valid for all $d < \dim_H X$, so $\sup_{U\in\mathcal{U}} \dim_H \partial U \ge \dim_H X$ for all basis $\mathcal{U}$, thus $\dim_{tH}(X \times [0,1]) \ge \dim_H X + 1$ by the definition of topological Hausdorff dimension. □

Remark 3.21. We cannot drop separability here. Indeed, if X is an uncountable discrete metric space then it is not difficult to see that $\dim_{tH}(X \times [0,1]) = 1$ and $\dim_H(X \times [0,1]) = \dim_H X = \infty$.

Separability is a rather natural assumption throughout the paper. First, the Hausdorff dimension is only meaningful in this context (it is always infinite for non-separable spaces), secondly for the theory of topological dimension this is the most usual framework.

Corollary 3.22. *If X is a non-empty separable metric space then*

$$\dim_{tH}\left(X \times [0,1]^d\right) = \dim_H\left(X \times [0,1]^d\right) = \dim_H X + d.$$

The possible values of $(\dim_t X, \dim_{tH} X, \dim_H X)$.

The following theorem provides a complete description of the possible values of the triple $(\dim_t X, \dim_{tH} X, \dim_H X)$. Moreover, all possible values can be realized by compact spaces as well.

Theorem 3.23. *For a triple $(d, s, t) \in [0, \infty]^3$ the following are equivalent.*

(i) There exists a compact metric space K such that $\dim_t K = d$, $\dim_{tH} K = s$, and $\dim_H K = t$.

(ii) There exists a separable metric space X such that $\dim_t X = d$, $\dim_{tH} X = s$, and $\dim_H X = t$.

(iii) There exists a metric space X such that $\dim_t X = d$, $\dim_{tH} X = s$, and $\dim_H X = t$.

(iv) $d = s = t = -1$, *or* $d = s = 0$, $t \in [0, \infty]$, *or* $d \in \mathbb{N}^+ \cup \{\infty\}$, $s, t \in [1, \infty]$, $d \leq s \leq t$.

Proof. The implications $(i) \implies (ii)$ and $(ii) \implies (iii)$ are obvious, and $(iii) \implies (iv)$ can easily be checked using Fact 3.1 and Theorem 3.4.

It remains to prove that $(iv) \implies (i)$. First, the empty set takes care of the case $d = s = t = -1$. Let now $d = s = 0$, $t \in [0, \infty]$. For $t \in [0, \infty]$ let K_t be a Cantor set with $\dim_H K_t = t$. Such sets are well-known to exist already in $[0, 1]^n$ for large enough n in case $t < \infty$, whereas if C is the middle-thirds Cantor set then $C^{\mathbb{N}}$ is such a set for $t = \infty$. Then clearly $\dim_t K_t = \dim_{tH} K_t = 0$ and $\dim_H K_t = t$, so we are done with this case.

Finally, let $d \in \mathbb{N}^+ \cup \{\infty\}$, $s, t \in [1, \infty]$, $d \leq s \leq t$. We may assume $d < \infty$, otherwise the Hilbert cube provides a suitable example. (Indeed, clearly $\dim_t [0, 1]^{\mathbb{N}} = \dim_{tH} [0, 1]^{\mathbb{N}} = \dim_H [0, 1]^{\mathbb{N}} = \infty$.) Define $K_{d,s,t} = (K_{s-d} \times [0, 1]^d) \cup K_t$ (this can be understood as the disjoint sum of metric spaces, but we may also assume that all these spaces are in the Hilbert cube, so the union is well defined). Since $\dim_t (X \times Y) \leq \dim_t X + \dim_t Y$ for non-empty spaces (see e.g. [3]), we obtain $\dim_t (K_{s-d} \times [0, 1]^d) = 0 + d = d$. Hence, by the stability of the topological dimension for closed sets, $\dim_t K_{d,s,t} = \max \left\{\dim_t \left(K_{s-d} \times [0, 1]^d\right), \dim_t K_t\right\} = \max\{d, 0\} = d$. Using Corollary 3.22 and the stability of the topological Hausdorff dimension for closed sets we infer that $\dim_{tH} K_{d,s,t} = \max \left\{\dim_{tH} \left(K_{s-d} \times [0, 1]^d\right), \dim_{tH} K_t\right\} = \max\{s - d + d, 0\} = s$. Again by Corollary 3.22 and by the stability of the Hausdorff dimension we obtain that $\dim_H K_{d,s,t} = \max \left\{\dim_H \left(K_{s-d} \times [0, 1]^d\right), \dim_H K_t\right\} = \max\{s - d + d, t\} = \max\{s, t\} = t$. This completes the proof. □

The topological Hausdorff dimension is not a function of the topological and the Hausdorff dimension.

As a particular case of the above theorem we obtain that there are compact metric spaces X and Y such that $\dim_t X = \dim_t Y$ and $\dim_H X = \dim_H Y$ but $\dim_{tH} X \neq \dim_{tH} Y$. This immediately implies the following, which shows that the topological Hausdorff dimension is indeed a genuinely new concept.

Corollary 3.24. $\dim_{tH} X$ *cannot be calculated form* $\dim_t X$ *and* $\dim_H X$*, even for compact metric spaces.*

4. Calculating the topological Hausdorff dimension

4.1. **Some classical fractals.** First we present certain natural examples of compact sets K with $\dim_t K = \dim_{tH} K < \dim_H K$. Let S be the Sierpiński triangle, then it is well-known that $\dim_t S = 1$ and $\dim_H S = \frac{\log 3}{\log 2}$.

Theorem 4.1. *Let S be the Sierpiński triangle. Then* $\dim_{tH}(S) = 1$.

Proof. Let $\varphi_i \colon \mathbb{R}^2 \to \mathbb{R}^2$ $(i = 1, 2, 3)$ be the three similitudes with ratio $1/2$ for which $S = \bigcup_{i=1}^3 \varphi_i(S)$. Sets of the form $\varphi_{i_n} \circ \dots \circ \varphi_{i_1}(S)$, $n \in \mathbb{N}$, $j \in \{1, \dots, n\}$, $i_j \in \{1, 2, 3\}$ are called the elementary pieces of S. It is not difficult to see that

$$\mathcal{U} = \{\operatorname{int}_S H : H \text{ is a finite union of elementary pieces of } S\}$$

is a basis of S such that $\#\partial_S U$ is finite for every $U \in \mathcal{U}$. Therefore $\dim_H \partial_S U \leq 0$, and hence $\dim_{tH} S \leq 1$. On the other hand, S contains a line segment, therefore $\dim_{tH} S \geq \dim_{tH} [0, 1] = 1$ by monotonicity. □

Now we turn to the von Koch snowflake curve K. Recall that $\dim_t K = 1$ and $\dim_H K = \frac{\log 4}{\log 3}$.

Fact 4.2. *If K is homeomorphic to $[0,1]$ then $\dim_{tH} K = 1$.*

Proof. By Corollary 3.3 we obtain that $\dim_{tH} K \geq 1$. On the other hand, since K is homeomorphic to $[0,1]$, there is a basis in K such that $\#\partial U \leq 2$ for every $U \in \mathcal{U}$. Thus $\dim_{tH} K \leq 1$. $\square$

Corollary 4.3. *Let K be the von Koch curve. Then $\dim_{tH} K = 1$.*

Next we take up a natural example of a compact set K with $\dim_t K < \dim_{tH} K < \dim_H K$. Let T be the Sierpiński carpet, then it is well-known that $\dim_t T = 1$ and $\dim_H T = \frac{\log 8}{\log 3}$.

Theorem 4.4. *Let T be the Sierpiński carpet. Then $\dim_{tH}(T) = \frac{\log 2}{\log 3} + 1 = \frac{\log 6}{\log 3}$.*

Proof. Let C denote the middle-thirds Cantor set. Observe that $C \times [0,1] \subseteq T$. Then monotonicity and Theorem 3.19 yield $\dim_{tH} T \geq \dim_{tH}(C \times [0,1]) = \dim_H C + 1 = \frac{\log 2}{\log 3} + 1$.

Let us now prove the opposite inequality. For $n \in \mathbb{N}$ and $i = 1, \dots, 3^n$ let $z_i^n = \frac{2i-1}{2(3^n)}$. Then clearly

$$\{z_i^n : n \in \mathbb{N},\, i \in \{1, \dots, 3^n\}\}$$

is dense in $[0,1]$. Let L be a horizontal line defined by an equation of the form $y = z_i^n$ or a vertical line defined by $x = z_i^n$. It is easy to see that $L \cap T$ consists of finitely many sets geometrically similar to the middle-thirds Cantor set. Using these lines it is not difficult to construct a rectangular basis $\mathcal{U}$ of T such that $\dim_H \partial_T U = \frac{\log 2}{\log 3}$ for every $U \in \mathcal{U}$, and hence $\dim_{tH} T \leq \frac{\log 2}{\log 3} + 1$. $\square$

Finally we remark that, by Theorem 3.19, $K = C \times [0,1]$ (where C is the middle-thirds Cantor set) is a natural example of a compact set with $\dim_t K < \dim_{tH} K = \dim_H K$.

4.2. **Kakeya sets.**

Definition 4.5. A subset of $\mathbb{R}^d$ is called a *Kakeya set* if it contains a non-degenerate line segment in every direction (some authors call these sets *Besicovitch sets*).

According to a surprising classical result, Kakeya sets of Lebesgue measure zero exist. However, one of the most famous conjectures in analysis is the Kakeya Conjecture stating that every Kakeya set in $\mathbb{R}^d$ has Hausdorff dimension d. This is known to hold only in dimension at most 2 so far, and a solution already in $\mathbb{R}^3$ would have a huge impact on numerous areas of mathematics.

It would be tempting to attack the Kakeya Conjecture using $\dim_{tH} K \leq \dim_H K$, but the following theorem, the main theorem of this section will show that unfortunately we cannot get anything non-trivial this way.

Theorem 4.6. *There exists a Kakeya set $K \subseteq \mathbb{R}^d$ of topological Hausdorff dimension 1 for every integer $d \geq 1$.*

This result is of course sharp, since if a set contains a line segment then its topological Hausdorff dimension is at least 1.

