PSEUDOARCS, PSEUDOCIRCLES, LAKES OF WADA AND GENERIC MAPS ON $S^2$

ZOLTÁN BUCZOLICH AND UDAYAN B. DARJI

**Abstract.** We prove a Bruckner-Garg type theorem for the fiber structure of a generic map from a continuum $X$ into the unit interval $I$. We also study the specific case of $X = S^2$. We show that each nondegenerate component of each fiber of a generic map in $C(S^2, I)$ is figure-eight-like. This together with a result by Krasinkiewicz and Levin gives that each nondegenerate component of each fiber of a generic map in $C(S^2, I)$ is hereditarily indecomposable and figure-eight-like. We also show that pseudoarcs, pseudocircles and Lakes of Wada appear in abundance in fibers of a generic map in $C(S^2, I)$. We also exhibit a general method for proving when a $P$-like hereditarily indecomposable continuum is $Q$-like when $Q$ is a certain graph containing $P$.

1. **Introduction**

The fiber structure of a generic continuous function on a closed interval has been extensively studied in real analysis by various authors. The classical Bruckner-Garg Theorem [1] describes the fiber structure of a generic continuous function. Morayne and the second author described the fiber structure of a generic smooth function in [6]. D’Aniello and the second author described the “worst case” behavior of the fiber structure of smooth functions in [5]. Kirchheim in [8] described the Hausdorff dimension of fibers of a generic map from $\mathbb{R}^n$ into $\mathbb{R}^m$. Further measure theoretic properties of generic continuous functions and monotone continuous functions defined on an interval were recently studied by the first author in [2] and [3]. In this paper, we study the fiber structure of a generic map from a continuum $X$ into the unit interval $I$ from a topological viewpoint.

Our first result shows that when viewed appropriately, the Bruckner-Garg Theorem holds in a very general setting. Let us first state the Bruckner-Garg Theorem. In the following, $X$ is the unit interval.

**Theorem 1.1.** A generic $f \in C(X, I)$ has the property that there is a countable dense set $D \subseteq (\min f, \max f)$ such that

- $f^{-1}(y)$ is a singleton set if $y \in \{\min f, \max f\}$,
- $f^{-1}(y)$ is homeomorphic to a Cantor set when $y \in (\min f, \max f) \setminus D$,
- $f^{-1}(y)$ is homeomorphic to the union of a Cantor set and an isolated point when $y \in D$.

The first listed author received support from the Hungarian National Foundation for Scientific Research Grants T032012 and T049727. The second listed author acknowledges the support of the Fulbright Foundation as well as the hospitality of Eötvös Loránd University.

The authors would like to thank Wayne Lewis and Tom Ingram for bringing several references as well as classical results to authors’ attention.

The authors would like thank the referee for a careful reading of the paper.
Theorem 3.10 shows that the above theorem holds for a nondegenerate continuum $X$ when $f^{-1}(y)$ is replaced by $\text{Comp}(f^{-1}(y))$, the space whose elements are components of $f^{-1}(y)$ and the topology is the upper semicontinuous topology. Moreover, the isolated points of $\text{Comp}(f^{-1}(y))$ are actually degenerate continua where $f$ attains local extrema. Since it is easy to show that a generic function from the interval into the interval is nowhere constant, the Bruckner-Garg Theorem is a special case of Theorem 3.10. Our proof of Theorem 3.10 makes an extensive use of the Boundary Bumping Theorem.

Krasinkiewicz [10] and Levin [11] independently showed that a generic map from a compactum $X$ into $I$ has the property that each of its fibers is a Bing compactum, a compactum all of whose components are hereditarily indecomposable. This gives us information about individual components of a generic map while Theorem 3.10 gives us information about the global structure of the fibers. As the case of $X = I$ is completely described, we investigate the higher dimensional phenomenon. To avoid unnecessary technical problems at the boundary and separation of cases, we work with $X = S^2$, the 2-sphere in $\mathbb{R}^3$, instead of $X = I^2$.

We know the global structure of fibers of a generic map $f \in C(S^2, I)$ and we know that each component of each fiber must be hereditarily indecomposable. However, there are many nonhomeomorphic hereditarily indecomposable continua on $S^2$. Our first step in understanding these components is to show that each such component must be either a point or figure-eight-like. As an intermediate step, we show using Krasinkiewicz’s characterization of hereditarily indecomposable continua [9] that every hereditarily indecomposable continuum which is $P$-like is $Q$-like where $Q$ is figure-eight and $P$ is any nondegenerate subcontinuum of $Q$. The technique developed for this result is general and may be useful elsewhere.

We next show that the pseudoarc, and the pseudocircle occur naturally as the components of the fibers of a generic map $f \in C(S^2, I)$. Indeed, we show that for a generic map $f \in C(S^2, I)$, for almost all $y \in f(S^2)$, all components of $f^{-1}(y)$ are either points, pseudoarcs or pseudocircles. Perhaps, the more surprising fact is that the Lakes of Wada continuum appears as component. Recall that a Lakes of Wada continuum is a hereditarily indecomposable continuum $M$ which separates $S^2$ into three pieces and is the boundary of each component of $S^2 \setminus M$. A “smooth” map from $S^2$ into $I$ may have saddle points and these saddle points are stable under small perturbation. They translate into Lakes of Wada continua in the generic case. Indeed, for a generic map $f \in C(S^2, I)$ there is a countable dense set $D \subseteq f(S^2)$ such that for each $y \in D$, there is a component of $f^{-1}(y)$ which is a Lakes of Wada continuum.

2. Definitions and Background Information

In this section we state background terminology and theorems, most of which can be found in [12] and [14].

Let $X$ be a complete metric space. The boundary and the closure of a subset $A$ will be denoted by $\partial A$ and $\overline{A}$, respectively. A set $M \subseteq X$ is of first category if it is the countable union of nowhere dense sets. We say that a generic $x \in X$ satisfies property $P$ means that $\{x \in X : x \text{ does not satisfy property } P\}$ is of first category in $X$. Often, particularly in real analysis, the word “typical” is used instead of generic. We will also say that $M \subseteq X$ is residual in $X$. This simply means $X \setminus M$ is of first category.
Let $X$ be a Polish space, i.e. a complete, separable metric space, then $\mathcal{K}(X)$ denotes the set of all compact subsets of $X$ with the Hausdorff metric, $d_H$. Recall that $\mathcal{K}(X)$ is Polish and if $X$ is compact then so is $\mathcal{K}(X)$.

If $X$ is a compact metric space, then $C(X,I)$ denotes the set of continuous functions from $X$ into the unit interval $I$ endowed with sup norm. We recall that $C(X,I)$ is a Polish space. A map is simply a continuous function. We use $S^2$ to denote the 2-sphere in $\mathbb{R}^3$. We point out $C(S^2, I)$ is homeomorphic to the set of all continuous functions $f : \mathbb{R}^2 \to I$ which have a limit at infinity.

A continuum is a compact connected metric space. A continuum is degenerate if it has only one point. Otherwise, we say that it is nondegenerate. The following fact about continua will be used frequently and can be found in Chapter V of [14].

**Theorem 2.1. (Boundary Bumping Theorem)** Suppose that $X$ is a continuum, $U$ is open in $X$, $U \neq X$ and $p \in U$. Then, there is a connected set $C$ containing $p$ and contained in $U$ whose closure intersects the boundary of $U$.

We also use the following fact which follows easily from the Boundary Bumping Theorem. If $U$ is any nonempty open subset of a nondegenerate continuum $X$, then $U$ contains a nondegenerate continuum.

We say that a continuum $P$ is a graph (see Chapter IX, [14]) if there are arcs $A_1, A_2, \ldots, A_n$ such that $P = \bigcup_{i=1}^n A_i$ and for $i \neq j$, $A_i \cap A_j$ has at most one point and this point is an endpoint of $A_i$ as well as an endpoint of $A_j$. We call such $A_1, A_2, \ldots, A_n$ a defining sequence for the graph $P$. If $x \in P$ is an endpoint of some $A_i$ and $x$ belongs to no other $A_j$'s, we call $x$ an endpoint of $P$. We note that this is equivalent to the usual definition of a point being an endpoint of a continuum, i.e. a point where the continuum has order one. Again refer to Chapter IX of [14] for more details. If $a$ and $b$ are points of a given arc $A$ then we will use the notation $[a, b]$ for the subarc determined by $a$ and $b$.

A map $f$ from a metric space $X$ onto a metric space $Y$ is an $\epsilon$-map means that $\epsilon > 0$ and $diam(f^{-1}(y)) < \epsilon$ for every $y \in Y$. We say that continuum $X$ is $P$-like if for every $\epsilon > 0$, there is an $\epsilon$-map from $X$ onto $P$. We are generally interested in continuum which are $P$-like for some graph $P$. The reader may refer to Chapter II of [14] for basic facts about $P$-like continua. A continuum which is arc-like (circle-like) is often called chainable (circularly chainable). Figure-eight is a continuum homeomorphic to the union of two circles which intersect in exactly one point.

A continuum is decomposable if it is the union of two proper subcontinua. Otherwise we call it indecomposable. A continuum is hereditarily indecomposable if each of its subcontinua is indecomposable.

A subset $C$ of a continuum $X$ is a composant of $X$ means that there is $p \in X$ such that $C = \bigcup\{K : K$ is a proper subcontinuum of $X$ containing $p\}$. We recall that each composant of $X$ is dense in $X$. We also recall that a nondegenerate continuum is indecomposable iff it has uncountably many pairwise disjoint composants. We refer the reader to Chapter XI of [14] for proofs of these statements and further information on composants.

A pseudoarc is a hereditarily indecomposable continuum which is arc-like. We recall that up to homeomorphism the pseudoarc is unique. For more information on the pseudoarc, the reader may refer to Lewis’s comprehensive survey article [13]. A pseudocircle is a hereditarily indecomposable continuum which is circle-like. There are uncountably many nonhomeomorphic pseudocircles. However, in the plane or $S^2$, up to homeomorphism, there is only one pseudocircle. We again refer
the reader to [13] for further information. We call a continuum \( X \subseteq S^2 \) a Lakes of Wada continuum if it is hereditarily indecomposable, \( S^2 \setminus X \) has exactly three components and \( X \) is the boundary of each of these components.

We next recall the definition and the basic properties of upper semicontinuous decompositions of a space and the corresponding topology.

Suppose that \( X \) is a compact metric space and \( D \) is a decomposition of \( X \) into closed sets. For each open set \( U \) in \( X \), we define

\[
[U] = \{ A \in D : A \subseteq U \}.
\]

If \( \bigcup [U] \) is open in \( X \) for each open set \( U \subseteq X \), then we say that \( D \) is an upper semicontinuous decomposition of \( X \). The collection \( \{ [U] : U \text{ is open in } X \} \) is a basis for a topology on \( D \) and the corresponding topology will be referred to as the upper semicontinuous topology on \( X \) generated by \( D \). We remark that the condition in the definition of an upper semicontinuous decomposition is equivalent to saying that for each closed set \( F \) in \( X \), the following set is closed

\[
\bigcup \{ A \in D : A \cap F \neq \emptyset \}.
\]

For notational convenience, we use \( [F] \) to denote the set \( \{ A \in D : A \cap F \neq \emptyset \} \). The following is a standard fact about upper semicontinuous topology.

\[ \text{Proposition 2.2.} \quad \text{(Theorem 3.9, [14]) Suppose that } X \text{ is a compact metric space and } D \text{ is an upper semicontinuous decomposition of } X. \text{ Then, } D \text{ is a compact metric space.} \]

Throughout the rest of this section, we assume that \( X \) is a compact metric space.

We use \( \text{Comp}(X) \) to denote the set of components of \( X \).

\[ \text{Proposition 2.3.} \quad \text{Comp}(X) \text{ is an upper semicontinuous decomposition of } X \text{ and hence a compact metric space.} \]

\[ \text{Proof.} \quad \text{Let } F \text{ be a closed set in } X \text{ and let } p \text{ be a limit point of } \bigcup [F]. \text{ Let } \{ p_i \} \text{ be a sequence of points in } \bigcup [F] \text{ which converges to } p \text{ and let } A_i \in [F] \text{ be such that } p_i \in A_i. \text{ As each } A_i \text{ is a continuum in the compact space } X, \text{ some subsequence of } \{ A_i \} \text{ converges in the Hausdorff metric to a continuum } A. \text{ Since each } A_i \cap F \neq \emptyset, \text{ we have that } A \cap F \neq \emptyset. \text{ Since } p \text{ is the limit of } \{ p_i \}, \text{ } p \in A. \text{ Therefore, } B, \text{ the component of } X \text{ containing } A, \text{ is in } [F]. \text{ Hence, } p \in \bigcup [F] \text{ which, in turn, implies that } \bigcup [F] \text{ is closed and } \text{Comp}(X) \text{ is upper semicontinuous.} \]

\[ \text{Theorem 2.4.} \quad \text{Comp}(X) \text{ is totally disconnected, i.e. the only components of } \text{Comp}(X) \text{ are singletons.} \]

\[ \text{Proof.} \quad \text{Let } M \subseteq \text{Comp}(X) \text{ be a continuum. It will suffice to show that } \bigcup M \text{ is a continuum in } X. \text{ It is easy to verify that } \bigcup M \text{ is closed. To obtain a contradiction, assume that } \bigcup M \text{ is not connected. Let } H, K \text{ be two disjoint nonempty closed sets in } X \text{ such that } \bigcup M = H \cup K. \text{ As each } A \in M \text{ is connected, we have that } A \subseteq H \text{ or } A \subseteq K. \text{ Let } M_H = \{ A \in M : A \subseteq H \} \text{ and } M_K = \{ A \in M : A \subseteq K \}. \text{ Then, } M_H \text{ and } M_K \text{ are two disjoint nonempty closed sets in } \text{Comp}(X) \text{ whose union is } M, \text{ contradicting that } M \text{ is connected.} \]

We make a frequent use of the Tietze Extension Theorem. We state it below for a compact metric space, the context in which it is used.
Theorem 2.5. (Tietze Extension Theorem [12]) Suppose that $X$ is a compact metric space, $M$ is a closed subset of $X$ and $f : M \to \mathbb{R}$ is a continuous function. Then, there is a continuous extension $g$ of $f$ to $X$ such that the range of $g$ is a subset of $[\min f, \max f]$.

3. Bruckner-Garg Theorem for Continua

In this section we prove the Bruckner-Garg Theorem for continua. The following result is well-known and simply follows from the fact that $X$ is separable. Throughout this section, $X$ is at least a compact metric space. After the next several results, $X$ is assumed to be a nondegenerate continuum.

Theorem 3.1. For each $f \in C(X, I)$, there are at most countably many values in $I$ where $f$ attains local extrema.

Lemma 3.2. Let $U = (u_1, u_2) \subseteq I$, $f \in C(X, I)$, and $\epsilon > 0$. Then, there is a $g \in C(X, I)$ and $\gamma > 0$ such that

1. $\|f - g\| < \epsilon$,
2. either $g(X) \cap U = \emptyset$, or $g(X) \cup U$ contains a local maximum as well as a local minimum of $g$.

Moreover, if $h \in C(X, I)$ and $\|g - h\| < \gamma$, then $h$ satisfies the above properties as well.

Proof. Suppose that $f(X) \cap U = \emptyset$. Then, $f$ can be slightly modified to a continuous function $g$ so that $g(X) \cap \overline{U} = \emptyset$. Let $\gamma > 0$ be less than the distance which separates $g(X)$ and $\overline{U}$.