We will actually prove somewhat more, since we will essentially show that the generic element of a carefully chosen space is a Kakeya set of topological Hausdorff dimension 1. This idea, as well as most of the others in this section are already present in [11] by T. W. Körner. However, he only works in the plane and his space slightly differs from ours. For the sake of completeness we provide the rather short proof details.

Let $(\mathcal{K}, d_H)$ be the set of compact subsets of $\mathbb{R}^{d-1} \times [0,1]$ endowed with the Hausdorff metric, that is for each $K_1, K_2 \in \mathcal{K}$

$$d_H(K_1, K_2) = \min \{r : K_1 \subseteq B(K_2, r) \text{ and } K_2 \subseteq B(K_1, r)\},$$

where $B(K,r) = \left\{x \in \mathbb{R}^{d-1} \times [0,1] : \operatorname{dist}(x,K) \leq r\right\}$. It is well-known that $(\mathcal{K}, d_H)$ is a complete metric space, see e.g. [9].

Let

$$\Gamma = \{(x_1, \dots, x_{d-1}, 1) : 1/2 \leq x_i \leq 1, \quad i = 1, \dots, d-1\}$$

denote a subset of directions in $\mathbb{R}^d$. A closed line segment w connecting $\mathbb{R}^{d-1} \times \{0\}$ and $\mathbb{R}^{d-1} \times \{1\}$ is called a standard segment.

Let us denote by $\mathcal{F} \subseteq \mathcal{K}$ the system of those compact sets in $\mathbb{R}^{d-1} \times [0,1]$ in which for each $v \in \Gamma$ we can find a standard segment w parallel to v. First we show that $\mathcal{F}$ is closed in $\mathcal{K}$. Let us assume that $F_n \in \mathcal{F}$, $K \in \mathcal{K}$ and $F_n \to K$ with respect to d_H. We have to show that $K \in \mathcal{F}$. Let $v \in \Gamma$ be arbitrary. Since $F_n \in \mathcal{F}$, there exists a $w_n \subseteq F_n$ parallel to v for every n. It is easy to see that $\bigcup_{n \in \mathbb{N}} F_n$ is bounded, hence we can choose a subsequence n_k such that w_{n_k} is convergent with respect to d_H. But then clearly $w_{n_k} \to w$ for some standard segment $w \subseteq K$, and w is parallel to v. Hence $K \in \mathcal{F}$ indeed.

Therefore, $(\mathcal{F}, d_H)$ is a complete metric space and hence we can use Baire category arguments.

The next lemma is based on [11, Thm. 3.6.].

Lemma 4.7. *The generic set in $(\mathcal{F}, d_H)$ is of topological Hausdorff dimension 1.*

Proof. The rational cubes form a basis of $\mathbb{R}^d$, and their boundaries are covered by the rational hyperplanes orthogonal to one of the usual basis vectors of $\mathbb{R}^d$. Therefore, it suffices to show that if S is a fixed hyperplane orthogonal to one of the usual basis vectors then $\{F \in \mathcal{F} : \dim_H(F \cap S) = 0\}$ is co-meager.

For $n \in \mathbb{N}^+$ define

$$\mathcal{F}_n = \left\{ F \in \mathcal{F} : \mathcal{H}^{\frac{1}{n}}_{\frac{1}{n}}(F \cap S) < \frac{1}{n} \right\}.$$

In order to show that $\{F \in \mathcal{F} : \dim_H(F \cap S) = 0\} = \bigcap_{n \in \mathbb{N}^+} \mathcal{F}_n$ is co-meager, it is enough to prove that each $\mathcal{F}_n$ contains a dense open set.

For $p \in \mathbb{R}^d$, $v \in \Gamma$ and $0 < \alpha < \pi/2$ we denote by $C(p, v, \alpha)$ the following doubly infinite closed cone

$$C(p, v, \alpha) = \{x \in \mathbb{R}^d : \text{ the angle between the lines of } v \text{ and } x - p \text{ is at most } \alpha\}.$$

We denote by $V(C(p, v, \alpha))$ the set of those vectors $u = (u_1, ..., u_{d-1}, 1)$ for which there is a line in $\operatorname{int}(C(p, v, \alpha)) \cup \{p\}$ parallel to u. Then $V(C(p, v, \alpha))$ is relatively open in $\mathbb{R}^{d-1} \times \{1\}$.

The sets of the form $C'(p, v, \alpha) = C(p, v, \alpha) \cap \left(\mathbb{R}^{d-1} \times [0,1]\right)$ will be called truncated cones, and the system of truncated cones will be denoted by $\mathcal{C}'$. A truncated cone $C'(p, v, \alpha)$ is S-compatible if either $C'(p, v, \alpha) \cap S = \{p\}$, or $C'(p, v, \alpha) \cap S = \emptyset$.

The set of S-compatible truncated cones is denoted by $\mathcal{C}'_S$. Define $\mathcal{F}_S$ as the set of those $F \in \mathcal{F}$ that can be written as the union of finitely many S-compatible truncated cones and finitely many points in $\mathbb{R}^{d-1} \times [0,1]$.

Next we check that $\mathcal{F}_S$ is dense in $\mathcal{F}$.

Suppose $F \in \mathcal{F}$ is arbitrary and $\varepsilon > 0$ is given. First choose finitely many points $\{y_i\}_{i=1}^t$ in F such that $F \subseteq B\left(\{y_i\}_{i=1}^t, \varepsilon\right)$. Let $v \in \Gamma$ be arbitrary, then there exists a standard segment $w_v \subseteq F$ parallel to v. By the choice of S and Γ, clearly $w_v \nsubseteq S$, hence we can choose p_v and α_v such that $C'(p_v, v, \alpha_v) \in \mathcal{C}'_S$ and $d_H(C'(p_v, v, \alpha_v), w_v) \leq \varepsilon$. Obviously $v \in V(C(p_v, v, \alpha_v))$, so $\{V(C(p_v, v, \alpha_v))\}_{v\in\Gamma}$ is an open cover of the compact set Γ. Therefore, there are $\{C'(p_{v_i}, v_i, \alpha_{v_i})\}_{i=1}^m$ in $\mathcal{C}'_S$ such that $\Gamma \subseteq \bigcup_{i=1}^m V(C(p_{v_i}, v_i, \alpha_{v_i}))$. Put $F' = \bigcup_{i=1}^m C'(p_{v_i}, v_i, \alpha_{v_i}) \cup \{y_1, \dots, y_t\}$, then $F' \in \mathcal{F}_S$. It is easy to see that $\bigcup_{i=1}^m C'(p_{v_i}, v_i, \alpha_{v_i}) \subseteq B(F, \varepsilon)$, and combining this with $\{y_i\}_{i=1}^t \subseteq F$ we obtain that $F' \subseteq B(F, \varepsilon)$. By the choice of $\{y_i\}_{i=1}^t$ we also have $F \subseteq B(F', \varepsilon)$. Thus $d_H(F, F') \leq \varepsilon$.

Now using our dense set $\mathcal{F}_S$ we verify that $\mathcal{F}_n$ contains a dense open set $\mathcal{U}$. We construct for all $F_0 \in \mathcal{F}_S$ a ball in $\mathcal{F}_n$ centered at F_0. By the definition of S-compatibility $F_0 \cap S$ is finite. Hence we can easily choose a relatively open set $U_0 \subseteq S$ such that $F_0 \cap S \subseteq U_0$ and $\mathcal{H}^{\frac{1}{n}}_{\frac{1}{n}}(U_0) < \frac{1}{n}$. Let us define

$$\mathcal{U} = \{F \in \mathcal{F} : F \cap S \subseteq U_0\}.$$

Clearly $F_0 \in \mathcal{U}$, $\mathcal{U} \subseteq \mathcal{F}_n$ and it is easy to see that $\mathcal{U}$ is open in $\mathcal{F}$. This completes the proof. $\square$

From this we obtain the main theorem of the section as follows.

Proof of Theorem 4.6. By the above lemma we can choose $F_0 \in \mathcal{F}$ such that $\dim_{tH} F_0 = 1$. Then F_0 contains a line segment in every direction of Γ, hence we can choose finitely many isometric copies of it, $\{F_i\}_{i=1}^n$ such that the compact set $K = \bigcup_{i=0}^n F_i$ contains a line segment in every direction. By the Lipschitz invariance of the topological Hausdorff dimension $\dim_{tH} F_i = \dim_{tH} F_0$ for all i, and by the stability of the topological Hausdorff dimension for closed sets $\dim_{tH} K = 1$. $\square$

4.3. **Brownian motion.** One of the most important stochastic processes is the Brownian motion (see e.g. [15]). Its trail and graph also serve as important examples of fractal sets in geometric measure theory. Since the graph is always homeomorphic to $[0,\infty)$, Fact 4.2 and countable stability for closed sets yield that its topological Hausdorff dimension is 1. Hence we focus on the trail only.

Each statement in this paragraph is to be understood to hold with probability 1 (almost surely). Clearly, in dimension 1 the trail is a non-degenerate interval, so it has topological Hausdorff dimension 1. Moreover, if the dimension is at least 4 then the trail has no multiple points ([15]), so it is homeomorphic to $[0,\infty)$, which in turn implies as above that the trail has topological Hausdorff dimension 1 again. However, the following question is open.

Problem 4.8. Let $d = 2$ or 3. Determine the almost sure topological Hausdorff dimension of the trail of the d-dimensional Brownian motion.

5. Application I: Mandelbrot's fractal percolation process

In this section we take up one of the most important random fractals, the limit set M of the fractal percolation process defined by Mandelbrot in [12].

His original motivation was that this model captures certain features of turbulence, but then this random set turned out to be very interesting in its own right. For example, M serves as a very powerful tool for calculating Hausdorff dimension. Indeed, J. Hawkes has shown ([7]) that for a fixed Borel set B in the unit square (or analogously in higher dimensions) $M \cap B = \emptyset$ almost surely iff the sum of the co-dimensions exceeds 2, (that is, $(2 - \dim_H M) + (2 - \dim_H B) > 2$), and this formula can be used in certain applications to determine $\dim_H B$. Moreover, it can be shown that the trail of the Brownian motion is so called *intersection-equivalent* to a percolation fractal (roughly speaking, they intersect the same sets with positive probability), and this can be used to deduce numerous dimension related results about the Brownian motion, see the works of Y. Peres, e.g. in [15].