Let us now consider the case when $f(X) \cap U \neq \emptyset$. Let $p \in X$ be such that $f(p) \in U$. Let $\delta > 0$ be small enough so that $\mathrm{osc}(f, B_\delta(p)) < \frac{\delta}{4}$ and $[\alpha, \beta] \subseteq U$ where $\alpha = \inf(f(B_\delta(p)))$ and $\beta = \sup(f(B_\delta(p)))$. Let $a - \frac{\delta}{4} < r < a$ and $b < s < b + \frac{\delta}{4}$ be such that $[r, s] \subseteq U$. If $B_\delta(p)$ contains only one point of $X$, then we let $g = f$ and $\gamma = \frac{1}{4} \cdot \min\{\epsilon, |u_1 - r|, |u_2 - s|\}$ and we are done. So let us assume that $B_\delta(p)$ contains at least two points. Let $p_1, p_2 \in B_\delta(p)$ and $\tilde{g} : (X \setminus B_\delta(p)) \cup \{p_1, p_2\} \to I$ be defined by $\tilde{g}(x) = f(x)$ if $x \in X \setminus B_\delta(p)$, $\tilde{g}(p_1) = r$ and $\tilde{g}(p_2) = s$. Let $g$ be a continuous extension of $\tilde{g}$ to $g$ so that $g(B_\delta(p)) \subseteq [r, s]$. Note that $\|g - f\| < \frac{\delta}{4}$ and $g(p_1)$ and $g(p_2)$ are local minimum and local maximum values of $g$, respectively. Let $\gamma = \frac{1}{4} \min\{|r-a|, |s-b|, |r-u_1|, |s-u_2|, \epsilon\}$. Then, $g, \gamma$ are the desired objects. \qed

Theorem 3.3. A generic $f \in C(X, I)$ has the property that its extreme values are dense in $f(X)$.

Proof. For each open interval $U_i$ with rational endpoints, let $\mathcal{G}_i = \{f \in C(X, I) : \text{either } f(X) \cap U_i = \emptyset, \text{ or } f(X) \cap U_i \text{ contains an extreme value}\}$. By Lemma 3.2 we have that $\mathcal{G}_i$ contains a dense open subset of $C(X, I)$. Hence, $\mathcal{G} = \bigcap_{i=1}^{\infty} \mathcal{G}_i$ is the desired residual set. \qed

The following theorem follows in a fashion similar to the above.

Theorem 3.4. A generic $f \in C(X, I)$ has the property that there is a dense subset of $X$ (depending on $f$) where $f$ attains a local extremum.

Throughout the rest of this section, $X$ is assumed to be a nondegenerate continuum in addition to being a compact metric space.
Lemma 3.5. Suppose that \( f \in C(X, I) \) and \( \epsilon > 0 \). Then, there is a finite collection \( U = U[f, \epsilon] \) consisting of pairwise disjoint open balls in \( X \), \( g = g[f, \epsilon] \in C(X, I) \) and \( \gamma = \gamma[f, \epsilon] > 0 \) such that

1. if \( U \in \mathcal{U} \) then the diameter of \( U \) is less than \( \epsilon \) and the boundary of \( U \) is nonempty,
2. \( \|f - g\| < \epsilon \),
3. if \( p \in X \) then there are at least two points, \( p_1, p_2 \) in \( X \) such that \( f \) equals one of them then \( \|p - q\| < \epsilon \) and \( f(p) = g(q) \),
4. if \( p \in X \setminus \bigcup U \), then there is \( q \) such that \( 0 < \|p - q\| < \epsilon \) and \( g(p) = g(q) \),
5. if \( p \in U \in \mathcal{U} \) and \( g(p) \in [\min g(\partial(U)), \max g(\partial(U))] \), then there is \( q \in X \) such that \( 0 < \|p - q\| < \epsilon \) such that \( g(p) = g(q) \), and
6. if \( U \in \mathcal{U} \), then the extrema of \( g \) relative to \( \partial U \) are attained in the interior of \( U \).

Moreover, if \( h \in C(X, I) \) and \( \|h - g\| < \gamma \), then \( h \) satisfies the above properties as well.

Proof. As \( f \) is uniformly continuous, we may choose \( 0 < \delta < \frac{\epsilon}{5} \) such that if \( x, y \in X \) and \( |x - y| < \delta \), then \( |f(x) - f(y)| < \frac{\epsilon}{5} \). We may also assume that \( \delta \) is less than the diameter of \( X \) which is greater than zero as \( X \) has at least two elements. Let \( x_1, x_2, \ldots, x_n \) be finitely many points so that if \( x \in X \), then there are at least two \( i \)'s such that \( |x - x_i| < \frac{\delta}{5} \). Now, choose open balls, \( U_i \)'s, centered at \( x_i \)'s with radii less than \( \frac{\delta}{5} \) so that if \( i \neq j \) then \( U_i \cap U_j = \emptyset \). By the Boundary Bumping Theorem, we may choose a nondegenerate continuum \( C_i \subseteq U_i \). Let us first define \( g \) on \( C_i \) so that \( g(C_i) = [a_i - \frac{\delta}{5}, b_i + \frac{\delta}{5}] \) where \( a_i = \min f(\overline{U_i}) \) and \( b_i = \max f(\overline{U_i}) \). If \( x \notin \bigcup_{i=1}^n U_i \), then let \( g(x) = f(x) \). Using the Tietze Extension Theorem, extend \( g \) to all of \( X \) so that \( g(U_i) = [a_i - \frac{\delta}{5}, b_i + \frac{\delta}{5}] \). Let \( \mathcal{U} = \{U_i : i = 1, \ldots, n\} \), and \( \gamma = \frac{\delta}{5} \). We claim that \( \mathcal{U}, g, \) and \( \gamma \) are the desired objects. By the fashion in which \( \delta \) was chosen and the fact that \( X \) is connected, we have that Condition 1 is satisfied. Let \( h \in C(X) \) with \( \|g - h\| < \gamma \). By construction, \( \|f - g\| < \frac{\epsilon}{5} + \frac{\delta}{5} \). Therefore, \( \|f - h\| \leq \|f - g\| + \|g - h\| < \frac{\epsilon}{5} + \frac{\delta}{5} < \epsilon \). Hence, Condition 2 is satisfied for \( g \) and \( h \). Let us proceed to Condition 3. Let \( p \in X \). Then, there are \( i \neq j \) such that \( |p - x_i| < \frac{\delta}{5} \) and \( |p - x_j| < \frac{\delta}{5} \). We also have that \( |f(p) - f(x_i)| < \frac{\delta}{5} \) for \( t = i, j \). As \( f(p) \in [a_i - \frac{\delta}{5}, b_i + \frac{\delta}{5}] \subseteq [a_t - \frac{\delta}{5}, b_t + \frac{\delta}{5}] \subseteq h(U_t) \) for \( t = i, j \), there is an element \( g \) in each of \( U_i \) and \( U_j \) such that \( h(q) = f(p) \). Similarly, Condition 4 follows from the fact that if \( p \notin \bigcup U \), then \( h(p) \in [a_t - \frac{\delta}{5}, b_t + \frac{\delta}{5}] \) for a suitable \( t \). To verify Condition 5 assume that \( p \in U_j \) for some \( j \). There is \( i \neq j \) such that \( |p - x_i| < \frac{\delta}{5} \). Since the radius of \( U_j \) is less than \( \frac{\delta}{5} \), for each \( r \) on the boundary of \( U_j \) we have that \( |r - x_i| < \delta \) and hence \( |f(r) - f(x_i)| < \frac{\delta}{5} \). Since \( f(r) = g(r) \) and \( \gamma = \frac{\delta}{5} \), we have that \( h(r) \in [a_t - \frac{\delta}{5}, b_t + \frac{\delta}{5}] \subseteq [a_t - \frac{\delta}{5}, b_t + \frac{\delta}{5}] \subseteq h(U_i) \). To verify Condition 6 we only need to observe that if \( p \) is on the boundary of \( U_i \), then \( h(p) \in [a_t - \frac{\delta}{5}, b_t + \frac{\delta}{5}] \subseteq [a_t - \frac{\delta}{5}, b_t + \frac{\delta}{5}] \subseteq h(U_i) \). \( \square \)

Theorem 3.6. A generic \( g \in C(X, I) \) has the property that if \( p \in X \) and \( p \) is an isolated point of \( g^{-1}(g(p)) \) then \( g \) has a local extremum at \( p \).

Proof. We have assumed that \( X \) is nondegenerate and hence it has at least two points. Let \( \{f_i\} \) be a sequence dense in \( C(X, I) \) and let \( \epsilon_{n,k} = \frac{1}{n+1} \). By Lemma 3.5 we may choose \( U_{n,k}, g_n = g[f_k, \epsilon_{n,k}] \) and \( \gamma_{n,k} = \gamma[f_k, \epsilon_{n,k}] \) which satisfy the conclusion of the lemma with respect to \( f_k \) and \( \epsilon_{n,k} \). Let \( B_{n,k} \) be the ball in
C(X, I) centered at \( g_{n, k} \) with radius \( \gamma_{n, k} \). Then, \( \mathcal{G}_n = \bigcup_{k=1}^{\infty} B_{n, k} \) is dense and open in C(X, I). Let \( \mathcal{G} = \bigcap_{n=1}^{\infty} \mathcal{G}_n \). We show that \( g \in \mathcal{G} \) satisfies the desired property. Let \( \{j_i\} \) be a sequence such that \( g \in B_{j_i} \) for all \( i \). Let \( U_i = U[j_i, \epsilon, j_i] \). Let \( p \in X \).

It will suffice to show that if \( g \) does not have a local extremum at \( p \), then \( g(p) \) is a limit point of \( g^{-1}(g(p)) \). Let us assume that \( g \) does not have a local extremum at \( p \) and let \( \eta > 0 \). Let \( n \) be large enough so that \( \frac{1}{n} < \frac{4}{3} \). If \( p \notin \bigcup U_n \), then by Condition 4 of Lemma 3.5 we know that there is a \( q \in X \) with \( 0 < |p - q| < \epsilon_n < \eta \) so that \( g(q) = g(p) \). For the second case assume that \( p \in \bigcup U_n \) and \( p \in U \) for some \( U \in \mathcal{U}_n \). If \( g(p) \notin \bigcup B_{p, \epsilon} \), then by Condition 5 we have that there is \( q \) with \( 0 < |p - q| < \epsilon_n < \eta \) and \( g(p) = g(q) \). So let us assume that \( g(p) \notin \bigcup B_{p, \epsilon} \). Without loss of generality, we may assume that \( g(p) > \max g(\partial(U)) \). By Condition 6 of Lemma 3.5 the maximum of \( g \) on \( \overline{U} \) must occur in \( U \). Let \( q \in U \) where this maximum occurs. Then, \( g(q) > g(p) \) as \( g \) does not have a local extremum at \( p \). By the Boundary Bumping Theorem, we may obtain a continuum \( C \subseteq \overline{U} \) containing \( q \) which intersects the boundary of \( U \). We note that \( \min g(C) < g(p) < \max g(C) \) as \( q \in C \) and some point of the boundary of \( U \) is also in \( C \). As \( \{f_{j_i}\}_{i=1}^{\infty} \) converges uniformly to \( g \) on \( X \), there exists \( m > n \) such that \( f_{j_m}(C) \) contains \( g(p) \). Since \( g \in B_{m, j_m} \), we have by Condition 3 of Lemma 3.5 that there are at least two \( q \)'s within \( \epsilon_{m, j_m} \) of \( p \) so that \( g(q) = f_{j_m}(p) \). Let \( q \) be such \( q \) one point distinct from \( p \). Then, \( 0 < |p - q| < |p - p'| + |p'| - q| < \epsilon_{n, j_n} + \epsilon_{m, j_m} < \frac{n}{2} + \frac{n}{2} = \eta \) and \( g(q) = f_{j_m}(p') = g(p) \). Hence, we have shown that if \( p \) is not a local extremum of \( g \) and \( \eta > 0 \) then, there is \( q \) with \( 0 < |p - q| < \eta \) with \( g(p) = g(q) \), completing the proof of the theorem.

**Definition 3.7.** Suppose that \( f \in C(X, I) \) and \( \epsilon > 0 \). We call the fiber \( f^{-1}(y) \) \( \epsilon \)-fine if for each \( A \in \text{Comp}(f^{-1}(y)) \) with \( \text{diam}(A) \geq \epsilon \), there is \( B \in \text{Comp}(f^{-1}(y)) \) such that \( \text{diam}(B) < \epsilon \), and points \( a, b \) in \( A, B \), respectively, such that \( d(a, b) < \epsilon \). Note that if \( 0 < \epsilon' \leq \epsilon \) and a fiber is \( \epsilon' \)-fine, then it is \( \epsilon \)-fine.

**Lemma 3.8.** Suppose that \( f \in C(X, I) \) and \( \epsilon > 0 \). Then, there is \( g \in C(X, I) \) and \( \gamma > 0 \) such that

1. \( \|f - g\| < \epsilon \),
2. \( g \) is constant on no ball of radius \( \epsilon \),
3. if \( p_1, p_2 \in X \) with \( d(p_1, p_2) \geq 3\epsilon \) and \( g \) restricted to \( B_\epsilon(p_i) \) has an extremum at \( p_i \) for \( i = 1, 2 \), then \( g(p_1) \neq g(p_2) \), and
4. each fiber of \( g \) is \( \epsilon \)-fine.

Moreover, if \( h \) is such that \( \|h - g\| < \gamma \), then \( h \) satisfies the above properties.

**Proof.** Using the uniform continuity of \( f \), choose \( 0 < \delta < \frac{\epsilon}{3} \) so that if \( x, y \in X \) with \( d(x, y) < \delta \), then \( |f(x) - f(y)| < \frac{\epsilon}{3} \). Using the compactness of \( X \), choose distinct points \( x_1, x_2, \ldots, x_n \) in \( X \) so that \( d(x, x_i) < \delta \). Let \( U_1, U_2, \ldots, U_n \) be pairwise disjoint open balls of radii less than \( \delta \) centered at \( x_1, x_2, \ldots, x_n \), respectively. For each \( i \), let \( V_i \) and \( W_i \) be balls centered at \( x_i \) such that \( W_i \subseteq V_i \subseteq U_i \). As \( X \) is a nondegenerate continuum, by the Boundary Bumping Theorem we may choose a nondegenerate continuum \( C \subseteq W_i \). Let \( a_i = \min f(\overline{V_i}) \) and \( b_i = \max f(\overline{V_i}) \). Let \( \tilde{g} : [X \setminus (\bigcup_{i=1}^{n} U_i)] \cup \bigcup_{i=1}^{n} (C \cup (\overline{V_i} \setminus W_i)) \to \mathbb{R} \) be a continuous function such that

i. \( \tilde{g}(x) = f(x) \) if \( x \in X \setminus (\bigcup_{i=1}^{n} U_i) \).
ii. \( \hat{g}(C_i) = [r_i, s_i] \) where \([a_i - \frac{\epsilon}{4}, b_i + \frac{\epsilon}{4}] \subseteq (r_i, s_i) \subseteq [a_i - \frac{\epsilon}{2}, b_i + \frac{\epsilon}{2}] \),

iii. \( g(\bigcup_i W_i) = r_i \), and

iv. for all \( i \neq j, r_i \neq r_j, s_i \neq s_j \), and for all \( i, j, r_i \neq s_j \).

Now using the Tietze Extension Theorem, let \( g \) be an extension of \( \hat{g} \) such that \( g(\bigcup_i W_i) \subseteq [r_i, s_i] \). Then, \( |f(x) - g(x)| < \frac{\epsilon}{2} \) for all \( x \in X \). Hence, Condition 1 of the lemma is satisfied. That \( g \) is constant on no ball of radius \( \epsilon \) follows from the fact that each such ball contains some \( C_i \). Let us now verify Condition 3. Suppose that \( p_1, p_2 \in X \) with \( d(p_1, p_2) \geq 3 \cdot \epsilon \) and \( g \) restricted to \( B_r(p_i) \) has an extremum at \( p_i \) for each \( i \). We first show that \( p_i \in \bigcup_{k=1}^n U_k \). Suppose that \( p \notin \bigcup_{k=1}^n U_k \). Then, there is \( k \) such that \( d(p, x_k) < \delta \). This implies that \( g(p) = f(p) \) \( \in [a_k - \frac{\delta}{3}, b_k + \frac{\delta}{3}] \). However, \( C_k \) is a subset of \( B_r(p) \) and \( g(C_k) = [r_k, s_k] \) with \( r_k < a_k - \frac{\delta}{3} \) and \( s_k > b_k + \frac{\delta}{3} \). Hence, \( g \) restricted to \( B_r(p) \) does not have an extremum at \( p \). Therefore, \( p_1 \in U_{k_1} \) and \( p_2 \in U_{k_2} \) for some \( k_1 \neq k_2 \). Since \( g(p_i) \subseteq [r_k, s_k] \) \( g \) restricted to \( B_r(p_i) \) has a local extremum at \( p_i \) and \( B_r(p_i) \supseteq U_k \), we have that \( g(p_i) \in \{ r_i, s_i \} \). Hence from property iv it follows that \( g(p_1) \neq g(p_2) \). Let us now show that each fiber of \( g \) is \( \epsilon \)-fine. Let \( g \in g(X) \) and \( A \in \text{Comp}(g^{-1}(y)) \) with \( \text{diam}(A) \geq \epsilon \). As \( \{ U_i \} \) is a pairwise disjoint sequence of open balls with diameter less than \( \frac{\epsilon}{4} \), there is \( p \in A \) such that \( p \in X \setminus \bigcup_{i=1}^n U_i \). Let \( k \) be such that \( d(p, x_k) < \delta \). As before, we have that \( g(p) \in [a_k - \frac{\delta}{3}, b_k + \frac{\delta}{3}] \). We know that there is \( q \in C_k \) such that \( g(q) = g(p) > r_k \). Then, \( B \), the component of \( g^{-1}(y) \) containing \( q \) is contained in \( W_k \) as \( g(\bigcup_k W_k) = r_k \). Now we have that \( B \) is the desired component as \( d(p, q) < \epsilon \) and \( \text{diam}(B) < \epsilon \). Now let \( \gamma > 0 \) be less than

\[
\frac{1}{8} \cdot \min \left\{ \epsilon, |r_i - (a_i - \frac{\epsilon}{4})|, |s_i - (b_i + \frac{\epsilon}{4})|, |r_i - r_j|, |s_i - s_j|, |r_i - s_i|, |r_i - s_j| : 1 \leq i, j \leq n \text{ and } i \neq j \right\}.
\]

Then, \( \gamma \) is the desired constant.

\[\square\]

**Theorem 3.9.** A generic \( f \in C(X, I) \) satisfies the following conditions:

1. Each fiber of \( f \) is nowhere dense.
2. No fiber of \( f \) contains two points where the function has a local extremum.
3. If \( g \in f(X) \), and \( A \in \text{Comp}(f^{-1}(y)) \) is an isolated point of \( \text{Comp}(f^{-1}(y)) \) in the upper semicontinuous topology, then \( A \) is a singleton set.

**Proof.** For each \( \epsilon > 0 \), consider the collection \( \mathcal{G}_\epsilon \) of those functions \( g \in C(X, I) \) which satisfy Conditions 2-4 of Lemma 3.8. Then, \( \mathcal{G}_\epsilon \) contains a dense open set in \( C(X, I) \). Let \( \mathcal{G} = \bigcap_{\epsilon = 1}^{\infty} \mathcal{G}_\epsilon \). Let \( f \in \mathcal{G} \). It is clear that \( f \) satisfies Conclusions 1 and 2 of the theorem. To see the last conclusion observe that each fiber of \( f \) is \( \epsilon \)-fine for every \( \epsilon > 0 \). \[\square\]

**Theorem 3.10.** For a generic \( f \in C(X, I) \) there is a countable dense set \( D \subseteq (\min f(X), \max f(X)) \) such that

1. if \( g \in \{ \min f(X), \max f(X) \} \), then \( \text{Comp}(f^{-1}(y)) \) is a singleton set,
2. if \( g \in D \), then \( \text{Comp}(f^{-1}(y)) \) is homeomorphic to the Cantor set union an isolated point, and
are arcs in $P$. Suppose that $[a, b]$ is an arc in $P$ with $a$ as one of its endpoints and $P \cap A = \{a\}$.

**Definition 4.2.** Suppose that $P$ is a graph and $b$ is an endpoint of $P$ and $[a, b]$ is an arc in $P$ such that $[a, b]$ is open in $P$. We say that graph $Q$ is a method $II$ extension of $P$, denoted by $P \prec_2 Q$, if $Q = P \cup A$ where $A$ is an arc with endpoints $a, b$ and $P \cap A = \{a, b\}$.

**Definition 4.3.** Suppose that $P$ is a graph, $b_1, b_2$ are endpoints of $P$, and $[a, b_1], [a, b_2]$ are arcs in $P$ such that $[a, b_1] \cap [a, b_2] = \{a\}$ and $(a, b_i)$ is open in $P$ for $i = 1, 2$.

4. When $P$-likeness Implies $Q$-likeness

In this section we develop a general technique which can be used to show that a $P$-like hereditarily indecomposable continuum is $Q$-like when $Q$ is a certain type of graph containing $P$. However, we first need some terminology.

**Definition 4.1.** Suppose that $P$ is a graph and $x \in P$. We say that the graph $Q$ is a method $I$ extension of $P$, denoted by $P \prec_1 Q$, if $Q = P \cup A$ where $A$ is an arc with $x$ as one of its endpoints and $P \cap A = \{x\}$.
say that graph $Q$ is a method III extension of $P$, denoted by $P \prec Q$, if $Q = P \cup A$ where $A$ is an arc with endpoints $b_1, b_2$ such that $A \cap P = \{b_1, b_2\}$.

**Theorem 4.4.** Suppose that $P$ is a graph, and $M$ is a continuum which is $P$-like. If $P \prec Q$, then $M$ is $Q$-like.

**Proof.** Let $Q = P \cup A$ where $A$ is from the definition of $P \prec Q$. Let $A_1, A_2, \ldots, A_n$ be a defining sequence for $P$. Without loss of generality, we may assume that $A \cap P$ is an endpoint of some $A_i$. Let $A_i = [a, b]$ and $b = A \cap P$. Let $\epsilon > 0$. Let $f$ be a map from $M$ onto $P$ and $\delta > 0$ so that if $U$ is a set with diameter less than $\delta$, then $f^{-1}(U)$ has diameter less than $\epsilon$. Let $J = [a', b'] \subseteq A_i$ be an arc with diameter less than $\delta$. Now, let $h$ be a continuous map from $J$ onto $J \cup A$ so that $h(a') = a'$ and $h(b) = b$. Extend $h$ to all of $P$ by making it identity on $P \setminus J$. Then, $h$ is a $\delta$-map from $P$ onto $Q$ and $h \circ f$ is an $\epsilon$-map from $M$ onto $Q$. \hfill $\Box$

The following lemma was generalized in [4]. We include it here because the technique of the proof given below is used in Lemma 4.6.

**Lemma 4.5.** Suppose that $P$ is a graph, $\epsilon > 0$, $M$ is a continuum and $f : M \to P$ is an $\epsilon$-map onto $P$. Then, there is $\eta > 0$ such that if $N$ is a continuum and the Hausdorff distance from $\epsilon$, $P$ be a defining sequence for $N$. Then, $\eta > 0$ such that if $N$ is a continuum and $\eta > 0$ such that $diam(f^{-1}(A_i)) < \epsilon$ for all $i$.

Now choose a sequence of open sets $V_1, V_2, \ldots, V_n$ in $M$ such that for all $1 \leq i \leq j \leq n$, $f^{-1}(A_i) \subseteq V_i$, the diameter of $V_i$ is less than $\epsilon$, $V_i \cap V_j \neq \emptyset$ if $A_i \cap A_j \neq \emptyset$, and $U_i = V_i \setminus (\bigcup_{j \neq i} V_j) \neq \emptyset$ and $U_i \cap f^{-1}(A_i) \neq \emptyset$.

Such $V_i$’s may be chosen in the following fashion. Let $\delta > 0$ be small enough so that for each $i$ we have that if $f^{-1}(A_i) \cap f^{-1}(A_i) = \emptyset$, then $d(a, b) > \delta$ for all $a \in f^{-1}(A_i)$ and $b \in f^{-1}(A_j)$. For each $i$, let $x_i \in f^{-1}(A_i)$ be such that $f(x_i)$ is not an endpoint of $A_i$. Since $p \in A_i \cap A_j$ if $p$ is an endpoint of $A_i$ and $A_j$, we can make $\delta$ small enough so that for all $a \in \bigcup_{j \neq i} f^{-1}(A_j)$, we have that $d(a, x_i) > \delta$, we can also assume that the same $\delta$ works for all $i$. Since the diameter of $f^{-1}(A_i)$ is less than $\epsilon$, we can choose an open set $V_i$ which contains $f^{-1}(A_i)$ such that the diameter of $V_i$ is less than $\epsilon$ and $V_i \subseteq \bigcup_{a \in f^{-1}(A_i)} B_{\frac{\delta}{2}}(a')$. Then, these $V_i$’s have the desired properties.

Let $\eta > 0$ be small enough so that if $N$ is a continuum within $\eta$ of $M$ in the Hausdorff metric, then $N \subseteq \bigcup_{i=1}^n V_i$ and $N \cap U_i \neq \emptyset$ for all $i$. In order to finish the proof, it will suffice to define a $2\epsilon$-map $g$ from $\bigcup_{i=1}^n V_i$ onto $P$ such that $g(N) = P$.

Let us first make an observation. Let $x \in V_i \cap V_j$ for some $i \neq j$ and let $x \in V_i \cap V_j'$ for some $j' \neq j$. Then, $A_i \cap A_j = A_{i'} \cap A_{j'}$ and $A_i \cap A_{j'}$ has exactly
one element. That $A_i \cap A_j$ has exactly one element follows from Properties 2) and 3) above. If $A_i \cap A_j \neq A_i \cap A_j$, then $A_i \cup A_j \cup A_i \cup A_j$ would contain a simple closed curve, contradicting Property 3) above.

Let $x \in \overline{V_i} \cap \overline{V_j}$ for $i \neq j$. Define $g(x)$ to be the point common to $A_i \cap A_j$. By the observation above, $g$ is well defined on $\bigcup_{(i,j), i \neq j} \overline{V_i} \cap \overline{V_j}$. From the properties of $V_i$’s and the observation above, we also have that $g$ is continuous. Since $N$ is a continuum, by the Boundary Bumping Theorem, there is a nondegenerate continuum $N_i \subseteq N \cap U_i$. Define $g$ continuously on $N_i$ so that $g(N_i) = A_i$. Since arcs are homeomorphic to $[0, 1]$, we can use the Tietze Extension Theorem to extend $g$ continuously from $N_i \cup (\bigcup_{j \neq i} \overline{V_i} \cap \overline{V_j})$ to $\overline{V_i}$ so that $g(\overline{V_i}) = A_i$. Doing this for all $1 \leq i \leq n$, we have a $2\epsilon$-map $g$ from $\bigcup_{i=1}^n \overline{V_i}$ onto $P$ so that $g(N) = P$.

**Lemma 4.6.** Suppose that $M$ is a nondegenerate indecomposable continuum, $P$ is a graph, $M$ is $P$-like, $\epsilon > 0$ and $n > 0$ is an integer. Then, there are pairwise disjoint, nowhere dense subcontinua $K_1, K_2, \ldots, K_n$ of $M$ and an $\epsilon$-map $g$ from $M$ onto $P$ such that $g(K_i) = P$ for all $1 \leq i \leq n$.

**Proof.** This essentially follows from the proof of Lemma 4.5 and that $M$ is an indecomposable continuum. Let $f$ be an $\epsilon$-map from $M$ onto $P$. Let $V_i$’s, and $\eta$ be as in the proof of Lemma 4.5. Using the fact that composants of a continuum are dense in the continuum and a nondegenerate indecomposable continuum has uncountably many pairwise disjoint composants, we may chose pairwise disjoint continua $K_1, K_2, \ldots, K_n$ so that the Hausdorff distance from any $K_i$ to $M$ is less than $\eta$. Now we define $g$ as in the earlier proof considering $K_1, K_2, \ldots, K_n$. Then, $g$ restricted to $M$ is the desired function.

**Theorem 4.7.** (Krasinkiewicz, [9]) The following are equivalent for a continuum $M$:

- $M$ is hereditarily indecomposable.
- If $C, D$ are two disjoint closed subsets of $M$ and $U$ is an open set intersecting each component of $C$, then there are closed sets $H, K$ such that $C \subseteq H$, $D \subseteq K$, $M = H \cup K$, and $H \cap K \subseteq U \setminus (C \cup D)$.

**Theorem 4.8.** Let $P$ be a graph and $M$ be a nondegenerate hereditarily indecomposable continuum which is $P$-like. Suppose that $Q$ is such that $P \prec_3 Q$. Then, $M$ is $Q$-like.

**Proof.** Let $b_1, b_2$ be two distinct endpoints of $P$ and $A$ be an arc with endpoints $b_1, b_2$ such that $A \cap P = \{b_1, b_2\}$. Let $A_1 = [a, b_1]$, $A_2 = [a, b_2]$ be two arcs of $P$ such that $A_1 \cap A_2 = \{a\}$ and $(a, b_1]$ and $(a, b_2]$ are open in $P$. Let $Q = P \cup A$. Let $\epsilon > 0$. Let $\delta > 0$ and $f$ be a continuous map from $M$ onto $P$ so that if $U$ is a set with diameter less than $\delta$, then $f^{-1}(U)$ has diameter less than $\frac{\delta}{3}$. As $M$ is a nondegenerate hereditarily indecomposable continuum, by Lemma 4.6 we may assume that there are three pairwise disjoint continua $K_1, K_2, K_3$ in $M$ such that $f(K_i) = P$, $1 \leq i \leq 3$. For $i = 1, 2$, let $T_i = [a_i, b_i] \subseteq A_i$ be an arc such that $a_i \neq a$ and the diameter of $[a_i, a_1] \cup [a, a_2]$ is less than $\frac{\delta}{3}$.

Let $b \in A \setminus \{b_1, b_2\}$. Let $h_i$ be a homeomorphism from $T_i$ onto the subarc of $A$ determined by $b_i$ and $b$ such that $h_i(a_i) = b$ and $h_i(b_i) = b_i$. Let $h = h_1 \cup h_2$. Extend $h$ to $A_1 \cup A_2$ by letting $h(x) = b$ for $x \in [a, a_1] \cup [a, a_2]$. Then, $h : (A_1 \cup A_2) \to A$ is a continuous map.
Let $R = P \setminus ((a, b_1] \cup (a, b_2))$. Let $D = K_3 \cup f^{-1}(R)$. Then, $D$ is a compact subset of $M$ since $R$ is closed. Note that for each $i = 1, 2$, $O_i = f^{-1}((a, b_i]) \cap K_i$ is a relatively open subset of $K_i$. Let $p_i \in O_i$ be such that $f(p_i) = b_i$. Now by the Boundary Bumping Theorem we may choose a continuum $N_i \subseteq O_i$ such that $p_i \in N_i$ and $N_i \cap \partial(O_i) \neq \emptyset$. Then, $f(N_i) = [a_i, b_i] = T_i$. We let $C = N_1 \cup N_2$. We observe that $C \cap D = \emptyset$.

For $i = 1, 2$, let $J_i = [a_i', b_i] \subseteq T_i$ be such that $\text{diam}(J_i) < \delta$ and let $U_i = f^{-1}([a_i', b_i])$. Then, $\text{diam}(U_i) < \frac{\delta}{2}$ and $U_1 \cap U_2 = \emptyset$. Let $U = U_1 \cup U_2$. Note that $U$ intersects each component of $C$ since $p_i \in U_i \cap N_i$ for $i = 1, 2$.