Let us now formally describe the fractal percolation process. Let $p \in (0, 1)$ and $n \geq 2$, $n \in \mathbb{N}$ be fixed. Set $M_0 = M_0^{(p,n)} = [0, 1]^2$. We divide the unit square into n^2 equal closed subsquares of side-length $1/n$ in the natural way. We keep each subsquare independently with probability p (and erase it with probability $1 - p$), and denote by $M_1 = M_1^{(p,n)}$ the union of the kept subsquares. Then each square in M_1 is divided into n^2 squares of side-length $1/n^2$, and we keep each of them independently (and also independently of the earlier choices) with probability p, etc. After k steps let $M_k = M_k^{(p,n)}$ be the union of the kept k^{th} level squares with side-length $1/n^k$. Let

$$M = M^{(p,n)} = \bigcap_{k=1}^{\infty} M_k. \tag{5.1}$$

The process we have just described is called *Mandelbrot's fractal percolation process*, and M is called its *limit set*.

Percolation fractals are not only interesting from the point of view of turbulence and fractal geometry, but they are also closely related to the (usual, graph-theoretic) percolation theory. In case of the fractal percolation the role of the clusters is played by the connected components. Our starting point will be the following celebrated theorem. Recall that a space is *totally disconnected* if every connected component is a singleton.

Theorem 5.1 (Chayes-Chayes-Durrett, [2]). *There exists a critical probability $p_c = p_c^{(n)} \in (0, 1)$ such that if $p < p_c$ then M is totally disconnected almost surely, and if $p > p_c$ then M contains a nontrivial connected component with positive probability.*

They actually prove more, the most powerful version states that in the supercritical case (i.e. when $p > p_c$) there is actually a unique unbounded component if the process is extended to the whole plane, but we will only concentrate on the most surprising fact that the critical probability is strictly between 0 and 1.

The main goal of the present section will be to prove the following generalization of the above theorem.

Theorem 5.2. *For every $d \in [0, 2)$ there exists a critical probability $p_c^{(d)} = p_c^{(d,n)} \in (0, 1)$ such that if $p < p_c^{(d)}$ then $\dim_{tH} M \leq d$ almost surely, and if $p > p_c^{(d)}$ then $\dim_{tH} M > d$ almost surely (provided $M \neq \emptyset$).*

In order to see that we actually obtain a generalization, just note that a compact space is totally disconnected iff $\dim_t M = 0$ ([3]), also that $\dim_{tH} M = 0$ iff $\dim_t M = 0$, and use $d = 0$. Theorem 5.1 basically says that certain curves show up at the critical probability, and our proof will show that even 'thick' families of curves show up, where the word thick is related to large Hausdorff dimension.

In the rest of this section first we do some preparations in the first subsection, then we prove the main theorem (Theorem 5.2) in the next subsection, and finally give an upper bound for $\dim_{tH} M$ and conclude that $\dim_{tH} M < \dim_H M$ almost surely in the non-trivial cases.

5.1. **Preparation.** For the proofs of the statements in the next two remarks see e.g. [2].

Remark 5.3. It is well-known from the theory of branching processes that $M = \emptyset$ almost surely iff $p \leq \frac{1}{n^2}$, so we may assume in the following that $p > \frac{1}{n^2}$.

If $\frac{1}{n^2} < p \leq \frac{1}{\sqrt{n}}$ then $\dim_t M = 0$ almost surely. Hence Fact 3.1 implies that $\dim_{tH} M = 0$ almost surely. (In fact, the same holds even for $p < p_c$, see Theorem 5.1.)

Remark 5.4. As for the Hausdorff dimension, for $p > \frac{1}{n^2}$ we have

$$\dim_H M = 2 + \frac{\log p}{\log n}$$

almost surely, provided $M \neq \emptyset$.

We will also need the 1-dimensional analogue of the process (intervals instead of squares). Here $M^{(1D)} = \emptyset$ almost surely iff $p \leq \frac{1}{n}$, and for $p > \frac{1}{n}$ we have

$$\dim_H M^{(1D)} = 1 + \frac{\log p}{\log n}$$

almost surely, provided $M^{(1D)} \neq \emptyset$.

Now we check that the almost sure topological Hausdorff dimension of M also exists.

Lemma 5.5. *For every $p > \frac{1}{n^2}$ and $n \geq 2$, $n \in \mathbb{N}$ there exists a number $d = d^{(p,n)} \in [0, 2]$ such that*

$$\dim_{tH} M = d$$

almost surely, provided $M \neq \emptyset$.

Proof. Let N be the random number of squares in M_1. Let us set $q = P(M = \emptyset)$. Then $q < 1$ by $p > \frac{1}{n^2}$, and [4, Thm. 15.2] gives that q is the least positive root of the polynomial

$$f(t) = -t + \sum_{k=0}^{n^2} P(N = k)t^k.$$

First we show that $P(\dim_{tH} M \leq x)$ is a root of f for every $x \in \mathbb{R}$.

Let $M_1 = \{Q_1, \dots, Q_N\}$, where the Q_i's are the first level subsquares, and fix $x \in \mathbb{R}$. (Define $Q_1 = \emptyset$ if $N = 0$.) For every i and k let $M_k^{Q_i}$ be the union of those squares in M_k that are in Q_i, and let $M^{Q_i} = \bigcap_k M_k^{Q_i}$. (Note that this is not the same as $M \cap Q_i$, since in this latter set there may be points on the boundary of Q_i 'coming from squares outside of Q_i'.) Then M^{Q_i} has the same distribution as a similar copy of M (this is called statistical self-similarity), and hence for every i

$$P\left(\dim_{tH} M^{Q_i} \leq x\right) = P\left(\dim_{tH} M \leq x\right).$$

Using the stability of the topological Hausdorff dimension for closed sets and the fact that the M^{Q_i}'s are independent and have the same distribution under the condition $N = k$, this implies

$$\begin{aligned} P\left(\dim_{tH} M \leq x \,|\, N = k\right) &= P\left(\dim_{tH} M^{Q_i} \leq x \text{ for each } 1 \leq i \leq k \,|\, N = k\right) \\ &= \left(P\left(\dim_{tH} M^{Q_1} \leq x\right)\right)^k \\ &= \left(P\left(\dim_{tH} M \leq x\right)\right)^k. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} P(\dim_{tH} M \leq x) &= \sum_{k=0}^{n^2} P(N = k) P(\dim_{tH} M \leq x \,|\, N = k) \\ &= \sum_{k=0}^{n^2} P(N = k) \left(P\left(\dim_{tH} M \leq x\right)\right)^k, \end{aligned}$$

and thus $P(\dim_{tH} M \leq x)$ is indeed a root of f for every x.

As mentioned above, $q \neq 1$ and q is also a root of f. Moreover, 1 is obviously also a root, and it is easy to see that f is strictly convex, hence there are at most two roots. Hence q and 1 are the only roots, therefore $P(\dim_{tH} M \leq x) = q$ or 1 for every x.

Then the distribution function $F(x) = P(\dim_{tH} M \leq x \,|\, M \neq \emptyset)$ only attains the values 0 and 1, moreover, $F(0) = 0$, $F(2) = 1$, thus there is a value d where it 'jumps' from 0 to 1, and this concludes the proof. □

5.2. **Proof of Theorem 5.2; the lower estimate of** $\dim_{tH} M$**.** Set

$$p_c^{(d,n)} = \sup\left\{p : \dim_{tH} M^{(p,n)} \leq d \text{ almost surely}\right\}.$$

First we need some lemmas. The following one is analogous to [6, p. 387].

Lemma 5.6. *For every $d \in \mathbb{R}$ and $n \in \mathbb{N}$, $n \geq 2$*

$$p_c^{(d,n)} < 1 \quad \Longleftrightarrow \quad p_c^{(d,n^2)} < 1.$$

Proof. Clearly, it is enough to show that

$$p_c^{(d,n)}\left(1 - \left(1 - p_c^{(d,n)}\right)^{\frac{1}{n^2}}\right) \leq p_c^{(d,n^2)} \leq p_c^{(d,n)}. \tag{5.2}$$

We say that the random construction X is dominated by the random construction Y if they can be realized on the same probability space such that $X \subseteq Y$ almost surely.

Let us first prove the second inequality in (5.2). It clearly suffices to show that

$$\dim_{tH} M^{(p,n^2)} \leq d \text{ almost surely} \implies \dim_{tH} M^{(p,n)} \leq d \text{ almost surely}.$$

But this is rather straightforward, since $M_{2k}^{(p,n)}$ is easily seen to be dominated by $M_k^{(p,n^2)}$ for every k, hence $M^{(p,n)}$ is dominated by $M^{(p,n^2)}$.

Let us now prove the first inequality in (5.2). Set $\varphi(x) = 1 - (1-x)^{1/n^2}$. We need to show that

$$0 < p < p_c^{(d,n)} \varphi(p_c^{(d,n)}) \implies \dim_{tH} M^{(p,n^2)} \leq d \text{ almost surely}. \tag{5.3}$$

Since $x\varphi(x)$ is an increasing homeomorphism of the unit interval, $p = q\varphi(q)$ for some $q \in (0,1)$. Then clearly $q < p_c^{(d,n)}$, so $\dim_{tH} M^{(q,n)} \leq d$ almost surely. Therefore, in order to prove (5.3) it suffices to check that

$$M^{(p,n^2)} \text{ is dominated by } M^{(q,n)}. \tag{5.4}$$

First we check that

$$M_k^{(\varphi(q),n^2)} \text{ is dominated by } M_k^{(q,n)} \text{ for every } k, \tag{5.5}$$

and consequently $M^{(\varphi(q),n^2)}$ is dominated by $M^{(q,n)}$. Indeed, in the second case we erase a subsquare of side length $\frac{1}{n}$ with probability $1-q$ and keep it with probability q, while in the first case we *completely* erase a subsquare of side length $\frac{1}{n}$ with the same probability $(1-\varphi(q))^{n^2} = 1-q$ and hence keep *at least a subset of it* with probability q.