We now apply the Kuratowski theorem to $M, C, D$, and $U$. Let $H, K$ be closed sets such that $M \cap K \subseteq H, D \subseteq K$ and $H \cap K \subseteq U$. Note that $H \setminus U$ and $K \setminus U$ are disjoint closed sets whose union is $M \setminus U$. Let us proceed to define an $\epsilon$-map $g$ from $M$ onto $Q$. We do this by considering three cases.

The first case is that $x \in (H \setminus U)$. In this case, $f(x) \in A_1 \cup A_2$, since $f^{-1}(R) \subseteq D \subseteq K$ and $H \cap K \subseteq U$. Define $g(x) = h(f(x))$. Note that $g(H \setminus U) \subseteq A$. The second case is $x \in (K \setminus U)$. In this case, let $g(x) = f(x)$. Finally, consider the case $x \in U$. We define $g$ on $U_1$ first. Note that $g$ is well defined on $\partial(U_1)$ and $g|_{M \setminus U_1}$ is continuous. Since $f(U_1) \subseteq J_1$, we have that $g(\partial(U_1)) \subseteq J_1 \cup h(J_1)$. Note that $J_1 \cup b(J_1)$ is an arc. By the Boundary Bumping Theorem, we know that $U_1$ contains a non-degenerate continuum. Hence, by the Tietze Extension Theorem, we can extend $g$ continuously from the boundary of $U_1$ to all of $U_1$, so that $g(U_1) = J_1 \cup h(J_1)$. We define $g$ analogously on $U_2$. Now we have a continuous function $g$ from $M$ into $Q$.

Let us first show that $g$ maps onto $Q$. Note that

$$P \setminus (J_1 \cup J_2) \subseteq f(K_3 \setminus U) = g(K_3 \setminus U).$$

Also,

$$\bigcup_{i=1}^{2} (J_i \cup h(J_i)) \subseteq g(U).$$

Finally,

$$A \setminus \left( \bigcup_{i=1}^{2} h(J_i) \right) \subseteq \bigcup_{i=1}^{2} h(f(N_i \setminus U_i)) \subseteq g(C \setminus U).$$

Hence it follows that $g$ maps onto $Q$. Now we want to verify that $g$ is an $\epsilon$-map. Let $y \in Q$. We first note that

$$g^{-1}(y) = (g|_{H \setminus U})^{-1}(y) \cup (g|_{K \setminus U})^{-1}(y) \cup (g|_{U})^{-1}(y)$$

$$= [h \circ f|_{H \setminus U}]^{-1}(y) \cup (f|_{K \setminus U})^{-1}(y) \cup (g|_{U})^{-1}(y).$$

If $y \in P \setminus (J_1 \cup J_2)$, then $g^{-1}(y) = (f|_{K \setminus U})^{-1}(y) \subseteq f^{-1}(y)$ and hence has diameter less than $\epsilon$. Now suppose that $y \in (J_i \cup h(J_i))$ for $i = 1$ or $i = 2$. Then, $g^{-1}(y) \subseteq f^{-1}(J_i) \cup U_i$. Since $p_i \in f^{-1}(J_i) \cup U_i$ and each of $f^{-1}(J_i)$ and $U_i$ has diameter less than $\frac{\delta}{2}$, we have that $f^{-1}(y)$ has diameter less than $\epsilon$. Now we consider the case that $y \in A \setminus (h(J_1) \cup h(J_2))$. If $y \neq b$, then $g^{-1}(y) = (h \circ f|_{H \setminus U})^{-1}(y) \subseteq f^{-1}(h^{-1}(y))$ and hence has diameter less than $\epsilon$. Finally, consider the case $y = b$. Then, $g^{-1}(y) = f^{-1}([a, a_1] \cup [a, a_2])$. Since the diameter of $[a, a_1] \cup [a, a_2]$ is less than $\delta$, we have that the diameter of $f^{-1}([a, a_1] \cup [a, a_2]) = g^{-1}(y)$ is less than $\epsilon$. \qed
Theorem 4.9. Let $P$ be a graph and $M$ be a nondegenerate hereditarily indecomposable continuum which is $P$-like. Suppose that $Q$ is such that $P \prec_2 Q$. Then, $M$ is $Q$-like.

Proof. Assume the hypothesis. Let $[a, b]$ be an arc of $P$ with $b$ an endpoint of $P$ and $(a, b]$ open in $P$. Let $A$ be an arc with endpoints $a, b$ such that $A \cap P = \{a, b\}$. Let $Q = P \cup A$. Let $c \in A \setminus \{a, b\}$. By Theorem 4.4, $M$ is $P \cup A_1$ like where $A_1$ is the subarc of $A$ determined by $a$ and $c$. Now applying Theorem 4.8 to $P \cup A_1$ and its endpoints $c, b$, we get that $M$ is $Q$-like.

Corollary 4.10. Suppose that $M$ is a nondegenerate hereditarily indecomposable continuum which is $P$-like for some graph $P$ and $Q$ is a graph which can be obtained from $P$ by applying a finite sequence of extensions of method I, II or III. Then, $M$ is $Q$-like.

Proof. This simply follows from applying Theorems 4.4, 4.9, and 4.8.

Corollary 4.11. Let $P$ be a nondegenerate subcontinuum of the figure-eight. If $M$ is a hereditarily indecomposable continuum which is $P$-like, then $M$ is figure-eight like.

5. The Fiber Structure of a Generic Map in $C(S^2, I)$

In this section we give a more precise description of the fibers of a generic map $f \in C(S^2, I)$. This section is divided into four subsections. In Subsection 5.1, we construct a well-behaved class of continuous functions in $C(S^2, I)$. We also show in this section that the saddle points are “stable” in $C(S^2, I)$. These results are used in later subsections to determine the fiber structure of a generic map in $C(S^2, I)$. In Subsection 5.2 we show that a generic $f \in C(S^2, I)$ has the property that each component of each fiber of $f$ is either a point or figure eight like. In Subsection 5.3 we show that a generic $f \in C(S^2, I)$ has the property that there is a countable dense set $D \subseteq f(S^2)$ such that for each $y \in D$, $f^{-1}(y)$ has a component which is a Lakes of Wada continuum. In Subsection 5.4 we show that all components of almost every fiber of a generic map in $C(S^2, I)$ are either points, pseudoarcs or pseudocircles.

5.1. A well-behaved countable dense subset of $C(S^2, I)$. We use the following parametrization of $S^2$. A point $p \in S$ can be represented as

$$p = \psi(\phi, \theta) = (\cos(\theta) \cos(\phi), \cos(\theta) \sin(\phi), \sin(\theta)).$$

We denote by $P_{North} = (0, 0, 1)$, and $P_{South} = (0, 0, -1)$, the North and South Poles of our sphere.

In the sequel we need to consider some triangulations of $S^2$. Our triangles on $S^2$ will be either one-to-one $\psi$ images of a triangle $T$ in $\mathbb{R}^2$, or when $P_{North}$ (or $P_{South}$) is one of the vertices, images of the form $\psi([\phi_1, \phi_2] \times [\theta_0, \pi/2])$ (or $\psi([\phi_1, \phi_2] \times [-\pi/2, \theta_0])$) such that $\psi$ is one-to-one on $[\phi_1, \phi_2] \times [\theta_0, \pi/2]$ (or on $[\phi_1, \phi_2] \times (-\pi/2, \theta_0)$).

Let us fix an equilateral triangle $S$ in $\mathbb{R}^2$ of side length one with vertices $S_1, S_2, S_3$. For each triangle $T$ of our triangulation of $S^2$ with vertices $V_1, V_2, V_3 \in S^2$ we fix a homeomorphism $\Phi_T$ from $T$ onto $S$ so that if $x \in V_iV_j$, then $\Phi_T(x) \in S_iS_j$ and the Euclidean distance between $\Phi_T(x)$ and $S_i$ is $\frac{\text{arc length of } V_iV_j}{\text{arc length of } V_iV_j}$. We say that a function $f : T \to \mathbb{R}$ is linear if $f \circ \Phi_T^{-1}$ is linear on $S$. 
Let $K > 1$ be a large odd integer. Next we want to define a triangulation $T = T_K$ of $S^2$.

The North cap and the South cap of $S^2$, respectively, are

$$B_{North} = \{ \Psi(\phi, \theta) : \phi \in [0, 2\pi), \frac{\pi}{2} - \frac{\pi}{2K} \leq \theta \leq \frac{\pi}{2} \},$$

and

$$B_{South} = \{ \Psi(\phi, \theta) : \phi \in [0, 2\pi), -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} + \frac{\pi}{2K} \}.$$  

Denote by $C^2$ the closure of $S^2 \setminus (B_{North} \cup B_{South})$. Clearly, $C^2$ is homeomorphic to a cylinder. First, we divide $C^2$ into non-overlapping “rectangles”. If $i \in \{0, ..., K - 1\}$ and $j$ is even, $-K < j < K - 1$, we put

$$R(i, j) = \{ \Psi(\phi, \theta) : 2\pi i \frac{\pi}{K} \leq \phi \leq 2\pi (i + 1) \frac{\pi}{K}, \frac{\pi}{2} j \frac{\pi}{2K} \leq \theta \leq \frac{\pi}{2} (j + 1) \frac{\pi}{2K} \}.$$  

If $i \in \{0, ..., K - 1\}$ and $j$ is odd, $-K < j < K - 1$, we put

$$R(i, j) = \{ \Psi(\phi, \theta) : 2\pi (i + \frac{1}{2}) \frac{\pi}{K} \leq \phi \leq 2\pi (i + \frac{3}{2}) \frac{\pi}{K}, \frac{\pi}{2} j \frac{\pi}{2K} \leq \theta \leq \frac{\pi}{2} (j + 1) \frac{\pi}{2K} \}.$$  

The center (the intersection of its “diagonals”) of $R(i, j)$ will be denoted by $C(i, j)$. If $j$ is even we call the rectangle even, if $j$ is odd we call the rectangle odd.

![Figure 1. Rectangle R and its neighbors](attachment:image1.png)

First, we triangulate $C^2$. Given a rectangle $R = R(i, j)$ with vertices $V_k$, $k = 1, ..., 4$ we assume that $V_1$ and $V_2$, (and hence $V_3$ and $V_4$) have the same $\theta$ coordinate, see Figure 1. We can also assume that the $\theta$ coordinate of $V_1$ is less than that of $V_3$. Denote by $V_5$ and $V_6$ the midpoint of the side $V_1V_2$ and $V_3V_4$, respectively. Taking a component of $\Psi^{-1}(R(i, j))$ consider the triangles determined by $\Psi^{-1}(V_j)$ ($j = 1, ..., 6$) and $\Psi^{-1}(C)$, where $C$ is the center of $R(i, j)$. Use the $\Psi$ image of these triangles for the triangulation of $R(i, j)$. We do this on all rectangles $R(i, j)$.

Next we triangulate $B_{North}$ by using the non-overlapping triangles determined by $P_{North}$ and the points $\{ \Phi(\frac{\pi}{K}, \frac{\pi}{2} - \frac{\pi}{2K}) : i = 0, ..., 2K - 1\}$. We triangulate $B_{South}$ similarly.

Now we have a triangulation of the whole sphere $S^2$. Denote the vertices of the graph determined by this triangulation by $V$ and the edges of this graph by $E$. We use $(V, E)$ to denote this graph. Observe that apart from the North and South Pole the above graph has degree no more than six at each vertex in $V$. We denote by $T$ the set of all closed spherical triangles determined by our triangulation. A triangulation is $\epsilon$-fine if $diam(T) < \epsilon$ for all $T \in T$. 


Lemma 5.1. The set of vertices $V$ can be divided into three sets: $V_p$ is the set consisting of the North and the South pole, $V_4$ is the set of points $V \in V$ where $(V, E)$ has degree four at $V$, and $V_6$ is the set of points $V \in V$ where $(V, E)$ has degree six at $V$.

Proof. It follows from the construction that if $V$ is a vertex which is not adjacent to one of the poles, then $(V, E)$ has degree six at $V$. If $V$ is adjacent to one of the poles and $V$ is a corner of some rectangle, then $(V, E)$ has degree six at $V$ as well. If $V$ is adjacent to one of the poles and $V$ is not a corner of some rectangle, then $(V, E)$ has degree four at $V$. $\square$

A function $f : S^2 \to \mathbb{R}$ is a triangular function if there is a triangulation $T$ of $S^2$ so that $f$ is linear on each $T \in T$, and

$$(1) \quad V, V' \in V, V \neq V' \Rightarrow f(V) \neq f(V').$$

We use $TF$ to denote the class of triangular functions. The following property holds for all triangular functions $f$.

Lemma 5.2. Suppose that $f \in TF$, $T$ is the triangulation associated to $f$, and $f^{-1}(y) \cap T \neq \emptyset$ for some $T \in T$. Then, one of the following is true:

- $f^{-1}(y) \cap T$ has exactly one point and this point is a vertex of $T$.
- $f^{-1}(y) \cap T$ is an arc, contains exactly one vertex, and the endpoints of $f^{-1}(y) \cap T$ are the vertex and a point on the side of $T$ opposite to this vertex.
- $f^{-1}(y) \cap T$ is an arc intersecting only two sides of $T$ and containing no vertex of $T$.

Proof. To see this, consider $f \circ \Phi_T^{-1}$. Function $f \circ \Phi_T^{-1}$ is linear on $S$, and different vertices of $S$ map to different numbers. Hence, the lemma is satisfied for $f \circ \Phi_T^{-1}$ with respect to $S$. The validity of the lemma now follows from this fact and that $\Phi_T$ is an appropriate homeomorphism from $T$ onto $S$. $\square$

We say that $f \in TF$ is proper if the following properties hold.

- If $T \in T$ and $V \in T$ is either the North pole or the center of an even rectangle, then $f$ restricted to $T$ has a maximum at $V$.
- If $T \in T$ and $V \in T$ is either the South pole or the center of an odd rectangle, then $f$ restricted to $T$ has a minimum at $V$.

Lemma 5.3. Suppose that $f$ is a proper triangular function and $T$ is the triangulation associated to $f$. If $V$ is a vertex of $T$ which is not one of the poles nor the center of some rectangle, then there are vertices $V_i \in V, i = 1, 2$ distinct from $V$, adjacent to $V$ and $T_i \in T, i = 1, 2$ with $\{V, V_i\} \subseteq T_i$, such that $f|T_1$ has maximum at $V_1$ and $f|T_2$ has minimum at $V_2$.

Proof. If $V$ is not adjacent to one of the poles, then this is clear from the definition of $f$ being proper. If $V$ is adjacent to one of the poles, then the lemma follows from the fact that $K$ is an odd integer. $\square$

Lemma 5.4. Suppose that $f$ is a proper triangular function. Then, the following properties are true.

1. For all $y \in f(S^2)$, $f^{-1}(y)$ has finitely many components.
(2) For each \( y \in f(S^2) \), every component of \( f^{-1}(y) \) is homeomorphic to a circle, with possibly one exception; the exception is either a point or is homeomorphic to the figure eight.

(3) There are only finitely many points \( y \) in the range of \( f \) such that \( f^{-1}(y) \) has a component which is a point, or homeomorphic to the figure eight.

(4) \( f \) has a local extremum at \( (x, f(x)) \) iff \( x \) is an isolated point of \( f^{-1}(f(x)) \).

Proof. To see the first conclusion, let \( y \in f(S^2) \) and \( f^{-1}(y) \cap T \neq \emptyset \) for some \( T \in \mathcal{T} \).
Then, by Lemma 5.2, we have that \( f^{-1}(y) \cap T \) is either a point or an arc. Since \( \mathcal{T} \) is finite, we have that \( f^{-1}(y) \) has finitely many components.

We now proceed to the second conclusion of the lemma. Let \( y \in f(S^2) \) and \( M \) be a component of \( f^{-1}(y) \). We consider three cases. The first case is that \( M \) contains no point of \( V \). Let \( M \cap T \neq \emptyset \) for some \( T \in \mathcal{T} \). Then, by Lemma 5.2 we have that \( M \cap T \) is an arc which intersects two edges of \( T \). Hence, it follows that \( M \) is a graph which has degree two. By Corollary 9.6 [14], it follows that \( M \) is homeomorphic to a circle.