But this will easily imply (5.4), which will complete the proof. Indeed, after each step of the processes $M^{(\varphi(q),n^2)}$ and $M^{(q,n)}$ let us perform the following procedures. For $M^{(\varphi(q),n^2)}$ let us keep every existing square independently with probability q and erase it with probability $1-q$ (we do not do any subdivisions in this case). For $M^{(q,n)}$ let us take one more step of the construction of $M^{(q,n)}$. Using (5.5) this easily implies that $M_k^{(q\varphi(q),n^2)}$ is dominated by $M_{2k}^{(q,n)}$ for every k, hence $M^{(q\varphi(q),n^2)}$ is dominated by $M^{(q,n)}$, but $q\varphi(q) = p$, and hence (5.4) holds. $\square$

From now on let N be a fixed (large) positive integer to be chosen later. Recall that a square of level k is a set of the form $\left[\frac{i}{N^k}, \frac{i+1}{N^k}\right] \times \left[\frac{j}{N^k}, \frac{j+1}{N^k}\right] \subseteq [0,1]^2$.

Definition 5.7. A *walk of level* k is a sequence $(S_1, \dots, S_l)$ of non-overlapping squares of level k such that S_r and S_{r+1} are abutting for every $r = 1, \dots, l-1$, moreover $S_1 \cap (\{0\} \times [0,1]) \neq \emptyset$ and $S_l \cap (\{1\} \times [0,1]) \neq \emptyset$.

In particular, the only walk of level 0 is $([0,1]^2)$.

Definition 5.8. We say that $(S_1, \dots, S_l)$ is a *turning walk (of level* 1*)* if it satisfies the properties of a walk of level 1 except that instead of $S_l \cap (\{1\} \times [0,1]) \neq \emptyset$ we require that $S_l \cap ([0,1] \times \{1\}) \neq \emptyset$.

Lemma 5.9. *Let $\mathcal{S}$ be a set of $N-2$ distinct squares of level 1 intersecting $\{0\} \times [0,1]$, and let $\mathcal{T}$ be a set of $N-2$ distinct squares of level 1 intersecting $\{1\} \times [0,1]$. Moreover, let F^* be a square of level 1 such that the row of F^* does not intersect $\mathcal{S} \cup \mathcal{T}$. Then there exist $N-2$ non-overlapping walks of level 1 not containing F^* such that the set of their first squares coincides with $\mathcal{S}$ and the set of their last squares coincides with $\mathcal{T}$.*

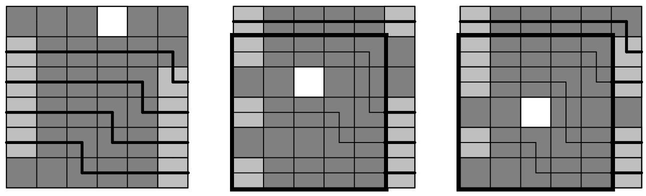
Proof. The proof is by induction on N. The case $N = 2$ is obvious.

Case 1. *F^* is in the top or bottom row.*

By simply ignoring this row it is straightforward how to construct the walks in the remaining rows.

Case 2. *F^* is not in the top or bottom row, and both top corners or both bottom corners are in $\mathcal{S} \cup \mathcal{T}$.*

Without loss of generality we may suppose that both top corners are in $\mathcal{S}\cup\mathcal{T}$. Let the straight walk connecting these two corners be one of the walks to be constructed. Then let us shift the remaining members of $\mathcal{T}$ to the left by one square, and either



Case 1. Case 2. Case 3.

FIGURE 1. Illustration to Lemma 5.9

we can apply the induction hypothesis to the $(N-1) \times (N-1)$ many squares in the bottom left corner of the original $N \times N$ many squares, or F^* is not among these $(N-1) \times (N-1)$ many squares and then the argument is even easier. Then one can see how to get the required walks.

Case 3. *Neither Case 1 nor Case 2 holds.*

Since there are only two squares missing on both sides, and F^* cannot be the top or bottom row, we infer that both $\mathcal{S}$ and $\mathcal{T}$ contain at least one corner. Since Case 2 does not hold, we obtain that both the top left and the bottom right corners or both the bottom left and the top right corners are in $\mathcal{S} \cup \mathcal{T}$. Without loss of generality we may suppose that both the top left and the bottom right corners are in $\mathcal{S} \cup \mathcal{T}$. By reflecting the picture about the center of the unit square if necessary, we may assume that F^* is not in the rightmost column. We now construct the first walk. Let it run straight from the top left corner to the top right corner, and then continue downwards until it first reaches a member of $\mathcal{T}$. Then, as above, we can similarly apply the induction hypothesis to the $(N-1) \times (N-1)$ many squares in the bottom left corner, and we are done. □

Lemma 5.10. *Let $\mathcal{S}$ be a set of $N-2$ distinct squares of level 1 intersecting $\{0\} \times [0,1]$, and let $\mathcal{T}$ be a set of $N-2$ distinct squares of level 1 intersecting $[0,1] \times \{1\}$ (the sets of starting and terminal squares). Moreover, let F^* be a square of level 1 (the forbidden square) such that the row of F^* does not intersect $\mathcal{S}$ and the column of F^* does not intersect $\mathcal{T}$. Then there exist $N-2$ non-overlapping turning walks not containing F^* such that the set of their first squares coincides with $\mathcal{S}$ and the set of their last squares coincides with $\mathcal{T}$.*

Proof. Obvious, just take the simplest 'L-shaped' walks. □

The last two lemmas will almost immediately imply the following.

Lemma 5.11. *Let $(S_1, \dots, S_l)$ be a walk of level k, and $\mathcal{F}$ a system of squares of level $k+1$ such that each S_r contains at most 1 member of $\mathcal{F}$. Then $(S_1, \dots, S_l)$ contains $N-2$ non-overlapping subwalks of level $k+1$ avoiding $\mathcal{F}$.*

Proof. We may assume that each S_r contains exactly 1 member of $\mathcal{F}$. Let us denote the member of $\mathcal{F}$ in S_r by F_r^*. The subwalks will be constructed separately in each S_r, using an appropriately rotated or reflected version of either Lemma 5.10 or Lemma 5.9. It suffices to construct $\mathcal{S}_r$ and $\mathcal{T}_r$ for every r (compatible with F_r^*) so that for every member of $\mathcal{T}_r$ there is an abutting member of $\mathcal{S}_{r+1}$. (Of course we

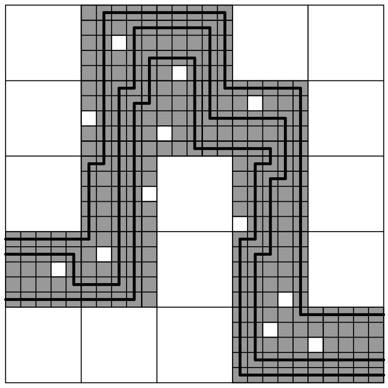


FIGURE 2. Illustration to Lemma 5.11

also have to make sure that every member of $\mathcal{S}_1$ intersects $\{0\} \times [0,1]$ and every member of $\mathcal{T}_l$ intersects $\{1\} \times [0,1]$.) For example, the construction of $\mathcal{T}_r$ for $r < l$ is as follows. The squares S_r and S_{r+1} share a common edge E. Assume for simplicity that E is horizontal. Then $\mathcal{T}_r$ will consist of those subsquares of S_r of level $k+1$ that intersect E and whose column differs from that of F_r^* and F_{r+1}^*. If these two columns happen to coincide then we can arbitrarily erase one more square. The remaining constructions are similar and the details are left to the reader. $\square$

Definition 5.12. We say that a square in M_k is 1-*full* if it contains at least $N^2 - 1$ many subsquares from M_{k+1}. We say that it is m-*full*, if it contains at least $N^2 - 1$ many $m-1$-full subsquares from M_{k+1}. We call M *full* if M_0 is m-full for every $m \in \mathbb{N}^+$.

The following lemma was the key realization in [2].

Lemma 5.13. *There exists a $p^{(N)} < 1$ such that for every $p > p^{(N)}$ we have $P\left(M^{(p,N)} \text{ is full}\right) > 0$.*

See [2] or [4, Prop. 15.5] for the proof.

Definition 5.14. Let $L \leq N$ be positive integers. A compact set $K \subseteq [0,1]$ is called (L,N)-regular if it is of the form $K = \bigcap_{i \in \mathbb{N}} K_i$, where $K_0 = [0,1]$, and K_{k+1} is obtained by dividing every interval I in K_k into N many non-overlapping closed intervals of length $1/N^{k+1}$, and choosing L many of them for each I.

The following fact is well-known, see e.g. the more general [4, Thm. 9.3].

Fact 5.15. *An (L,N)-regular compact set has Hausdorff dimension $\frac{\log L}{\log N}$.*

Next we prove the main result of the present subsection.

Proof of Theorem 5.2. Let $d \in [0,2)$ be arbitrary. First we verify that, for sufficiently large N, if $M = M^{(p,N)}$ is full then $\dim_{tH} M > d$. The strategy is as follows. We define a collection $\mathcal{G}$ of disjoint connected subsets of M such that if a set intersects each member of $\mathcal{G}$ then its Hausdorff dimension is larger than $d-1$. Then we show that for every countable open basis $\mathcal{U}$ of M the union of the boundaries, $\bigcup_{U\in\mathcal{U}} \partial_M U$ intersects each member of $\mathcal{G}$, which clearly implies $\dim_{tH} M > d$.

Let us fix an integer N such that

$$N \geq 6 \text{ and } \frac{\log(N-2)}{\log N} > d-1, \tag{5.6}$$

and let us assume that M is full. Using Lemma 5.13 at each step we can choose $N-2$ non-overlapping walks of level 1 in M_1, then $N-2$ non-overlapping walks of level 2 in M_2 in each of the above walks, etc. Let us denote the obtained system at step k by

$$\mathcal{G}_k = \left\{\Gamma_{i_1,\ldots,i_k} : (i_1,\ldots,i_k) \in \{1,\ldots,N-2\}^k\right\},$$

where $\Gamma_{i_1,\ldots,i_k}$ is the union of the squares of the corresponding walk. (Set $\mathcal{G}_0 = \{\Gamma_\emptyset\} = \{[0,1]^2\}$.) Let us also put

$$C_k = \left\{y \in [0,1] : (0,y) \in \bigcup \mathcal{G}_k\right\}$$

and define

$$C = \bigcap_{k\in\mathbb{N}} C_k.$$

Then clearly C is an $(N-2,N)$-regular compact set, therefore Fact 5.15 yields that $\dim_H C = \frac{\log(N-2)}{\log N} > d-1$. We will also need that $\dim_H(C\setminus\mathbb{Q}) > d-1$, but this is clear since $\dim_H C > 0$ and hence $\dim_H(C\setminus\mathbb{Q}) = \dim_H C$.

For every $y \in C\setminus\mathbb{Q}$ and every $k \in \mathbb{N}$ there is a unique $(i_1,\ldots,i_k)$ such that $(0,y) \in \Gamma_{i_1,\ldots,i_k}$. (For y's of the form $\frac{i}{N^l}$ there may be two such $(i_1,\ldots,i_k)$'s, and we would like to avoid this complication.) Put $\Gamma_k(y) = \Gamma_{i_1,\ldots,i_k}$ and $\Gamma(y) = \bigcap_{k=1}^{\infty} \Gamma_k(y)$. Since $\Gamma(y)$ is a decreasing intersection of compact connected sets, it is itself connected ([3]). (Actually, it is a continuous curve, but we will not need this here.) It is also easy to see that it intersects $\{0\}\times[0,1]$ and $\{1\}\times[0,1]$.

We can now define

$$\mathcal{G} = \{\Gamma(y) : y \in C\setminus\mathbb{Q}\}.$$

Next we prove that $\mathcal{G}$ consists of disjoint sets. Let $y,y' \in C\setminus\mathbb{Q}$ be distinct. Pick $l \in \mathbb{N}$ so large such that $|y-y'| > \frac{6}{N^l}$. Then there are at least 5 intervals of level l between y and y'. Since we always chose $N-2$ intervals out of N along the construction, there can be at most 4 consecutive non-selected intervals, therefore there is a $\Gamma_{i_1,\ldots,i_l}$ separating y and y'. But then this also separates $\Gamma_l(y)$ and $\Gamma_l(y')$, hence $\Gamma(y)$ and $\Gamma(y')$ are disjoint.

Now we check that for every $y \in C\setminus\mathbb{Q}$ and every countable open basis $\mathcal{U}$ of M the set $\bigcup_{U\in\mathcal{U}} \partial_M U$ intersects $\Gamma(y)$. Let $z_0 \in \Gamma(y)$ and $U_0 \in \mathcal{U}$ such that $z_0 \in U_0$ and $\Gamma(y) \nsubseteq U_0$. Then $\partial_M U_0$ must intersect $\Gamma(y)$, since otherwise $\Gamma(y) = (\Gamma(y)\cap U_0) \cup (\Gamma(y)\cap \operatorname{int}_M(M\setminus U_0))$, hence a connected set would be the union of two non-empty disjoint relatively open sets, a contradiction.

Thus, as explained in the first paragraph of the proof, it is sufficient to prove that if a set Z intersects every $\Gamma(y)$ then $\dim_H Z > d-1$. This is easily seen to hold if we can construct an onto Lipschitz map

$$\varphi\colon \bigcup \mathcal{G} \to C \setminus \mathbb{Q}$$

that is constant on every member of $\mathcal{G}$, since Lipschitz maps do not increase Hausdorff dimension, and $\dim_H(C \setminus \mathbb{Q}) > d - 1$. Define

$$\varphi(z) = y \text{ if } z \in \Gamma(y),$$

which is well-defined by the disjointness of the members of $\mathcal{G}$.

Let us now prove that this map is Lipschitz. Let $y, y' \in C \setminus \mathbb{Q}$, $z \in \Gamma(y)$, and $z' \in \Gamma(y')$. Choose $l \in \mathbb{N}^+$ such that $\frac{1}{N^l} < |y - y'| \le \frac{1}{N^{l-1}}$. Then using $N \ge 6$ we obtain $|y - y'| > \frac{6}{N^{l+1}}$, thus, as above, there is a walk of level $l+1$ separating z and z'. Therefore $|z - z'| \ge \frac{1}{N^{l+1}}$, and hence

$$|\varphi(z) - \varphi(z')| = |y - y'| \le \frac{1}{N^{l-1}} = N^2 \frac{1}{N^{l+1}} \le N^2 |z - z'|,$$

therefore φ is Lipschitz with Lipschitz constant at most N^2.

To finish the proof, let n be given as in Theorem 5.2 and pick $k \in \mathbb{N}$ so large that $N = n^{2^k}$ satisfies (5.6). If $p > p^{(N)}$ then using Lemma 5.13 we deduce that

$$P\left(\dim_{tH} M^{(p,N)} > d\right) \ge P\left(M^{(p,N)} \text{ is full }\right) > 0,$$

which implies $p_c^{(d,N)} < 1$. Iterating k times Lemma 5.6 we infer $p_c^{(d,n)} < 1$.

Now, if $p > p_c^{(d,n)}$ then

$$P\left(\dim_{tH} M^{(p,n)} > d \,\middle|\, M^{(p,n)} \ne \emptyset\right) \ge P(\dim_{tH} M^{(p,n)} > d) > 0.$$

Combining this with Lemma 5.5 we deduce that

$$P\left(\dim_{tH} M^{(p,n)} > d \,\middle|\, M^{(p,n)} \ne \emptyset\right) = 1,$$

which completes the proof of the theorem. □

Remark 5.16. It is well-known and not difficult to see that $\lim\limits_{p \to 1} P(M^{(p,n)} = \emptyset) = 0$. Using this it is an easy consequence of the previous theorem that for every integer $n > 1$, $d < 2$ and $\varepsilon > 0$ there exists a $\delta = \delta^{(n,d,\varepsilon)} > 0$ such that for all $p > 1 - \delta$

$$P\left(\dim_{tH} M^{(p,n)} > d\right) > 1 - \varepsilon.$$

5.3. **The upper estimate of** $\dim_{tH} M$**.** The argument of this section will rely on some ideas from [2].

Theorem 5.17. *If* $p > \frac{1}{\sqrt{n}}$ *then almost surely*

$$\dim_{tH} M \le 2 + 2 \frac{\log p}{\log n}.$$

Proof. A segment is called a *basic segment* if it is of the form $\left[\frac{i-1}{n^k}, \frac{i}{n^k}\right] \times \{\frac{j}{n^k}\}$ or $\{\frac{j}{n^k}\} \times \left[\frac{i-1}{n^k}, \frac{i}{n^k}\right]$, where $k \in \mathbb{N}^+$, $i \in \{1, ..., n^k\}$ and $j \in \{1, ..., n^k - 1\}$.

It suffices to show that for every basic segment S and for every $\varepsilon > 0$ there exists (almost surely, a random) arc $\gamma \subseteq [0,1]^2$ connecting the endpoints of S in the ε-neighborhood of S such that $\dim_H(M \cap \gamma) \le 1 + 2\frac{\log p}{\log n}$. Indeed, we can almost surely construct the analogous arcs for all basic segments, and hence obtain

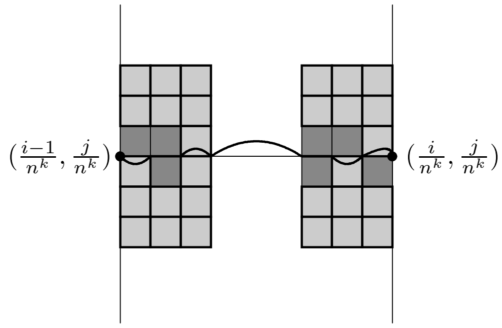


Figure 3. Construction of the arc γ connecting the endpoints of S

a basis of M consisting of 'approximate squares' whose boundaries are of Hausdorff dimension at most $1+2\frac{\log p}{\log n}$, therefore $\dim_{tH} M \leq 2+2\frac{\log p}{\log n}$ almost surely.

Let us now construct such an arc γ for S and $\varepsilon > 0$. We may assume that S is horizontal, hence it is of the form $S = \left[\frac{i-1}{n^k}, \frac{i}{n^k}\right] \times \{\frac{j}{n^k}\}$ for some $k \in \mathbb{N}^+$, $i \in \{1, ..., n^k\}$ and $j \in \{1, ..., n^k - 1\}$.

We divide S into n subsegments of length $\frac{1}{n^{k+1}}$, and we call a subsegment $\left[\frac{m-1}{n^{k+1}}, \frac{m}{n^{k+1}}\right] \times \{\frac{j}{n^k}\}$ *bad* if both the adjacent squares $\left[\frac{m-1}{n^{k+1}}, \frac{m}{n^{k+1}}\right] \times [\frac{j}{n^k} - \frac{1}{n^{k+1}}, \frac{j}{n^k}]$ and $\left[\frac{m-1}{n^{k+1}}, \frac{m}{n^{k+1}}\right] \times [\frac{j}{n^k}, \frac{j}{n^k} + \frac{1}{n^{k+1}}]$ are in M_{k+1}. Otherwise we say that the subsegment is *good*. Let B_1 denote the union of the bad segments. Then inside every bad segment we repeat the same procedure, and obtain B_2 and so on. It is easy to see that this process is (a scaled copy of) the 1-dimensional fractal percolation with p replaced by p^2. Let $B = \bigcap_l B_l$ be its limit set. Then by Remark 5.4 (note that $p^2 > \frac{1}{n}$) we obtain $\dim_H B = 1 + \frac{\log p^2}{\log n} = 1 + 2\frac{\log p}{\log n}$ or $B = \emptyset$ almost surely. So it suffices to construct a γ connecting the endpoints of S in the ε-neighborhood of S such that $\gamma \cap M = B$ (except perhaps some endpoints, but all the endpoints form a countable set, hence a set of Hausdorff dimension 0).