If \( M \) contains the center of some rectangle, the North pole or the South pole, then \( f \) has a strict local extremum at this point and hence \( M \) is a singleton set.

Finally, let us assume that \( M \) contains a vertex \( V \) which is not one of the poles nor the center of some rectangle. Since different vertices map to different values, \( M \) contains exactly one vertex. Therefore, if \( x \in M \setminus \{V\} \), then \( M \) has degree two at \( x \). Let us now determine the degree of \( M \) at \( V \). By Lemma 5.1, it follows that \( (V, E) \) has degree six or four at \( V \). We will argue that if \( (V, E) \) has degree six at \( V \), then \( M \) has degree four or two at \( V \). An analogous argument will show that if \( (V, E) \) has degree four at \( V \), then the degree of \( M \) at \( V \) is two. Hence, we will obtain that \( M \) is homeomorphic either to the figure-eight, or to the circle.

Denote the edges meeting at \( V \) by \( E_j \) (\( j = 1, \ldots, 6 \)), and denote by \( V_j \) the endpoint of \( E_j \) which is different from \( V \), see Figure 2. It is not difficult to see that among the \( V_j \)’s three are rectangle centers (\( P_{\text{North}} \) and \( P_{\text{South}} \) are regarded to be “rectangle centers” in this context). Without limiting generality we can assume that \( E_j \) (\( j = 1, \ldots, 6 \)) are labelled in a counterclockwise order and \( V_1, V_3, \) and \( V_5 \) are the rectangle centers. Then, these are strict local extrema of \( f \) and by our choice of even-odd rectangles there are both local minima and maxima among them. Without limiting generality we assume \( V_1 \) is a strict local maximum, \( V_4 \) and \( V_5 \) are local minima. Note that \( f(V_1) > y \), \( f(V_3) < y \) and \( f(V_5) < y \).
Lemma 5.6. \[ \text{if} \quad \text{satisfies the conclusion of Lemma 5.4.} \]

To verify the third conclusion of the lemma we notice that \( f^{-1}(y) \) has a component which is a point, or homeomorphic to figure-eight if and only if \( y \in f(V) \). As \( f(V) \) is finite, the third conclusion holds.

Finally, let us verify the fourth conclusion. If \( x \in S^2 \setminus V \), then \( f \) does not have a local extremum at \( x \) as different vertices of \( V \) map to different numbers under \( f \), and \( f \) is linear on some \( T \in T \) which contains \( x \). If \( x \in V \) but \( x \) is not the center of some rectangle \( R(i,j) \), then by Lemma 5.3 there are vertices \( V,V' \) different from \( x \) and \( T,T' \in T \) with \( \{x,V\} \subseteq T \) and \( \{x,V'\} \subseteq T' \) such that \( f[T] \) has a maximum at \( V \) and \( f[T'] \) has a minimum at \( V' \). Hence, \( f \) does not have a local extremum at \( x \). Therefore, \( f \) has a local extremum only at a center of some rectangle and, therefore, it is an isolated point of \( f^{-1}(f(x)) \).

\[ \square \]

**Definition 5.5.** We say that \( f \in C(S^2,I) \) is well-behaved, denoted by \( f \in \text{WB} \), if \( f \) satisfies the conclusion of Lemma 5.4.

**Lemma 5.6.** The set of proper triangular functions is dense in \( C(S^2,I) \). Hence, \( \text{WB} \) is dense in \( C(S^2,I) \).

**Proof.** We know that \( C(S^2,I) \) is a separable metric space. Given Lemma 5.4, it will suffice to show that for each \( h \in C(S^2,I) \) and \( \epsilon > 0 \), there is a proper \( f \in \text{TF} \) with \( \|f - h\| < 2 \cdot \epsilon \).

Let us proceed to define \( f \). As \( h \) is uniformly continuous, we may choose \( \delta > 0 \) so that \( d(p,q) < \delta \), \( p,q \in S^2 \) implies \( |h(p) - h(q)| < \frac{\epsilon}{8} \). We choose a triangulation \( T \) of \( S^2 \) which is \( \frac{\delta}{4} \)-fine. We define \( f \) on the vertices \( V \) so that if \( V,V' \in V \) with \( V \neq V' \), then \( f(V) \neq f(V') \) and the following inequalities hold. If \( V = P_{North} \) then we choose \( f(P_{North}) \) so that

\[ h(P_{North}) + \frac{\epsilon}{2} < f(P_{North}) < h(P_{North}) + \epsilon. \]

If \( V = P_{South} \) then we choose \( f(P_{South}) \) so that

\[ h(P_{South}) - \epsilon < f(P_{South}) < h(P_{South}) - \frac{\epsilon}{2}. \]

If \( V \) is the vertex of a rectangle \( R \) then we choose \( f(V) \) so that

\[ h(V) - \frac{\epsilon}{8} < f(V) < h(V) + \frac{\epsilon}{8}. \]

If \( V \) is the center of an even rectangle \( R \) then we choose \( f(V) \) so that

\[ h(V) + \frac{\epsilon}{2} < f(V) < h(V) + \epsilon. \]

If \( V \) is the center of an odd rectangle \( R \) then we choose \( f(V) \) so that

\[ h(V) - \epsilon < f(V) < h(V) - \frac{\epsilon}{2}. \]

If \( T \in T \), then we extend \( f \) to \( T \) so that \( f \) is linear on \( T \). From the linearity of \( f \) on each \( T \) and the definition of \( \Phi_T \), it follows that \( f \) is a well-defined continuous
function on all of $S^2$. By our choice of $f$ on $V$ and by the linearity of $f$, it follows that $f$ is a proper triangular function and $\|f - h\| < 2 \cdot \epsilon$. \hfill \Box

We next discuss the stability of the figure-eight in level sets of functions in class $\mathbf{WB}$. However, we first need some definitions, notation and background terminology.

**Definition 5.7.** Suppose that $M \subseteq S^2$ is a compact set and $n \geq 2$. We say that $M$ separates $S^2$ into at least $n$ pieces if $S^2 \setminus M$ has at least $n$ components. If $A_1, A_2, \ldots, A_n$ are pairwise disjoint subsets of $S^2$, then we say that $M$ separates $\mathcal{A}$.

**Remark 5.8.** We remark here that for a closed set $M \subseteq S^2$, the components of $S^2 \setminus M$ are open sets which are arcwise connected. Therefore, $M$ separates $A_1, A_2, \ldots, A_n$ if $\bigcup_{i=1}^n A_i \subseteq S^2 \setminus M$ and there is no arc in $S^2 \setminus M$ which intersects two of the $A_i$’s.

**Lemma 5.9.** Suppose that $U_1, U_2, \ldots, U_n$ are pairwise disjoint nonempty open subsets of $S^2$ and $\{M_k\}_{k=1}^\infty$ is a sequence of compact sets each of which separates $U_1, U_2, \ldots, U_n$. Furthermore, assume that $\{M_k\}$ converges to $M$ in the Hausdorff metric. Then, $M$ separates $U_1, U_2, \ldots, U_n$ and if $M$ is a subset of the compact set $N \subseteq S^2$ then $N$ separates $S^2$ into at least $n$ pieces or $N$ contains at least one of $U_1, U_2, \ldots, U_n$.

**Proof.** To show that $M$ separates $U_1, U_2, \ldots, U_n$, we use Remark 5.8. We first note that $M \cap (\bigcup_{i=1}^n U_i) = \emptyset$ as $\{M_k\}$ converges to $M$ in the Hausdorff metric and $U_i$’s are open sets. Let $1 \leq i < j \leq n$. Let $A$ be an arc in $S^2$ which intersects $U_i$ and $U_j$. Since $M_k$ separates $U_i$ from $U_j$, we have that $M_k \cap A \neq \emptyset$ for all $k$. Since $\{M_k\}$ is a sequence of compact sets which converges to $M$ in the Hausdorff metric and $A$ is compact as well, we have that $M \cap A \neq \emptyset$. Hence, $M$ separates $U_1, U_2, \ldots, U_n$. Now let $N$ be a compact set such that $M \subseteq N$. Let $V_i = U_i \setminus N$ for $1 \leq i \leq n$. If $V_i = \emptyset$ for some $i$, then we have that $U_i \subseteq N$ for some $i$. Hence, assume that no $V_i$ is empty. Then, $\bigcup_{i=1}^n V_i \subseteq S^2 \setminus N$ and every arc in $S^2$ intersecting $V_i$ and $V_j$ with $i \neq j$ must intersect $N$ because each such arc must intersect $M$. Hence, we have that $N$ separates $V_1, V_2, \ldots, V_n$ and $S^2 \setminus N$ has at least $n$ components. \hfill \Box

**Remark 5.10.** Let $f \in \mathbf{WB}$. Consider the function $F : f(S^2) \to \mathcal{K}(S^2)$ defined by $F(y) = f^{-1}(y)$. Then, $F$ is continuous at all $y$ which are not local extrema of $f$. Hence, $F$ is continuous except at a finite set. At each such exception $y$, we have that $\lim_{t \to y^-} F(t)$ and $\lim_{t \to y^+} F(t)$ exist the symmetric difference of these two sets is a singleton set where $f$ attains a local extrema.

**Definition 5.11.** Suppose that $f \in \mathbf{WB}$, $y \in f(S^2)$, and $A_i \subseteq S^2 \setminus f^{-1}(y)$, $1 \leq i \leq 3$ are disjoint continua. Let $\mathcal{M}_{A_2}$ be the set of those components $M$ of $f^{-1}(y)$ which separate $A_2$ from $A_3$ and $A_2$ from $A_1$. Similarly, let $\mathcal{M}_{A_1}$ be the set of those components $M$ of $f^{-1}(y)$ which separate $A_3$ from $A_2$ and separate $A_3$ from $A_1$. Furthermore, assume that $\mathcal{M}_{A_2} \neq \emptyset$ and $\mathcal{M}_{A_3} \neq \emptyset$. Then, we let $\delta(f, y, A_1, A_2, A_3) = \inf\{d_H(M_2, M_3) : M_2 \in \mathcal{M}_{A_2} \text{ and } M_3 \in \mathcal{M}_{A_3}\}$.

**Remark 5.12.** As $f^{-1}(y)$ has finitely many components, we have that the inf in the definition of $\delta(f, y, A_1, A_2, A_3)$ is actually realized. Hence, if $\delta(f, y, A_1, A_2, A_3) = 0$, then $f$ attains a local extrema at $y$. \hfill \Box
then we have that there is a component of $f^{-1}(y)$ which separates $A_1, A_2, A_3$. As $f \in \text{WB}$, it follows that $f^{-1}(y)$ contains a component homeomorphic to the figure-eight.

**Remark 5.13.** If $\delta(f, y, A_1, A_2, A_3) > 0$, then we have $\delta(f, y', A_1, A_2, A_3) > 0$ for all $y'$ sufficiently close to $y$. To see this, let us first recall the fact that the set of all $t$ such that $f^{-1}(t)$ contains a component not homeomorphic to the circle is finite. Therefore, for $t$ sufficiently close to $y$ with $t \neq y$, we have that each component of $f^{-1}(t)$ is homeomorphic to the circle. Therefore, it separates $S^2$ into exactly two pieces. Hence, for $y'$ sufficiently close to $y$, we have that $\delta(f, y', A_1, A_2, A_3)$ is well-defined by Remark 5.10. By Remark 5.12, we have that $\delta(f, y', A_1, A_2, A_3) \neq 0$.

**Lemma 5.14.** Suppose that $f \in \text{WB}$ and $f^{-1}(y)$ contains a component $M$ homeomorphic to the figure-eight for some $y \in \mathbb{R}$ and $\gamma > 0$. Then, there is $0 < \epsilon < \gamma$ such that if $y \in C(S^2, I)$ and $\|f - g\| < \epsilon$, then there is $y' \in \mathbb{R}$ with $|y - y'| \leq 2\epsilon$ such that $y'^{-1}(y')$ contains a component which separates $S^2$ into at least three pieces.

**Proof.** Let $M$ be the component of $f^{-1}(y)$ which is homeomorphic to the figure-eight. Then, $S^2 \setminus M$ has three components which we denote by $W_i$, $1 \leq i \leq 3$. As $f \in \text{WB}$, we have that $f^{-1}(y)$ has finitely many components and $M$ contains no point of local extremum of $f$. Hence, we can choose an open set $U$ such that

- $M \subseteq U$,
- $\overline{U} \cap (f^{-1}(y) \setminus M) = \emptyset$, and
- for each $1 \leq i \leq 3$, we have that $f(W_i \cap U) \subseteq (y, \infty)$ or $f(W_i \cap U) \subseteq (-\infty, y)$.

(Imagine $U$ as a small “fattening up” of $M$.) Since $f$ does not have a local extremum at any point of $M$, we have that for some $i$, $f(W_i \cap U) \subseteq (y, \infty)$ and for another $i$, $f(W_i \cap U) \subseteq (-\infty, y)$. Without loss of generality, we assume that $f(W_1 \cap U) \subseteq (y, \infty)$ and $f(W_2 \cap U) \subseteq (-\infty, y)$. Using Tietze’s Extension Theorem, we define a function $f_0$ such that $f_0 = f$ on $U$ and $f_0(W_1) \subseteq (y, \infty)$ and $f_0(W_2 \cup W_3) \subseteq (-\infty, y)$. Let $\rho_0 = \inf\{|f(x) - y| : x \in \partial U\} > 0$. Let $U_i \subseteq U$ be an open ball (hence $U_i$ is a continuum) such that $U_i \subseteq W_i$. Set $\rho_1 = \inf\{|f_0(x) - y| : x \in \cup_{i=1}^3 W_i \setminus U\}$ and $\rho = \min\{\rho_0, \rho_1\}$. Note that $\rho_1 > 0$ and hence $\rho > 0$. Let

$$\epsilon = \frac{1}{8} \min \{\rho, \rho, \inf f(U_1) - y, y - \sup f(U_2), y - \sup f(U_3)\}.$$ 

Then, $\epsilon > 0$.

Let us first prove the lemma for $f_0$ and $g_0 \in \text{WB}$. Let $g_0 \in \text{WB}$ such that $\|f_0 - g_0\| < \epsilon$. Let $\epsilon < c < 2\epsilon$ be such that $g_0^{-1}(y - c)$ contains no component homeomorphic to the figure-eight or to a point. By our choice of $\epsilon$, we have that for $y - 2\epsilon \leq z \leq y + 2\epsilon$, $g_0^{-1}(z) \cap \cup_{i=1}^3 U_i = \emptyset$. As $g_0(M) \subseteq (y - \epsilon, y + \epsilon)$, each arc intersecting $U_2$ and $U_3$ intersects $M$ and each arc intersecting $U_2$ and $U_1$ intersects $M$, we have that $g^{-1}(y - c)$ separates $U_2$ from $U_3$ and $g^{-1}(y - c)$ separates $U_2$ from $U_1$. Let $C_2$ be the boundary of the component of $S^2 \setminus g^{-1}(y - c)$ containing $U_2$. Note that $C_2$ is homeomorphic to a circle. Hence, $C_2$ separates $U_2$ from $U_3$ and separates $U_2$ from $U_1$. Similarly, there is a component $C_3$ of $g_0^{-1}(y - c)$ which separates $U_3$ from $U_2$ and separates $U_3$ from $U_1$. Hence, $\delta(g_0, y - c, U_1, U_2, U_3) > 0$.