But this is easily done. Indeed, for every good subsegment I let γ_I be an arc connecting the endpoints of I in a small neighborhood of I such that γ is disjoint from M apart from the endpoints (this is possible, since either the top or the bottom square was erased from M). Then $\gamma = \left(\bigcup_{I \text{ is good}} \gamma_I\right) \cup B$ works. $\square$

Using Remarks 5.3 and 5.4 this easily implies

Corollary 5.18. *Almost surely*

$$\dim_{tH} M < \dim_H M \text{ or } M = \emptyset.$$

Remark 5.19. Calculating the exact value of $\dim_{tH} M$ seems to be difficult, since it would provide the value of the critical probability p_c of Chayes, Chayes and Durrett (where the phase transition occurs, see above), and this is a long-standing open problem.

6. Application II: The Hausdorff dimension of the level sets of the generic continuous function

Now we return to Problem 1.3. The main goal is to find analogues to Kirchheim's theorem, that is, to determine the Hausdorff dimension of the level sets of the generic continuous function defined on a compact metric space K.

Let us first note that the case $\dim_t K = 0$, that is, when there is a basis consisting of clopen sets is trivial because of the following well-known and easy fact.

Fact 6.1. *If K is a compact metric space with $\dim_t K = 0$ then the generic continuous function is one-to-one on K.*

Corollary 6.2. *If K is a compact metric space with $\dim_t K = 0$ then every non-empty level set of the generic continuous function is of Hausdorff dimension 0.*

Hence from now on we can restrict our attention to the case of positive topological dimension.

In the first part of this section we prove Theorem 6.20 and Corollary 6.21, our main theorems concerning level sets of the generic function defined on an arbitrary compact metric space, then we use this to derive conclusions about homogeneous and self-similar spaces in Theorem 6.22 and Corollary 6.23.

6.1. Arbitrary compact metric spaces.

The goal of this section is to prove Theorem 6.20. In order to do this we will need a sequence of equivalent definitions of the topological Hausdorff dimension. These equivalent definitions may be of some interest in their own right.

Let us fix a compact metric space K with $\dim_t K > 0$, and let $C(K)$ denote the space of continuous real-valued functions equipped with the supremum norm. Since this is a complete metric space, we can use Baire category arguments.

Definition 6.3. We say that a continuous function is *d-level narrow*, if there exists a dense set $S_f \subseteq \mathbb{R}$ such that $\dim_H f^{-1}(y) \le d-1$ for every $y \in S_f$. Let $\mathcal{N}_d$ be the set of d-level narrow functions. Define

$$P_n = \{d : \mathcal{N}_d \text{ is somewhere dense in } C(K)\},$$

and let $\dim_n K = \inf P_n$.

We repeat the definition of the topological Hausdorff dimension in an analogous form.

Definition 6.4. Define

$$P_{tH} = \{d : K \text{ has a basis } \mathcal{U} \text{ such that } \dim_H \partial U \le d-1 \text{ for every } U \in \mathcal{U}\},$$

then $\dim_{tH} K = \inf P_{tH}$.

For the next definition we need the following notation.

Notation 6.5. If $\mathcal{A}$ is a family of sets then let $T(\mathcal{A})$ denote the set of points covered by at least two members of $\mathcal{A}$, that is, $T(\mathcal{A}) = \bigcup_{A_1,A_2\in\mathcal{A},A_1\neq A_2}(A_1\cap A_2)$.

Definition 6.6. We say that $\mathcal{C}$ is a *d-dimensional small cover for $\varepsilon > 0$*, if $\mathcal{C}$ is a finite family of compact sets such that $\bigcup \mathcal{C} = K$, $\operatorname{diam} C \le \varepsilon$ for all $C \in \mathcal{C}$ and $\dim_H T(\mathcal{C}) \le d-1$. Define

$$P_s = \{d : \forall\varepsilon > 0,\ \exists \text{ a } d\text{-dimensional small cover for } \varepsilon\},$$

and let $\dim_s K = \inf P_s$.

Definition 6.7. For $d \geq 1$ we say that $\mathcal{C}$ is a a *d-dimensional pre-measure fat packing for* $\varepsilon > 0$ if $\mathcal{C}$ is a finite family of disjoint compact subsets of K such that $\operatorname{diam} C \leq \varepsilon$ for all $C \in \mathcal{C}$ and $\mathcal{H}_\infty^{d-1+\varepsilon}(K \setminus \bigcup \mathcal{C}) \leq \varepsilon$. Define

$$P_m = \{d \geq 1 : \forall \varepsilon > 0,\ \exists \text{ a } d\text{-dimensional pre-measure fat packing for } \varepsilon\},$$

and let $\dim_m K = \inf P_m$.

Definition 6.8. Define

$$P_l = \left\{d : \text{for the generic } f \in C(K),\ \forall y \in \mathbb{R}\ \dim_H f^{-1}(y) \leq d - 1\right\},$$

and let $\dim_l K = \inf P_l$ be the *generic level set dimension* of K.

We assume that by definition $\infty \in P_n, P_{tH}, P_s, P_m, P_l$.

Our goal is to show in Theorem 6.11 that if $\dim_t K > 0$ then our five notions of dimension coincide. One can verify that they may differ if $\dim_t K = 0$.

Remark 6.9. The restriction $d \geq 1$ in Definition 6.7 is not too artificial. It is easy to check directly that if $\dim_t K > 0$ then $P_n, P_{tH}, P_s, P_m, P_l \subseteq [1, \infty]$. However, it will also be a consequence of Fact 3.2 and Theorem 6.11.

First we need a technical lemma related to Definition 6.7.

Lemma 6.10. *Let X be a metric space, $0 \leq c < \infty$ and $c_n \searrow c$. If $\mathcal{H}_\infty^{c_n}(X) \to 0$ then $\dim_H X \leq c$.*

Proof. We may assume that $\mathcal{H}_\infty^{c_n}(X) < 1$. Fix $t > c$. Then $c_n < t$ for large enough n. It is not difficult to see that $c_n < t$ and $\mathcal{H}_\infty^{c_n}(X) < 1$ imply $\mathcal{H}_\infty^{t}(X) \leq \mathcal{H}_\infty^{c_n}(X)$. Therefore $\mathcal{H}_\infty^{c_n}(X) \to 0$ yields $\mathcal{H}_\infty^{t}(X) = 0$, which easily implies $\mathcal{H}^t(X) = 0$. Hence $\dim_H X \leq t$, and since $t > c$ was arbitrary, $\dim_H X \leq c$. □

Theorem 6.11. *If K is a compact metric space with $\dim_t K > 0$ then $P_n = P_{tH} = P_s = P_m = P_l$.*

This immediately yields.

Corollary 6.12. *If K is a compact metric space with $\dim_t K > 0$ then $\dim_n K = \dim_{tH} K = \dim_s K = \dim_m K = \dim_l K$.*

This result will be a technical tool in the sequel, but we believe that the equation $\dim_{tH} K = \dim_s K$ is particularly interesting in its own right. Let us reformulate it now.

Corollary 6.13. *If K is a compact metric space then $\dim_{tH} K$ is the smallest number d for which K can be covered by a finite family of compact sets of arbitrarily small diameter such that the set of points covered more than once has Hausdorff dimension $d - 1$.*

Next we prove Theorem 6.11. The proof will consist of five lemmas.

Lemma 6.14. *$P_n \subseteq P_{tH}$.*

Proof. Assume $d \in P_n$ and $d < \infty$. Let us fix $x_0 \in K$ and $r > 0$. To verify $d \in P_{tH}$ we need to find an open set U such that $x_0 \in U \subseteq U(x_0, r)$ and $\dim_H \partial U \leq d - 1$. We may assume $\partial U(x_0, r) \neq \emptyset$, otherwise we are done.

By $d \in P_n$ we obtain that $\mathcal{N}_d$ is dense in a ball $B(f_0, 6\varepsilon)$, $\varepsilon > 0$. By decreasing r if necessary, we may assume that $\operatorname{diam}(f_0(U(x_0, r))) \le 3\varepsilon$. Then Tietze's Extension Theorem provides an $f \in B(f_0, 6\varepsilon)$ such that $f(x_0) = f_0(x_0)$ and $f|_{\partial U(x_0,r)}(x) = f_0(x_0) + 3\varepsilon$ for every $x \in \partial U(x_0, r)$. Since $\mathcal{N}_d$ is dense in $B(f_0, 6\varepsilon)$, we can choose $g \in \mathcal{N}_d$ such that $||f - g|| \le \varepsilon$. By the construction of g it follows that $g(x_0) < \min\{g(\partial U(x_0, r))\}$. Hence in the dense set S_g (see Definition 6.3) there is an $s \in S_g$ such that

$$g(x_0) < s < \min\{g(\partial U(x_0, r))\}. \tag{6.1}$$

Let

$$U = g^{-1}((-\infty, s)) \cap U(x_0, r),$$

then clearly $x_0 \in U \subseteq U(x_0, r)$. By (6.1) we have $\partial g^{-1}((-\infty, s)) \cap \partial U(x_0, r) = \emptyset$, therefore $\partial U \subseteq \partial g^{-1}((-\infty, s)) \subseteq g^{-1}(s)$. Using $s \in S_g$ we infer that $\dim_H \partial U \le \dim_H g^{-1}(s) \le d - 1$. □

Lemma 6.15. $P_{tH} \subseteq P_s$.

Proof. Assume $d \in P_{tH}$ and $d < \infty$. Fix $\varepsilon > 0$, and let $\mathcal{U}$ be an open basis of K such that $\dim_H \partial U \le d - 1$ for all $U \in \mathcal{U}$. Then $\{U \in \mathcal{U} : \operatorname{diam} U \le \varepsilon\}$ covers K, hence by compactness there exists a finite subcover $\{U_i\}_{i=1}^k$. Then $\operatorname{diam} U_i \le \varepsilon$ and $\dim_H \partial U_i \le d - 1$ for all i. Let us now consider all the 2^k possible sets of the form $U_1^{\alpha_1} \cap \dots \cap U_k^{\alpha_k}$, where every $\alpha_i \in \{1, -1\}$, and $U_i^1 = \operatorname{cl} U_i$, and $U_i^{-1} = K \setminus U_i$. Let $\mathcal{C}$ be the family consisting of the sets of the above form, then it is easy to check that $\mathcal{C}$ covers K and also that $\operatorname{diam} C \le \varepsilon$ for every $C \in \mathcal{C}$ (note that $\operatorname{diam}(U_1^{-1} \cap \dots \cap U_k^{-1}) \le \varepsilon$ holds simply because $U_1^{-1} \cap \dots \cap U_k^{-1} = \emptyset$). Moreover, one can check that $T(\mathcal{C}) \subseteq \bigcup_{i=1}^k \partial U_i$, hence $\dim_H T(\mathcal{C}) \le \dim_H \bigcup_{i=1}^k \partial U_i \le d - 1$. Therefore, $\mathcal{C}$ is a d-dimensional small cover for $\varepsilon > 0$, hence $d \in P_s$. □

Lemma 6.16. $P_s \subseteq P_m$.

Proof. Assume $d \in P_s$ and $d < \infty$. First we check that $\dim_t K > 0$ implies $d \ge 1$. If $d < 1$ and for every $\varepsilon > 0$ there exists a d-dimensional small cover for ε then these covers are actually finite partitions (since $d - 1 < 0$) into compact sets, but then these sets are also open, hence K has a clopen basis, which is impossible.

Fix $\varepsilon > 0$. Let $\mathcal{C}$ be a d-dimensional small cover for ε. Since $\dim_H T(\mathcal{C}) \le d - 1$, we can choose an open set V such that $T(\mathcal{C}) \subseteq V$ and $\mathcal{H}^{d-1+\varepsilon}_\infty(V) \le \varepsilon$. Let $\mathcal{C}' = \{C \setminus V : C \in \mathcal{C}\}$, then $\mathcal{C}'$ is a finite family of disjoint compact sets. Clearly, $\operatorname{diam} C' \le \varepsilon$ for every $C' \in \mathcal{C}'$. Since $K \setminus \bigcup \mathcal{C}' = V$, we also have $\mathcal{H}^{d-1+\varepsilon}_\infty(K \setminus \bigcup \mathcal{C}') \le \varepsilon$, hence $\mathcal{C}'$ is a d-dimensional pre-measure fat packing for ε and thus $d \in P_m$. □

Lemma 6.17. $P_m \subseteq P_l$.

Proof. Assume $d \in P_m$ and $d < \infty$. By definition, $d \ge 1$. First assume $d > 1$. For $n \in \mathbb{N}^+$ let $\mathcal{C}^n = \{C_1^n, \dots, C_{k_n}^n\}$ be a d-dimensional pre-measure fat packing for $1/n$, that is, $\mathcal{C}^n$ consists of disjoint compact sets, for all $i \in \{1, \dots, k_n\}$

$$\operatorname{diam} C_i^n \le \frac{1}{n}, \tag{6.2}$$

and for the open set $V^n = K \setminus \bigcup_{i=1}^{k_n} C_i^n$ we have

$$\mathcal{H}^{d-1+\frac{1}{n}}_\infty(V^n) = \mathcal{H}^{d-1+\frac{1}{n}}_\infty\left(K \setminus \bigcup \mathcal{C}^n\right) \le \frac{1}{n}. \tag{6.3}$$

Let

$$\mathcal{R}(n) = \left\{(r_1, \dots, r_{k_n}) \in \mathbb{Q}^{k_n} : r_i \neq r_j \text{ if } i \neq j\right\},$$

and let $f_{n,r_1,\dots,r_{k_n}} \in C(K)$ be a continuous function that is constant r_i on C_i^n for every $i \in \{1, \dots, k_n\}$. It is not difficult to see (using that every element of $C(K)$ is uniformly continuous) that for every $N \in \mathbb{N}^+$ the set

$$\{f_{n,r_1,\dots,r_{k_n}} : n \geq N,\ (r_1, \dots, r_{k_n}) \in \mathcal{R}(n)\}$$

is dense in $C(K)$.

Set

$$\delta_{r_1,\dots,r_{k_n}} = \min\left\{\frac{1}{n}, \frac{|r_i - r_j|}{3} : i, j \in \{1, \dots, k_n\},\ i \neq j\right\} > 0,$$

then

$$\mathcal{G}(N) = \bigcup_{n \geq N} \bigcup_{(r_1,\dots,r_{k_n}) \in \mathcal{R}(n)} U(f_{n,r_1,\dots,r_{k_n}}, \delta_{r_1,\dots,r_{k_n}})$$

is dense open in $C(K)$ for all $N \in \mathbb{N}^+$. Therefore

$$\mathcal{G} = \bigcap_{N \in \mathbb{N}^+} \mathcal{G}(N)$$

is co-meager in $C(K)$.

It remains to prove that for all $f \in \mathcal{G}$ all level sets of f are of Hausdorff dimension at most $d-1$. Fix $f \in \mathcal{G}$ and $y \in \mathbb{R}$. By the definition of $\mathcal{G}$, there are infinitely many $n \in \mathbb{N}^+$ such that f is in one of the $U(f_{n,r_1,\dots,r_{k_n}}, \delta_{r_1,\dots,r_{k_n}})$'s. For every such n there exists $i = i(y, n)$ such that

$$f^{-1}(y) \subseteq C_i^n \cup V^n.$$

Using (6.2), (6.3) (C_i^n is covered by itself when estimating $\mathcal{H}_\infty^{d-1+\frac{1}{n}}(C_i^n)$) and finally $d > 1$ we obtain

$$\mathcal{H}_\infty^{d-1+\frac{1}{n}}\left(f^{-1}(y)\right) \leq \mathcal{H}_\infty^{d-1+\frac{1}{n}}(C_i^n) + \mathcal{H}_\infty^{d-1+\frac{1}{n}}(V^n) \leq \left(\frac{1}{n}\right)^{d-1+\frac{1}{n}} + \frac{1}{n} \to 0 \text{ as } n \to \infty. \tag{6.4}$$

Let us now fix a sequence n_k of integers for which (6.4) holds. By applying Lemma 6.10 for $c_k = d - 1 + \frac{1}{n_k} \searrow d - 1$ we obtain that for all $f \in \mathcal{G}$ and $y \in \mathbb{R}$

$$\dim_H f^{-1}(y) \leq d - 1.$$

Therefore, $d \in P_l$.

Let us now consider $d = 1$. Fix a sequence $d_n \searrow 1$. By the previous case we have $d_n \in P_l$ for all $n \in \mathbb{N}$, hence there exist co-meager sets $\mathcal{F}_n \subseteq C(K)$ such that $\dim_H f^{-1}(y) \leq d_n - 1$ for all $f \in \mathcal{F}_n$ and $y \in \mathbb{R}$. The set $\mathcal{F} = \bigcap_{n \in \mathbb{N}} \mathcal{F}_n$ is co-meager in $C(K)$, and obviously $\dim_H f^{-1}(y) \leq \lim_{n\to\infty} (d_n - 1) = 0$ for all $f \in \mathcal{F}$ and $y \in \mathbb{R}$. Thus $1 \in P_l$. $\square$

Lemma 6.18. $P_l \subseteq P_n$.

Proof. Assume $d \in P_l$ and $d < \infty$. By the definition of P_l, for the generic continuous function f we have $\dim_H f^{-1}(y) \leq d-1$ for all $y \in \mathbb{R}$. Hence $\mathcal{N}_d$ is co-meager, thus (everywhere) dense. Hence $d \in P_n$. $\square$

This concludes the proof of Theorem 6.11.

For the proof of the main theorem of this section we will need that the infima in the above definitions are actually attained.

Lemma 6.19. *If K is a compact metric space with $\dim_t K > 0$ then $\dim_n K = \min P_n$, $\dim_{tH} K = \min P_{tH}$, $\dim_s K = \min P_s$, $\dim_m K = \min P_m$, and $\dim_l K = \min P_l$.*

Proof. By Theorem 6.11 it suffices to prove that P_m has a minimal element. We may assume $P_m \neq \{\infty\}$. Let $d = \inf P_m$. Fix $\varepsilon > 0$. There exists $d' \in P_m$ such that $d' \geq d$ and $d' - d < \varepsilon$. Set $\varepsilon' = \varepsilon - (d' - d)$, then $0 < \varepsilon' \leq \varepsilon$ and $d' - 1 + \varepsilon' = d - 1 + \varepsilon$. By $d' \in P_m$ there exists a d'-dimensional pre-measure fat packing $\mathcal{C}$ for ε', that is, a finite disjoint family of compact sets such that $\operatorname{diam} C \leq \varepsilon'$ for all $C \in \mathcal{C}$ and $\mathcal{H}^{d'-1+\varepsilon'}_\infty(K \setminus \bigcup \mathcal{C}) \leq \varepsilon'$. But then by $\varepsilon' \leq \varepsilon$ and $d' - 1 + \varepsilon' = d - 1 + \varepsilon$ we obtain that $\mathcal{C}$ is also d-dimensional pre-measure fat packing for ε, and hence $d \in P_m$. $\square$

Now we are ready to describe the Hausdorff dimension of the level sets of generic continuous functions.

As already mentioned above, if $\dim_t K = 0$ then every level set of a generic continuous function on K consists of at most one point.

Theorem 6.20. *If K is a compact metric space with $\dim_t K > 0$ then for the generic $f \in C(K)$*

(i) $\dim_H f^{-1}(y) \leq \dim_{tH} K - 1$ *for every* $y \in \mathbb{R}$,
(ii) *for every* $\varepsilon > 0$ *there exists a non-degenerate interval* $I_{f,\varepsilon}$ *such that* $\dim_H f^{-1}(y) \geq \dim_{tH} K - 1 - \varepsilon$ *for every* $y \in I_{f,\varepsilon}$.

Proof. By Lemma 6.19 we have $\dim_l K = \min P_l$ and hence $\dim_l K \in P_l$. By the definition of P_l and using Corollary 6.12 we deduce that there is a co-meager set $\mathcal{F} \subseteq C(K)$ such that for every $f \in \mathcal{F}$ and $y \in \mathbb{R}$

$$\dim_H f^{-1}(y) \leq \dim_l K - 1 = \dim_{tH} K - 1,$$

therefore (i) holds.