Let us next observe that for $z \geq y + \epsilon$, $g_0^{-1}(z)$ does not separate $U_2$ from $U_3$. Indeed, this is the case as $g_0(W_2) \cup g_0(W_3) \cup g_0(M) \subseteq (-\infty, y + \epsilon)$ and $W_2 \cup W_3 \cup M$ is connected.
Let \( T = \{ t : \delta(g_0, t, \overline{U_1}, \overline{U_2}, \overline{U_3}) > 0 \} \). Then, \( y - y' \in T \) and \( y' = \sup T \) is less than \( y + \epsilon \). Hence \( |y - y'| < 2\epsilon \). By Remark 5.10, \( \delta(g_0, y', \overline{U_1}, \overline{U_2}, \overline{U_3}) \) is well-defined and by Remark 5.13, we have that \( \delta(g_0, y', \overline{U_1}, \overline{U_2}, \overline{U_3}) = 0 \). Hence, by Remark 5.12 \( g_0^{-1}(y') \) has a component which is homeomorphic to the figure-eight which separates \( \overline{U_1}, \overline{U_2}, \overline{U_3} \). By the choice of \( \rho \) and \( \epsilon \) we also have \( g_0^{-1}(y') \subseteq U \).

Let us now consider \( g_0 \in \mathcal{C}(S^2, I) \) with \( \|f_0 - g_0\| < \epsilon \). As \( \mathcal{WB} \) is dense in \( \mathcal{C}(S^2, I) \) (Lemma 5.6), we may choose a sequence \( \{g_n\} (n = 1, 2, \ldots) \) in \( \mathcal{WB} \) with \( \|f_0 - g_n\| < \epsilon \) and \( g_0 \) is the uniform limit of \( \{g_n\} \). By what we have just shown, there is \( y'_n \) with \( |y'_n - y| < 2\epsilon \) and a component \( N_3 \) of \( g_0^{-1}(y'_n) \) which separates \( U_1, U_2, U_3 \). By turning to a suitable subsequence, we may assume that \( \{N_n\} \) converges to some continuum \( M' \) in the Hausdorff metric and \( \{y'_n\} \) converges to some \( y' \). Note that \( g_0(M') = y' \) and \( y' \in [y - 2\epsilon, y + 2\epsilon] \). By Lemma 5.9, \( N \), the component of \( g_0^{-1}(y') \) containing \( M' \) either separates \( U_1, U_2, U_3 \) or it contains one of the \( U_i \)'s. As \( g_0(U_i) \cap [y - 2\epsilon, y + 2\epsilon] = \emptyset \) for \( 1 \leq i \leq 3 \), we have that \( N \) separates \( U_1, U_2, U_3 \). Again \( N \subseteq U \) by the choice of \( \rho \) and \( \epsilon \).

Finally, let us consider the general case of \( f \in \mathcal{WB} \) and \( g \in \mathcal{C}(S^2, I) \) with \( \|f - g\| < \epsilon \). Let \( U \) and \( f_0 \) be as above. Then, using Tietze’s Extension Theorem (applied to \( G = (g - f)(\overline{U}) \)) one can choose \( G_0 \) such that \( G_0 \) equals \( g - f \) on \( U \) and \( \|G_0\| < \epsilon \). Then set \( g_0 = f_0 + G_0 \). Now \( g_0 = g \) on \( U \) and \( \|g_0 - f_0\| < \epsilon \). By the above argument we can find a component, \( N \subseteq U \) of \( g_0^{-1}(y') \) which separates \( U_1, U_2, U_3 \). Since \( g_0 = g \) on \( U \) we obtain that \( N \) is a component of \( g^{-1}(y') \) as well.

5.2. Figure-eight-likeness in generic maps. In this subsection we show that a generic \( f \in \mathcal{C}(S^2, I) \) has the property that all components of each fiber of \( f \) are either points, or figure-eight-like.

Let us first note that up to homeomorphism, there are only finitely many sub-continua of figure-eight, namely a point, an arc, a circle, the letter \( T \), the letter \( X \), a circle with one hair, a circle with two hairs originating from the same point, and the figure-eight. We let \( T \) be this finite collection.

Let us also recall the following result which was proved independently by Krasinkiewicz [10] and Levin [11].

**Theorem 5.15.** (Krasinkiewicz-Levin) Let \( X \) be a compact metric space. Then, a generic \( f \in \mathcal{C}(X, I) \) has the property that each of its fibers is a Bing compactum, a compactum with all components hereditarily indecomposable.

**Lemma 5.16.** A generic \( f \in \mathcal{C}(S^2, I) \) has the property that if \( y \in f(S^2) \) and \( M \) is a nondegenerate component of \( f^{-1}(y) \), then \( M \) is hereditarily indecomposable and \( P \)-like for some \( P \in T \).

**Proof.** That \( M \) is hereditarily indecomposable follows from Theorem 5.15. Let us now show that \( M \) is \( P \)-like for some \( P \in T \). Let \( \epsilon > 0 \). Let \( \mathcal{G} \) be the collection of those \( f \in \mathcal{C}(S^2, I) \) for which there is \( y \in f(S^2) \) and a component \( M \) of \( f^{-1}(y) \) such that there is no \( \epsilon \)-map from \( M \) onto any element of \( T \). It will suffice to show that the closure of \( \mathcal{G} \) is nowhere dense. Let \( \{f_n\} \) be a sequence of elements in \( \mathcal{G} \) and let \( f \) be its limit in the sup norm. Let \( \{y_n\} \) and \( \{M_n\} \) be such that \( M_n \) is a component of \( f_n^{-1}(y_n) \) and there is no \( \epsilon \)-map from \( M_n \) onto any member of \( T \). Without loss
of generality we may assume that \( \{ y_n \} \) converges to some \( y \) and \( \{ M_n \} \) converges to some \( M \) in the Hausdorff metric. Then, \( f(M) = y \). Let \( N \) be the component of \( f^{-1}(y) \) which contains \( M \). We claim that there is no \( \frac{1}{2} \)-map from \( N \) onto any member of \( T \). To obtain a contradiction, assume there is an \( \frac{1}{2} \)-map from \( N \) onto some member of \( T \). As \( M \) is a subcontinuum of \( N \), there is an \( \epsilon \)-map from \( M_n \) onto some element of \( T \). By Lemma 4.5, for sufficiently large \( n \), there is an \( \epsilon \)-map from \( M_n \) onto some element of \( T \), yielding a contradiction. Hence, we have shown that if \( y \in \mathcal{G} \), then there is \( y \in g(S^2) \) and a component \( M \) of \( g^{-1}(y) \) which is \( P \)-like for no \( P \in T \). By Lemma 5.6, we have that \( \mathcal{G} \) is nowhere dense.

**Theorem 5.17.** A generic \( f \in C(S^2, I) \) has the property that each component of each fiber of \( f \) is either a point, or a hereditarily indecomposable continuum which is figure-eight-like.

**Proof.** This simply follows from Lemma 5.16 and Theorem 4.11. \( \square \)

5.3. **Existence of Lakes of Wada continua.** In this subsection we show that a generic \( f \in C(S^2, I) \) has the property that there is a countable dense set \( D \subseteq f(S^2) \) such that for all \( y \in D \), there is a component of \( f^{-1}(y) \) which is a Lakes of Wada continuum.

**Lemma 5.18.** A generic \( f \in C(S^2, I) \) has the property that if \( y \) is in the range of \( f \), then no component of \( f^{-1}(y) \) separates \( S^2 \) into more than three pieces.

**Proof.** Fix a countable basis \( \mathcal{B} \) for the topology on \( S^2 \). Let \( U_1, U_2, U_3, U_4 \) be pairwise disjoint elements of \( \mathcal{B} \). Let \( \mathcal{G} \) consist of those \( f \in C(S^2, I) \) for which there is a \( y \) in its range and a component \( M \) of \( f^{-1}(y) \) which separates \( U_1, U_2, U_3, U_4 \). It will suffice to show that the closure of \( \mathcal{G} \) in \( C(S^2, I) \) is a nowhere dense subset of \( C(S^2, I) \). Let \( \{ f_n \} \) be a sequence in \( \mathcal{G} \) and let \( f \) be its limit in the sup norm. Let \( \{ y_n \} \) and \( \{ M_n \} \) be such that for all \( n \) we have that \( M_n \) is a component of \( f_n^{-1}(y_n) \) and \( M_n \) separates \( U_1, U_2, U_3, U_4 \). Without loss of generality, we may assume that \( \{ y_n \} \) converges to some number \( y \) and \( \{ M_n \} \) converges to some continuum \( M \) in the Hausdorff metric. We note that \( f(M) = y \). Let \( N \) be the component of \( f^{-1}(y) \) containing \( M \). By Lemma 5.9 we have that \( N \) separates \( S^2 \) into at least four pieces, or \( N \) contains an open set. Therefore, we have that if \( y \) is in the closure of \( \mathcal{G} \), then there is \( y \) and a component of \( g^{-1}(y) \) which separates \( S^2 \) into at least four pieces, or some component of \( g^{-1}(y) \) contains an open set. By Lemma 5.6, it follows that the closure of \( \mathcal{G} \) is nowhere dense.

**Lemma 5.19.** Suppose that \( M_1, M_2 \subseteq S^2 \) are disjoint continua, separating \( S^2 \) into \( k_1, k_2 \) pieces, respectively. Then, \( M_1 \cup M_2 \) separates \( S^2 \) into at least \( k_1 + k_2 - 1 \) pieces.

**Proof.** We note that since \( M_2 \) is a continuum, \( M_2 \) is a subset of some component of \( S^2 \setminus M_1 \). Using this observation, choose components \( U_1, U_2, \ldots, U_k \) of \( S^2 \setminus M_1 \) and components \( V_1, V_2, \ldots, V_k \) of \( S^2 \setminus M_2 \) so that \( M_2 \subseteq U_1 \). We first note that for all \( 1 \leq i \leq k_2 \), \( \partial V_i \subseteq M_2 \subseteq U_1 \). Hence, \( V_i \cap U_1 \neq \emptyset \) for \( i = 1, 2, \ldots, k_2 \). Now we claim that \( M_1 \cup M_2 \) separates \( U_2, U_1 \cap V_1, U_1 \cap V_2, \ldots, U_1 \cap V_k \), Clearly, \( U_2 \cup \ldots \cup U_k \cup (U_1 \cap V_1) \cup (U_1 \cap V_2) \cup \ldots \cup (U_1 \cap V_k) \subseteq S^2 \setminus (M_1 \cup M_2) \). Let \( W_1, W_2 \) be two distinct elements of \( \{ U_2, \ldots, U_k, U_1 \cap V_1, U_1 \cap V_2, \ldots, U_1 \cap V_k \} \) and let \( A \) be an arc in \( S^2 \) intersecting \( W_1 \) and \( W_2 \). If \( \{ W_1, W_2 \} \subseteq \{ U_2, \ldots, U_k \} \), then \( A \cap M_1 \neq \emptyset \). If \( \{ W_1, W_2 \} \subseteq \{ U_1 \cap V_1, U_1 \cap V_2, \ldots, U_1 \cap V_k \} \), then \( A \cap M_2 \neq \emptyset \). If one
of \( \{ W_1, W_2 \} \) is in \( \{ U_2, \ldots, U_{n_k} \} \) and the other is in \( \{ U_1 \cap V_1, U_1 \cap V_2, \ldots, U_1 \cap V_{k_1} \} \), then we have that \( A \cap M_1 \neq \emptyset \). Hence, we have shown that \( A \cap (M_1 \cup M_2) \neq \emptyset \). Therefore, \( S^2 \setminus (M_1 \cup M_2) \) has at least \( k_1 + k_2 - 1 \) components. 

**Lemma 5.20.** A generic \( f \in \mathcal{C}(S^2, I) \) has the property that if \( y \) is in the range of \( f \), then \( f^{-1}(y) \) contains at most one component which separates \( S^2 \) into three pieces or more.

**Proof.** As earlier, fix a countable basis \( \mathcal{B} \) for the topology on \( S^2 \). For \( i = 1, 2 \), let \( U_1, U_2, U_3 \) be elements of \( \mathcal{B} \) so that \( U_i \cap U_j = \emptyset \) if \( i \neq j \). Let \( \mathcal{G} \) be the set of those functions \( f \in \mathcal{C}(S^2, I) \) for which there is \( y \) and two distinct components \( M^1 \) and \( M^2 \) of \( f^{-1}(y) \) such that \( M^1 \) separates \( \{ U_1, U_2, U_3 \} \). It will suffice to show that the closure of \( \mathcal{G} \) is nowhere dense. As before, let \( \{ f_n \} \) be a sequence in \( \mathcal{G} \) and let \( f \) be its limit in the sup norm. Let \( \{ y_n \} \) and \( \{ M^1_n \} \) (\( i = 1, 2 \)) be such that for all \( n \) and \( i \), we have that \( M^1_n \) is a component of \( f^{-1}_n(y_n) \); \( M^1_n \) separates \( U_1, U_2, U_3 \); furthermore \( M^1_n \) and \( M^2_n \) are two distinct components of \( f^{-1}_n(y_n) \). Without loss of generality, we may assume that \( \{ y_n \} \) converges to some number \( y \) and \( \{ M^1_n \} \) converges to some continuum \( M^1 \) in the Hausdorff metric. We note that \( f(M^1) = y \). Let \( N^1 \) be the component of \( f^{-1}(y) \) containing \( M^1 \). We have that either \( N^1 \) and \( N^2 \) are disjoint or \( N^1 = N^2 \).

Let us first consider the case \( N^1 \cap N^2 = \emptyset \). Then, by Lemma 5.9, we have that \( N^1 \) either separates \( S^2 \) into three pieces or contains an open set. In this case, we have that there is \( y \) and a component of \( f^{-1}(y) \) which contains an open set, or there are two components of \( f^{-1}(y) \) which separate \( S^2 \) into at least three pieces.

Let us now consider the case that \( N^1 = N^2 \). Note that for all \( n \), \( M^1_n \) and \( M^2_n \) are two disjoint continua each one of which separates \( S^2 \) into at least three pieces. Therefore, by Lemma 5.19, we have that \( M^1_n \cup M^2_n \) separates \( S^2 \) into at least five pieces. Since \( N^1 \supseteq M^1 \cup M^2 \), we have by Lemma 5.9 that \( N^1 \) separates the plane into at least five pieces or \( N^1 \) contains an open set. In this case, we have that there is \( y \) and a component of \( f^{-1}(y) \) which separates \( S^2 \) into five pieces or contains an open set.

Combining the two cases above, what we have is that if \( g \in \mathcal{G} \), then there is \( y \) such that one of the following happens:

- \( g^{-1}(y) \) has at least two distinct components each of which separates \( S^2 \) into three or more pieces,
- \( g^{-1}(y) \) has at least one component which separates \( S^2 \) into five or more pieces, or
- \( g^{-1}(y) \) has a component which contains an open set.

Now it follows from Lemma 5.6 that \( \overline{\mathcal{G}} \) is nowhere dense. 

**Lemma 5.21.** If \( \mathcal{M} \) is a pairwise disjoint collection of continua in \( S^2 \) and each element of \( \mathcal{M} \) separates \( S^2 \) into at least three components but no more than finitely many, then \( \mathcal{M} \) is countable.

**Proof.** To obtain a contradiction, assume that \( \mathcal{M} \) is an uncountable such collection. By taking an appropriate uncountable subcollection, we may assume that there is a positive integer \( n \geq 3 \) such that each element of \( \mathcal{M} \) separates \( S^2 \) into exactly \( n \) components. Furthermore, using the separability of \( S^2 \), we may assume that there is a sequence of distinct points \( x_1, x_2, \ldots, x_n \in S^2 \) which are separated by each element of \( \mathcal{M} \). Let \( M_1, M_2 \) be two distinct continua in \( \mathcal{M} \) which separate
Let \( U_1, U_2, \ldots, U_n \) be the components of \( S^2 \setminus M \) which contain \( x_1, x_2, \ldots, x_n \), respectively. Since \( M_2 \) is a continuum, \( M_2 \) is a subset of one of \( U_1, U_2, \ldots, U_n \). Without loss of generality, we may assume that it is \( U_1 \). Then, \( U_2 \cup U_3 \cup M_1 \) is a connected set which misses \( M_2 \). This contradicts that \( M_2 \) separates \( x_2 \) and \( x_3 \). Hence, \( M \) is countable.

**Lemma 5.22.** A generic \( f \in C(S^2, I) \) has the property that there is a dense set \( D \subseteq f(S^2) \) such that for each \( y \in D \), there is a component of \( f^{-1}(y) \) which separates \( S^2 \) into three pieces or more.

**Proof.** Let \( r \in I \) be a rational number and \( n \) be an integer. Let \( G_{r,n} \) be those functions \( f \) such that if \( f(S^2) \cap (r - 1/n, r + 1/n) \neq \emptyset \), then there is a \( y \in (r - 3/n, r + 3/n) \) such that \( f^{-1}(y) \) contains a component which separates the plane into at least three pieces. We claim that \( G_{r,n} \) contains a dense open set in \( C(S^2, I) \). Indeed, let \( f \in WB \) and \( \epsilon > 0 \). If \( f(S^2) \cap (r - 1/n, r + 1/n) \neq \emptyset \), then we can find \( g \in WB \) so that \( ||f - g|| < \epsilon \) and some \( y \in (r - 1/n, r + 1/n) \) so that \( g^{-1}(y) \) contains a component homeomorphic to the figure-eight. By Lemma 5.14, we have that there is an open ball containing \( g \) which is a subset of \( G_{r,n} \). Now, \( G = \cap_{r \in Q} \cap_{n=1}^{\infty} G_{r,n} \) is the desired dense \( G \) set.

**Theorem 5.23.** A generic \( f \in C(S^2, I) \) has the property that there is a countable dense set \( D \subseteq f(S^2) \) such that for each \( y \in f(S^2) \) every component of \( f^{-1}(y) \) separates \( S^2 \) into two pieces or less except when \( y \in D \). In the latter case, the same applies to each component with one exception which separates \( S^2 \) into exactly three pieces.

**Proof.** A generic function satisfies the conclusions of Lemmas 5.18, 5.20, and 5.22. Let \( f \) be such a function. By Lemma 5.18, for each \( y \in f(S^2) \), each component of \( f^{-1}(y) \) separates \( S^2 \) into three pieces or less. Let \( D \) be the set of \( y \in f(S^2) \) for which there is a component of \( f^{-1}(y) \) which separates \( S^2 \) into exactly three pieces. By Lemma 5.21, \( D \) is countable and by Lemma 5.22 it is dense in \( f(S^2) \). By Lemma 5.20, for each \( y \in D \), there is exactly one component of \( f^{-1}(y) \) which separates \( S^2 \) into three pieces.

**Definition 5.24.** Suppose that \( M, N \subseteq S^2 \), and \( \epsilon > 0 \). We say that \( M \) is \( \epsilon \)-approximated by \( N \) if for each \( x \in M \) there is \( y \in N \) such that \( d(x,y) < \epsilon \). We note that if \( M \) is \( \epsilon \)-approximated by \( N \), \( N \) is \( \epsilon \)-approximated by \( M \) and furthermore, \( M, N \) are compacta, then \( d_H(M,N) < \epsilon \).

**Definition 5.25.** Let \( M \subseteq S^2 \) be a continuum and \( \epsilon > 0 \). We say that \( M \) is \( \epsilon \)-approximated from the outside if \( M \) is \( \epsilon \)-approximated by each component \( U \) of \( S^2 \setminus M \).

**Proposition 5.26.** Let \( f \in C(S^2, I) \) and \( \gamma, \epsilon > 0 \). Then, there is \( g \in WB \) such that \( ||f - g|| < \gamma \) and for all \( y \in g(S^2) \) and \( M \in \text{Comp}(g^{-1}(y)) \), \( M \) is \( \epsilon \)-approximated from the outside.

**Proof.** By Lemma 5.6 we may choose \( h \in WB \) such that \( ||f - h|| < \frac{\epsilon}{2} \). We note that if \( y \in h(S^2) \), \( M \in \text{Comp}(h^{-1}(y)) \) which is not homeomorphic to the figure-eight, then \( M \) is \( \epsilon \)-approximated from the outside. We now suggest how \( h \) can be adjusted slightly so that all components of all fibers of \( h \) are \( \epsilon \)-approximated from the outside. Let \( y_1 < y_2 < \ldots < y_n \) be those reals for which \( h^{-1}(y_i) \) contains a component homeomorphic to the figure-eight.
Set $\phi(x, y) = ((x - 1)^2 + y^2)((x + 1)^2 + y^2)$. Then, $\phi^{-1}(1)$ is homeomorphic to the figure-eight.

Choose $\gamma' \in (0, \gamma/4)$ so that $\gamma' < \min\{|y_j - y_k|/4 : j \neq k\}$, and for any $j$ any two different components $M, M'$ of $f^{-1}(y_j)$ belong to different components of $h^{-1}(y_j - \gamma', y_j + \gamma')$. Assume now that $M$ is a component of $h^{-1}(y_j)$ homeomorphic to the figure eight. Denote by $\Omega_M$ the component of $h^{-1}((y_j - \gamma', y_j + \gamma'))$ which contains $M$. We can also assume that $\gamma'$ is chosen so small that $\Omega_M$ is homeomorphic to the “fat” figure eight $\Omega_{0.5, 1.5} = \{(x,y) : 0.5 < \phi(x,y) < 1.5\}$ and without limiting generality we can assume that $\partial\Omega_M$ contains one component of $h^{-1}(y_j + \gamma')$ and two components of $h^{-1}(y_j - \gamma')$. The first component is denoted by $M_+$ and the other two components are denoted by $M_{-1}$ and $M_{-2}$. Now $\Omega_M \setminus M$ can be split into three regions $\Omega_{M,+} \cup \Omega_{M,-1} \cup \Omega_{M,-2}$ so that one component of the boundary of these regions is $M_+$, $M_{-1}$, and $M_{-2}$, respectively. Hence $f > y_j$ on $\Omega_{M,+}$ and $f < y_j$ on $\Omega_{M,-1} \cup \Omega_{M,-2}$ and all these regions are homeomorphic to an annulus. Now, one can replace $M$ by another continuum $M' \subseteq \Omega_M$ so that $M'$ is homeomorphic to the figure-eight, it is $c$-approximated from the outside and $\Omega_M \setminus M'$ can be split into three regions $\Omega_{M,+} \cup \Omega_{M,-1} \cup \Omega_{M,-2}$ so that these regions are homeomorphic to an annulus, one component of their boundary is part of $M'$, and the other component of their boundary is $M_+, M_{-1}$, and $M_{-2}$, respectively. Choose homeomorphisms $\Psi_+: \overline{\Omega}_{M,+} \to \overline{\Omega}_{M,+}, \Psi_{-1}: \overline{\Omega}_{M,-1} \to \overline{\Omega}_{M,-1}, \Psi_{-2}: \overline{\Omega}_{M,-2} \to \overline{\Omega}_{M,-2}$ so that $\Psi_+|_{M_+}, \Psi_{-1}|_{M_{-1}}, \Psi_{-2}|_{M_{-2}}$ are identities. Set $g(x) = h(\Psi_+(x))$ for $x \in \overline{\Omega}_{M,+}$ and $g(x) = h(\Psi_{-1}(x))$ for $x \in \overline{\Omega}_{M,-1}$, $\ell = 1, 2$.

Define $g$ as above for all $j$ and for all components $M$ of $h^{-1}(y_j)$ homeomorphic to the figure eight. Since by our assumptions the sets $\Omega_M$ are disjoint $g$ is well-defined on these components. For those $x$ which do not belong to any of these components we put $g(x) = h(x)$.

Then, $g$ is the desired function. □

**Lemma 5.27.** A generic $f \in C(S^2, I)$ has the property that if $y \in f(S^2)$ and $M \in \text{Comp}(f^{-1}(y))$, then $M$ is the boundary of each component of $S^2 \setminus M$.

**Proof.** Let $\mathcal{B}$ be a countable base for the topology on $S^2$ consisting of open balls, $k \in \mathbb{N}$ and $V \in \mathcal{B}$. Let $\mathcal{G}_{V,k}$ be the set of those $f \in C(S^2, I)$ for which there exists $y \in f(S^2)$, a component $M$ of $f^{-1}(y)$ and $x \in M$ such that the following two conditions hold:

- $|f(p) - f(x)| = |f(p) - y| \geq \frac{1}{k}$ for all $p \in V$, and
- $d(x, \overline{U}) \geq \frac{1}{k}$ where $U$ is the component of $S^2 \setminus M$ with $V \subseteq U$.

It will suffice to show that $\mathcal{G}_{V,k}$ is nowhere dense and closed.

Let us first proceed to show that $\mathcal{G}_{V,k}$ is closed. Let $\{f_n\}$ be a sequence of functions in $\mathcal{G}_{V,k}$ which converges uniformly to some function $f \in C(S^2, I)$. Let $\{y_n\}, \{M_n\}, \{x_n\}, \{U_n\}$ be such that $y_n \in f_n(S^2), M_n \in \text{Comp}(f_n^{-1}(y_n)), x_n \in M_n, |f(p) - y_n| \geq 1/k$ for all $p \in V$ and $U_n$ is the component of $S^2 \setminus M_n$ containing $V$ and satisfying $d(x_n, \overline{U_n}) \geq 1/k$. Without loss of generality, we may assume that $\{y_n\}$ converges to some $y$, $\{M_n\}$ converges to some $M$, $\{x_n\}$ converges to some $x$, and $\{U_n\}$ converges in the Hausdorff metric to some set $K$. We note that $M$ and $K$ are continua, $f(M) = y$, $x \in M, d(x, K) \geq 1/k, |f(p) - y| \geq 1/k$ for all $p \in V$ and $V \subseteq K$. Let $N$ be the component of $f^{-1}(y)$ containing $M$. Denote by $U$ the component of $S^2 \setminus N$ with $V \subseteq U$. We need to show that $U \subseteq K$ to complete the proof of the fact that $\mathcal{G}_{V,k}$ is closed. Let $p \in U$. As $V \subseteq U$, there is an arc
A \subseteq U$ such that $p \in A$ and $A \cap V \neq \emptyset$. For sufficiently large $n$, we have that $A \cap M_n = \emptyset$; for otherwise, $A \cap M \neq \emptyset$ would imply $A \cap N \neq \emptyset$, contradicting that $A \subseteq U \subseteq S^2 \setminus N$. As $A$ is connected, $A \cap V \neq \emptyset$, and $A \subseteq S^2 \setminus M_n$ for sufficiently large $n$, we have that $A \subseteq U_n$ for sufficiently large $n$. As $\{U_n\}$ converges to $K$ in the Hausdorff metric, we have that $p \in K$. Hence, $\mathcal{G}_{V \cup k}$ is closed.

Now the fact that $\mathcal{G}_{V \cup k}$ is nowhere dense simply follows from Proposition 5.26. \hfill \Box

**Corollary 5.28.** A generic $f \in \mathcal{C}(S^2, I)$ has the property that if $y \in f(S^2)$ and $M \in \text{Comp}(f^{-1}(y))$ separates $S^2$ into three pieces, then $M$ is a Lakes of Wada continuum.

*Proof.* This simply follows from Theorem 5.17, and Theorem 5.27. \hfill \Box

**Corollary 5.29.** A generic function $f \in \mathcal{C}(S^2, I)$ has the property that there is a countable dense set $D \subseteq f(S^2)$ such that for all $y \in D$ there is a component $M$ of $f^{-1}(y)$ which is a Lakes of Wada continuum.

*Proof.* This follows from Corollary 5.28 and Theorem 5.23. \hfill \Box

### 5.4. Existence of pseudoarcs and pseudocircles.

In this subsection we show that for a generic $f \in \mathcal{C}(S^2, I)$, for almost all $y \in f(S^2)$, all components of $f^{-1}(y)$ are either points, pseudoarcs or pseudocircles. Furthermore, for a generic $f$, for all $y \in (\min f, \max f)$ there are components of $f^{-1}(y)$ which are pseudoarcs.

Given a set $E$ we denote its $\epsilon > 0$ neighborhood by $B_{\epsilon}(E) = \{x : d(x, E) < \epsilon\}$, the closure of $B_{\epsilon}(E)$ is denoted by $\overline{B_{\epsilon}(E)}$.

The following lemma is a standard fact from the plane topology.

**Lemma 5.30.** Let $M \subseteq S^2$ be a continuum, $U \subseteq S^2$ be a connected open set with $M \subseteq U$ and $\alpha, \beta > 0$. Then, there is an arc $\gamma \subseteq U$ and $0 < \rho < \beta$ such that

- $d_M(M, \gamma) < \alpha$,
- $\overline{B_{\rho}(\gamma)} \subseteq U$, and
- $\overline{B_{\rho}(\alpha)}$ and $\overline{B_{\rho/2}(\alpha)}$ admit $\alpha$ and $2\alpha$ maps onto $\gamma$, respectively.

*Proof.* The proof basically follows from the well-known fact that each continuum in $S^2$ can be approximated by an arc. \hfill \Box

For $A, B \subseteq S^2$ we put $\text{dist}(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$.

**Theorem 5.31.** A generic $f \in \mathcal{C}(S^2, I)$ has the property that if $y \in (\min(f(S^2)), \max(f(S^2)))$, then $f^{-1}(y)$ contains a component which is a pseudoarc.

*Proof.* In light of Theorem 5.15, it will suffice to show that a generic $f \in \mathcal{C}(S^2, I)$ has the property that for all $y \in (\min f, \max f)$ there is a component of $f^{-1}(y)$ which is arc-like.

Given $f \in \mathcal{C}(S^2, I)$, $y \in (\min f, \max f)$ and $\epsilon, \rho > 0$ we call a subset of $S^2$ an $\epsilon$-$\rho$-$y$-worm for $f$ and denote it by $W$ if

- $W = \overline{B_{\rho}(\gamma)}$ for some arc $\gamma$,
- $f(x) = y + \epsilon$ for all $x \in \partial W$,
- $f(x) > y - \epsilon$ for all $x \in \gamma$, and
- there exists a $4\rho$-map, $\Gamma_W : W \to \gamma$. 

It will suffice to show that for a generic \( f \in C(S^2, I) \) and \( y \in (\min f, \max f) \) there exists a sequence of \( \epsilon_k, \rho_k \)-y-worms \( \{ W_k \} \) for \( f \) such that

- \( W_{k+1} \subseteq W_k \),
- \( \epsilon_k \to 0, \rho_k \to 0 \), and
- \( \text{diam}(W_k) \neq 0 \).

Indeed, then it is not difficult to see that \( \bigcap_{k=1}^{\infty} W_k \) is an arc-like continuum which is a component of \( f^{-1}(y) \).

Let \( \phi \in \mathbf{WB} \), \( \mu, \nu \in \mathbb{N} \). We construct \( \phi^* = \phi^*(\mu, \nu) \) as follows. Partition \( \phi(S^2) \) into \( 2^\mu \) many equal pieces and let \( J \) consist of the endpoints of these partition intervals. Let \( J \) be the set of all components of \( \phi^{-1}(y) \) such that \( y \in J \). Note that since \( \phi \in \mathbf{WB} \), each element of \( J \) is either a point, a circle or a figure-eight.

Let \( 0 < \eta < 2^{-\nu} \) be small enough so that if \( M \) and \( M' \) are two distinct elements of \( J \), then \( \overline{B}_\eta(M) \cap \overline{B}_\eta(M') = \emptyset \) and if \( d(x, M) < \eta \) for some \( M \in J \), then \( |\phi(x) - \phi(M)| < 2^{-\mu} \lambda_1(\phi(S^2)) \), where \( \lambda_1(\phi(S^2)) \) denotes the Lebesgue measure of \( \phi(S^2) \).