Let us now prove (ii). Clearly, by Theorem 6.11, $\dim_{tH} K - \frac{1}{k} < \dim_{tH} K = \dim_n K$ for every $k \in \mathbb{N}^+$. Hence $\mathcal{N}_{\dim_{tH} K - \frac{1}{k}}$ is nowhere dense by the definition of $\dim_n K$. It follows from the definition of $\mathcal{N}_d$ that for every $f \in C(K) \setminus \mathcal{N}_{\dim_{tH} K - \frac{1}{k}}$ there exists a non-trivial interval $I_{f,\frac{1}{k}}$ such that $\dim_H f^{-1}(y) \geq \dim_{tH} K - 1 - \frac{1}{k}$ for every $y \in I_{f,\frac{1}{k}}$. But then (ii) holds for every $f \in C(K) \setminus (\bigcup_{k \in \mathbb{N}^+} \mathcal{N}_{\dim_{tH} K - \frac{1}{k}})$, and this latter set is clearly co-meager, which concludes the proof of the theorem. $\square$

This immediately implies

Corollary 6.21. *If K is a compact metric space with $\dim_t K > 0$ then $\sup\{\dim_H f^{-1}(y) : y \in \mathbb{R}\} = \dim_{tH} K - 1$ for the generic $f \in C(K)$.*

6.2. **Homogeneous and self-similar compact metric spaces.** In this section we show that if the compact metric space is sufficiently homogeneous, e.g. self-similar (see [4] or [14]) then we can say much more.

Theorem 6.22. *Let K be a compact metric space with $\dim_t K > 0$ such that $\dim_{tH} B(x,r) = \dim_{tH} K$ for every $x \in K$ and $r > 0$. Then for the generic $f \in C(K)$ for the generic $y \in f(K)$ we have*

$$\dim_H f^{-1}(y) = \dim_{tH} K - 1.$$

Before turning to the proof of this theorem we formulate a corollary. Recall that K is self-similar if there are contractive similitudes $\varphi_1, \dots, \varphi_k : K \to K$ such that $K = \bigcup_{i=1}^k \varphi_i(K)$. The sets of the form $\varphi_{i_1} \circ \varphi_{i_2} \circ \dots \varphi_{i_m}(K)$ are called the elementary pieces of K. It is easy to see that every ball in K contains an elementary piece. Moreover, by Corollary 3.8 the topological Hausdorff dimension of every elementary piece is $\dim_{tH} K$. Hence, using monotonicity as well, we obtain that if K is self-similar then $\dim_{tH} B(x,r) = \dim_{tH} K$ for every $x \in K$ and $r > 0$. This yields the following.

Corollary 6.23. *Let K be a self-similar compact metric space with $\dim_t K > 0$. Then for the generic $f \in C(K)$ for the generic $y \in f(K)$ we have*

$$\dim_H f^{-1}(y) = \dim_{tH} K - 1.$$

Before proving Theorem 6.22 we need a lemma.

Lemma 6.24. *Let $K_1 \subseteq K_2$ be compact metric spaces and*

$$R \colon C(K_2) \to C(K_1), \quad R(f) = f|_{K_1}.$$

If $\mathcal{F} \subseteq C(K_1)$ is co-meager then so is $R^{-1}(\mathcal{F}) \subseteq C(K_2)$.

Proof. The map R is clearly continuous. Using the Tietze Extension Theorem it is not difficult to see that it is also open. We may assume that $\mathcal{F}$ is a dense G_δ set in $C(K_1)$. The continuity of R implies that $R^{-1}(\mathcal{F})$ is also G_δ, thus it is enough to prove that $R^{-1}(\mathcal{F})$ is dense in $C(K_2)$. Let $\mathcal{U} \subseteq C(K_2)$ be non-empty open, then $R(\mathcal{U}) \subseteq C(K_1)$ is also non-empty open, hence $R(\mathcal{U}) \cap \mathcal{F} \neq \emptyset$, and therefore $\mathcal{U} \cap R^{-1}(\mathcal{F}) \neq \emptyset$. □

Proof of Theorem 6.22. Theorem 6.20 implies that for the generic $f \in C(K)$ for *every* $y \in \mathbb{R}$ we have $\dim_H f^{-1}(y) \leq \dim_{tH} K - 1$, so we only have to prove the opposite inequality.

For $f \in C(K)$ and $\varepsilon > 0$ let

$$L_{f,\varepsilon} = \{y \in f(K) : \dim_H f^{-1}(y) \geq \dim_{tH} K - 1 - \varepsilon\}.$$

First we show that it suffices to construct for every $\varepsilon \in (0,1)$ a co-meager set $\mathcal{F}_\varepsilon \subseteq C(K)$ such that for every $f \in \mathcal{F}_\varepsilon$ the set $L_{f,\varepsilon}$ is co-meager in $f(K)$. Indeed, then the set $\mathcal{F} = \bigcap_{k \in \mathbb{N}, k \geq 2} \mathcal{F}_{\frac{1}{k}} \subseteq C(K)$ is co-meager, and for every $f \in \mathcal{F}$ the set $L_f = \bigcap_{k \in \mathbb{N}, k \geq 2} L_{f, \frac{1}{k}} \subseteq f(K)$ is also co-meager. Since for every $y \in L_f$ clearly $\dim_H f^{-1}(y) \geq \dim_{tH} K - 1$, this finishes the proof.

Let us now construct such an $\mathcal{F}_\varepsilon$ for a fixed $\varepsilon \in (0,1)$. Let $\{B_n\}_{n \in \mathbb{N}}$ be a countable basis of K consisting of closed balls, and for all $n \in \mathbb{N}$ let $R_n \colon C(K) \to C(B_n)$ be defined as

$$R_n(f) = f|_{B_n}.$$

Let us also define

$$\mathcal{B}_n = \left\{ f \in C(B_n) : \exists I_{f,\varepsilon} \text{ s. t. } \forall y \in I_{f,\varepsilon} \ \dim_H f^{-1}(y) \geq \dim_{tH} K - 1 - \varepsilon \right\},$$

(where $I_{f,\varepsilon}$ is understood to be a non-degenerate interval). Finally, let us define

$$\mathcal{F}_\varepsilon = \bigcap_{n\in\mathbb{N}} R_n^{-1}(\mathcal{B}_n).$$

First we show that $\mathcal{F}_\varepsilon$ is co-meager. By our assumption $\dim_{tH} B_n = \dim_{tH} K > \dim_{tH} K - \varepsilon$ (which also implies $\dim_t B_n > 0$ by Fact 3.1, since $\dim_{tH} K \geq 1$ and $\varepsilon < 1$), thus Theorem 6.20 yields that $\mathcal{B}_n$ is co-meager in $C(B_n)$. Lemma 6.24 implies that $R_n^{-1}(\mathcal{B}_n)$ is co-meager in $C(K)$ for all $n \in \mathbb{N}$, thus $\mathcal{F}_\varepsilon$ is also co-meager.

It remains to show that for every $f \in \mathcal{F}_\varepsilon$ the set $L_{f,\varepsilon}$ is co-meager in $f(K)$. Let us fix $f \in \mathcal{F}_\varepsilon$. We will actually show that $L_{f,\varepsilon}$ contains an open set in $\mathbb{R}$ which is a dense subset of $f(K)$. So let $U \subseteq \mathbb{R}$ be an open set in $\mathbb{R}$ such that $f(K) \cap U \neq \emptyset$. It is enough to prove that $L_{f,\varepsilon} \cap U$ contains an interval. Since the B_n's form a basis, the continuity of f implies that there exists an $n \in \mathbb{N}$ such that $f(B_n) \subseteq U$. It is easy to see using the definition of $\mathcal{F}_\varepsilon$ that $f|_{B_n} \in \mathcal{B}_n$, so there exists a non-degenerate interval $I_{f|_{B_n},\varepsilon}$ such that for all $y \in I_{f|_{B_n},\varepsilon}$ we have

$$\dim_H f^{-1}(y) \geq \dim_H \left(f|_{B_n}\right)^{-1}(y) \geq \dim_{tH} K - 1 - \varepsilon.$$

Thus $I_{f|_{B_n},\varepsilon} \subseteq L_{f,\varepsilon}$. On the other hand, as we saw above, $\dim_{tH} K - \varepsilon > 0$. Hence, $\dim_{tH} K - 1 - \varepsilon > -1$ which implies $\left(f|_{B_n}\right)^{-1}(y) \neq \emptyset$ for every $y \in I_{f|_{B_n},\varepsilon}$, thus $I_{f|_{B_n},\varepsilon} \subseteq f(B_n)$. But it follows from $f(B_n) \subseteq U$ that $I_{f|_{B_n},\varepsilon} \subseteq U$. Hence $I_{f|_{B_n},\varepsilon} \subseteq L_{f,\varepsilon} \cap U$ and this completes the proof. □

7. Open Problems

First let us recall the most interesting open problem.

Problem 4.8. Determine the almost sure topological Hausdorff dimension of the trail of the d-dimensional Brownian motion for $d = 2$ or 3.

Now we collect a few more open problems.

Problem 7.1. Let $B \subseteq \mathbb{R}^d$ be a Borel set and $\varepsilon > 0$. Does there exist a compact set $K \subseteq B$ with $\dim_{tH} K \geq \dim_{tH} B - \varepsilon$?

Problem 7.2. Let $B \subseteq \mathbb{R}^d$ be a Borel set and $1 \leq c < \dim_{tH} B$ arbitrary. Does there exist a Borel set $B' \subseteq B$ with $\dim_{tH} B' = c$?

The next problem is somewhat vague. It is motivated by the proof of Theorem 5.2.

Problem 7.3. Is there some sort of structural characterization of the sets with topological Hausdorff dimension at least c? For example, is it true that a Borel set $B \subseteq \mathbb{R}^d$ satisfies $\dim_{tH} B \geq c$ iff it contains a disjoint family of non-degenerate connected sets such that each set meeting all members of this family is of Hausdorff dimension at least $c - 1$?

In a somewhat similar vein, is there some sort of analogue of Frostman's Lemma? (See e.g. [4] or [14].)

Moreover, it would also be interesting to know whether the theory of the topological packing dimension and topological box-counting dimension (defined analogously to $\dim_{tH} B$ in the obvious way) differs significantly from ours.

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