Fix \( M \in J \) with \( \phi(M) = y \). Using Lemma 5.30, choose an arc \( \gamma_M \subseteq B_\eta(M) \) and \( 0 < \rho_M < \eta \) so that

\[
\text{diam}(\gamma_M) - \rho_M > (1 - (1/2)^\nu) \text{diam}(M),
\]

\( \overline{B}_{\rho_M}(\gamma_M) \subseteq B_\eta(M) \), and \( \overline{B}_{\rho_M}(\gamma_M) \) and \( \overline{B}_{\rho_M/2}(\gamma_M) \) admit \( 4 \rho_M \) and \( 2 \rho_M \) maps onto \( \gamma_M \), respectively. Set \( V_M = \overline{B}_{\rho_M}(\gamma_M) \) and \( W_M = \overline{B}_{\rho_M/2}(\gamma_M) \). Assume that \( M \subseteq W_M \) is a continuum separating \( \partial W_M \) from \( \gamma_M \). For any \( x \in \partial W_M \) there exists \( x' \in \gamma_M \) and an arc \( \gamma' \subseteq W_M \) of length \( \rho_M/2 \) connecting \( x \) and \( x' \). This arc should intersect \( M \). Hence

\[
\text{diam}(\gamma_M - \rho_M) \geq \text{diam}(\gamma_M) - \rho_M.
\]

Now we define \( \phi^*_M \) on \( V_M \) as follows. If \( x \in \partial V_M \), then put \( \phi^*_M(x) = \phi(x) \). If \( x \in \gamma_M \), then set \( \phi^*_M(x) = y - 3 \cdot 2^{-\nu} \lambda_1(\phi(S^2)) \). If \( x \in \partial W_M \), then set \( \phi^*_M(x) = y + 3 \cdot 2^{-\nu} \lambda_1(\phi(S^2)) \). Now using Tietze Extension Theorem, extend \( \phi^*_M \) to all of \( V_M \) so that \( \phi^*_M(V_M) = [y - 3 \cdot 2^{-\nu} \lambda_1(\phi(S^2)), y + 3 \cdot 2^{-\nu} \lambda_1(\phi(S^2))] \).

We do this for all \( M \in J \) and obtain \( V_M, W_M \) and \( \phi^*_M \). We let \( \phi^* = \phi^*_M \) on \( V_M \) and \( \phi^* = \phi \) otherwise. We put \( \delta = 2^{-\mu} \lambda_1(\phi(S^2)) \). Let \( g \in C(S^2, I) \) be such that \( \| \phi^* - g \| \leq \delta \). We note that \( \phi^*, g \) satisfy the following properties.

1. \( \| \phi - \phi^* \| \leq 4 \cdot 2^{-\mu} \lambda_1(\phi(S^2)) \leq 4 \cdot 2^{-\mu} \),
2. \( \| \phi - g \| < 5 \cdot 2^{-\mu} \lambda_1(\phi(S^2)) \leq 5 \cdot 2^{-\mu} \),
3. if \( y \in \phi(S^2) \) and \( M \) is some component of \( \phi^{-1}(y') \) for some \( y' \in J \) nearest to \( y \), then \( W_M \) is a \( 2^{-\mu} \lambda_1(\phi(S^2)) \)-\( \rho_M \)-y worm for \( \phi^* \) with respect to some arc \( \gamma \),
4. Property (3) holds when \( \phi^* \) is replaced by \( g \).

Now we proceed to construct our desired dense \( G_\delta \) set. Choose \( \{ f_m \in \mathbf{WB} : m = 1, 2, \ldots \} \) dense in \( C(S^2, I) \). For each positive integer \( n \in \mathbb{N} \), obtain \( h_{m,n} = f_m(m + n, n) = \phi^*(\mu, \nu) \) from \( f_m = \phi \) in the above fashion using \( \mu = m + n \) and \( \nu = n \). Let \( \delta_{m,n} = 2^{-(m+n)} \lambda_1(f_m(S^2)) \).

Our desired dense \( G_\delta \) is \( G = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} B_{\delta_{m,n}}(h_{m,n}) \).

Now we fix a \( g \in G \) and a \( y \in (\min g, \max g) \). Let \( d = \min \{ y - \min g, \max g - y \} \). Let \( N \in \mathbb{N} \) be such that \( 2^{-N} < d/5 \). By Property (2) above, we have that for all \( n > N \) and \( m \in \mathbb{N} \), if \( g \in B_{\delta_{m,n}}(h_{m,n}) \), then \( y \in [\min f_m, \max f_m] \).
Now we choose \( n_1 > N \) and \( m_1 \in \mathbb{N} \) such that \( g \in B_{m_1,n_1}(h_{m_1,n_1}) \). Using the fact that \( g \in f_m(S^2) \) and Property (4) above, there is an \( \epsilon_1, \rho_1 \)-y worm, \( W_1 \) for \( g \) with respect to some arc \( \gamma_1 \) and some \( 0 < \epsilon_1 \leq 2^{-m_1+n_1}, 0 < \rho_1 < 2^{-n_1} \) such that (7) and (8) hold.  

Now suppose that \( k > 1 \), \( \{W_i\}_{i=1}^k, \{\gamma_i\}_{i=1}^k, \{n_i\}_{i=1}^k, \{m_i\}_{i=1}^k, \{\epsilon_i\}_{i=1}^k, \) and \( \{\rho_i\}_{i=1}^k \) have been constructed so that  

(a) \( \{n_i\}_{i=1}^k \) is an increasing sequence,  
(b) \( \epsilon_i \leq 2^{-(m_i+n_i)} \) for all \( 1 \leq i \leq k \),  
(c) \( \rho_i < 2^{-n_i} \) for all \( 1 \leq i < k \),  
(d) \( W_{i+1} \subseteq W_i \) for all \( 1 \leq i < k \),  
(e) for all \( 1 \leq i < k \), \( W_i \) is an \( \epsilon_i, \rho_i \)-y worm for \( g \) with respect to an arc \( \gamma_i \), and  
(f) for all \( 1 < i \leq k \),  

\[
\text{diam}(\gamma_i) - \rho_i > (1 - (1/2)^i)(\text{diam}(\gamma_{i-1}) - \rho_{i-1}) > 0,
\]

and hence \( \text{diam}(\gamma_i) - \rho_i > \prod_{j=2}^i (1 - (1/2)^j)(\text{diam}(\gamma_1) - \rho_1) > 0 \).  

For each \( n > n_k \), let \( m(n) \) be chosen so that \( g \in B_{m(n),n}(h_{m(n),n}) \). Then \( \{h_{m(n),n}\} \) and, by Property 2, \( \{f_{m(n)}\} \) converge uniformly to \( g \). Using the fact that \( \phi(g(W_k)) \subseteq (y + \epsilon_k, 1] \) and \( \phi(g_k) \subseteq [0, y - \epsilon_k] \), we have that for sufficiently large \( n \), \( f_{m(n)}(\partial W_k) \subseteq (y + \epsilon_k, 1] \) and \( \phi(g_{m(n)}(\gamma_k)) \subseteq [0, y - \epsilon_k] \).

Again, using the fact that \( \{f_{m(n)}\} \) converges uniformly to \( g \), we have that \( f_{m(n)}^{-1}(y) \) converges in the Hausdorff metric to some subset of \( g^{-1}(y) \). If \( \eta > 0 \) is such that \( \text{dist}(g^{-1}(y) \cap W_k, \partial W_k) > \eta/4 \) and \( \text{dist}(g^{-1}(y) \cap W_k, \gamma_k) > \eta/4 \), then for sufficiently large \( n \), we have that \( \text{dist}(f_{m(n)}^{-1}(y) \cap W_k, \partial W_k) > \eta/4 \) and \( \text{dist}(f_{m(n)}^{-1}(y) \cap W_k, \gamma_k) > \eta/4 \). Since the mesh of the partition of \( f_{m(n)}(S^2) \) is going to zero as \( n \to \infty \), we have the following property for sufficiently large \( n \): if \( y' \) is an endpoint of the partition interval of \( f_{m(n)}(S^2) \) closest to \( y \) and \( M \) is the component of \( f_{m(n)}^{-1}(y') \) contained in \( W_k \) and separating \( \gamma_k \) from \( \partial W_k \), then \( B_{2\eta}(M) \subseteq W_k \) and (8) can be applied to \( M \) and \( \gamma_k \).

Choose an \( n_k > n_k \) for which the previous property holds. Set \( m(n_k) = m(n_k+1) \). Now, let \( W_{k+1} \) be the worm associated with \( M \) in Properties (3) and (4) applied with \( \phi = f_{m(n_k+1)}, \phi^* = f_{m(n_k+1)}^*, h_{m(n_k+1),n_k+1}, n_k+1, \) and \( g = g \). We let \( \gamma_{k+1} \) be the arc associated with \( W_{k+1} \) and \( \epsilon_{k+1}, \rho_{k+1}, \phi_{k+1} \) be parameters associated with \( W_{k+1} \) and \( \gamma_{k+1} \). It is clear that Properties (a)-(e) of the induction hypothesis are satisfied. To see Property (f), we use that \( n_1 > 1 \), \( \{n_i\} \) is increasing, \( M \) separates \( \gamma_k \) from \( \partial W_k \), (7) and (8) can be applied for \( \gamma_{k+1} \), \( M \) and \( \gamma_k \). This implies  

\[
\text{diam}(\gamma_{k+1}) - \rho_{k+1} \geq (1 - (1/2)^{n_{k+1}}) \cdot \text{diam}(M) > (1 - (1/2)^{k+1}) \cdot (\text{diam}(\gamma_k) - \rho_k).
\]

Hence, we have constructed our desired sequence \( \{W_k\}, \{\gamma_k\}, \{n_k\}, \{m_k\}, \{\epsilon_k\} \) and \( \{\rho_k\} \).

\[\square\]

**Lemma 5.32.** Assume \( f \in \text{WB} \), \( y \in f(S^2) \) and \( M \) is a component of \( f^{-1}(y) \) homeomorphic to a circle. Given \( \epsilon > 0 \) there exists \( \eta \in (0, \epsilon) \) and an \( \epsilon \)-mapping \( \Delta_M \) from \( B_\eta(M) \) onto \( M \).

**Proof.** This follows from Lemma 4.5. \[\square\]

**Theorem 5.33.** All nondegenerate components of almost all fibers of a generic \( f \in C(S^2, I) \) are arc-like, or circle-like.
Proof. Assume that each component $f_m \in \mathcal{W}B$ is dense in $C(S^2, I)$.

Fix $m, n > 0$. First choose $y_1, ..., y_k$ such that if $y \in f_m(S^2) \setminus \{y_1, ..., y_k\}$ then all components of $f_m^{-1}(y)$ are homeomorphic to the circle. Choose $\epsilon_m, n > 0$ so that $4k\epsilon_m, n < 2^{-n}$. Put $E_{m,n} = \cup_{k=1}^k(y_k - 2\epsilon_m, n, y_k + 2\epsilon_m, n)$ and $E_{m,n} = \cup_{k=1}^k(y_k - \epsilon_m, n, y_k + \epsilon_m, n)$. Then, $\lambda_1(E_{m,n}) < 2^{-n}$.

Now fix $y \in f_m(S^2) \setminus \{y_1, ..., y_k\}$. By using Lemma 5.32 with $\epsilon = 1/n$ for each component $M$ of $f_m^{-1}(y)$ choose $\eta \in (0, 1/n)$ and a $1/n$-mapping $\Delta_M$ from $B_\eta(M)$ onto $M$. By taking minimum, we can assume that the same $\eta$ works for all components of $f_m^{-1}(y)$ and if $M, M'$ are different components of $f_m^{-1}(y)$ then $\overline{\Delta_M}(M) \cap \overline{\Delta_{M'}}(M') = \emptyset$ and $f_m(\overline{\Delta_M}(M)) \cap \{y_1, ..., y_k\} = \emptyset$. Choose $\rho_y \in (0, \epsilon_m, n)$ such that if $M$ is a component of $f_m^{-1}(y)$ then $|f_m(x) - y| > 3\rho_y$ for all $x \in \partial B_\eta(M)$. Let $G_y = f_m^{-1}((y - \rho_y, y + \rho_y)) \cap B_\eta(f_m^{-1}(y))$.

Do the above process for all $y \in f_m(S^2) \setminus \{y_1, ..., y_k\}$. The sets $f_m^{-1}((y_k - \epsilon_m, n, y_k + \epsilon_m, n))$ and the sets $G_z$ for $z \in f_m(S^2) \setminus \{y_1, ..., y_k\}$ form an open cover of $S^2$. Hence, there is a finite cover consisting of sets $G_{z, \ell} = 1, ..., t$ and of type $f_m^{-1}((y_k - \epsilon_m, n, y_k + \epsilon_m, n)), \ell = 1, ..., k$. Denote $\rho_m = \min\{\rho_z : \ell = 1, ..., t\}$.

Let $G_m = \cup_{n=1}^\infty B_{\rho_m, n}(f_m)$ and $G = \cap_{n=1}^\infty G_n$. Clearly, $G$ is a dense $G_\delta$ set in $C(S^2, I)$.

Let $f \in G$. For each $n$ choose $m_n$ such that $f \in B_{\rho_m, n}(f_{m_n})$. Let $E_f = \cup_{n=1}^\infty \cap_{n=K}^\infty E_{m_n}$. Then, clearly $\lambda_1(E_f) = 0$. We will show that for all $y \in (\min f, \max f) \setminus E_f$ all nondegenerate components of $f^{-1}(y)$ are either arc-like or circle-like. To this end, let $M$ be a nondegenerate component of $f^{-1}(y)$ and $\epsilon > 0$. Since $y \notin E_f$, there are infinitely many $n$ such that $y \notin E_{m_n}$. Hence, there is $n$ such that $1/n < \epsilon$ and $y \notin E_{m_n}$. Let $x \in M$ and $y' = f_{m_n}(x)$. Since $f \in B_{\rho_m, n}(f_{m_n})$, $|y' - y| < \rho_m, n < \epsilon_m, n$. Hence, $y' \notin E_{m_n}$. Using this fact, we may obtain $z$ such that $|y' - z| < \rho_z$ and a component $M''$ of $f_m^{-1}(z)$ such that $x \in B_{\eta}(M'')$. (Constants $\rho_z$ and $\eta$ are associated with the function $f_{m_n}$.) Now we have that for all $t \in M$,

$$|f_{m_n}(t) - z| = |f_{m_n}(t) - f(t)| + |f(t) - z| \leq \rho_m, n + |f(x) - f_{m_n}(x)| + |f_{m_n}(x) - z| \leq \rho_m, n + \rho_m, n + |y' - z| \leq 2\rho_z + \rho_z = 3\rho_z.$$  

Since for all $t \in \partial B_\eta(M'')$ we have $|f_{m_n}(t) - z| > 3\rho_z$ there is no point of $M$ on $\partial B_\eta(M'')$. Therefore, $M \subseteq B_\eta(M'')$. Hence, $\Delta_{M''}$, restricted to $M$, is a $1/n$-map of $M$ into $M''$. Hence, we have constructed an $\epsilon$-map from $M$ onto an arc or a circle.

\[\square\]

**Corollary 5.34.** A generic $f \in C(S^2, I)$ has the property that for almost all $y \in (\min f, \max f)$, all components of $f^{-1}(y)$ are either points, pseudoarcs or pseudocircles.

**Proof.** This follows from Theorem 5.33 and Theorem 5.15.  

\[\square\]

**Theorem 5.35.** A generic function $f \in C(S^2, I)$ has the property that almost all of its fibers contain pseudocircles as components.
Proof. This follows from Corollary 5.34, the fact that $f^{-1}(y)$ separates $S^2$ for any $f \in C(S^2, I)$ and $y \in (\min f, \max f)$, and the fact that no pseudoarc separates $S^2$. \square

References


Department of Analysis, Eötvös Loránd University, 1117 Budapest, Pázmány Péter sétány 1/c, Hungary
E-mail address: bucko@cs.elte.hu

Department of Mathematics, University of Louisville, Louisville, KY 40204
E-mail address: ubdarj01@athena.louisville.edu