MULTIFRACTAL SPECTRUM AND GENERIC PROPERTIES OF FUNCTIONS MONOTONE IN SEVERAL VARIABLES

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Abstract. We study the singularity (multifractal) spectrum of continuous functions monotone in several variables. We find an upper bound valid for all functions of this type, and we prove that this upper bound is reached for generic functions monotone in several variables. Let E_f^h be the set of points at which f has a pointwise exponent equal to h. For generic monotone functions $f : [0,1]^d \to \mathbb{R}$, we have that $\dim E_f(h) = d-1+h$ for all $h \in [0,1]$, and in addition, we obtain that the set E_f^h is empty as soon as $h > 1$. We also investigate the level set structure of such functions.

1. Introduction

The genericity of multifractal properties for functions and measures have been investigated in several contexts [3, 8, 4]. Recall that a property is typical, or generic in a complete metric space E, when it holds on a residual set, i.e. a set with a complement of first Baire category (a set is of first Baire category if it is the union of countably many nowhere dense sets). One of the first results in this direction was obtained by Buczolich and Nagy in [3] who proved that typical continuous increasing functions defined on the interval $[0,1]$ enjoy very specific multifractal properties. There are several ways to generalize this result in a higher-dimensional context. In [4] we investigated the multifractal properties of typical Borel measures on $[0,1]^d$, and we proved that typical measures satisfy a multifractal formalism. In this paper, we focus on a natural generalization in dimension $d \geq 1$ of one-dimensional monotone functions, which has been already studied by other authors [11, 5]: the so-called functions monotone increasing in several variables. This work is closely related to many other works (see the works mentioned above, and [9, 10] by L. Olsen where prevalent properties of measures are also studied).

Let d be an integer greater than one. A function $f : [0,1]^d \to \mathbb{R}$ is continuous monotone increasing in several variables (in short: MISV) if for all $i \in \{1, ..., d\}$, the functions

$$f^{(i)}(t) = f(x_1, ..., x_{i-1}, t, x_{i+1}, ..., x_d) \tag{1}$$

are continuous monotone increasing. We use the notation

$$\mathcal{M}^d = \{f \in C([0,1]^d) : f \text{ MISV}\}.$$

The space $\mathcal{M}^d$ is a separable complete metric space when equipped with the supremum, L^∞ norm for functions, that we denote by $\|.\|$.

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The multifractal properties of functions in $\mathcal{M}^1$ have been examined in [3]. In this paper, we study the case of higher dimensional functions $d \geq 2$. Let us recall the notions of Hölder exponent and singularity spectrum for a locally bounded function.

Definition 1. Let $f \in L^\infty([0,1]^d)$. For $h \geq 0$ and $x \in [0,1]^d$, the function f belongs to C^h_x if there are a polynomial P of degree less than $[h]$ and a constant C such that, for x' close to x,

$$|f(x') - P(x'-x)| \leq C|x'-x|^h. \tag{2}$$

The pointwise Hölder exponent of f at x is $h_f(x) = \sup\{h \geq 0 : \ f \in C^h_x\}$.

Observe that when $h_f(x) < 1$, the pointwise Hölder exponent of f at x is also given by the formula

$$h_f(x) = \liminf_{x' \to x} \frac{\log|f(x') - f(x)|}{\log|x'-x|}.$$

Definition 2. The singularity spectrum of f is defined by

$$d_f(h) = \dim_H E^h_f, \quad \text{where } E^h_f = \{x : h_f(x) = h\}.$$

Here $\dim_H$ denotes the Hausdorff dimension, and $\dim \emptyset = -\infty$ by convention. The singularity spectrum of f describes the geometric repartition of the singularities of f. This is the quantity we will mainly focus on in the following.

We will also use the sets

$$E^{h,\leq}_f = \{x : h_f(x) \leq h\} \supset E^h_f. \tag{3}$$

Remark 3. Clearly, if $0 \leq h < 1$ then $x \in E^{h,\leq}_f$ if and only if for every $\varepsilon > 0$ and $\delta > 0$ there exists y such that $|x-y| < \delta$ and $|f(x) - f(y)| > |x-y|^{h+\varepsilon}$.

In the one-dimensional case the typical properties of functions in $\mathcal{M}^1$ are summarized in the following theorem which is a consequence of results in [3].

Theorem 4. *Consider the space of monotone continuous functions $\mathcal{M}^1$ defined on $[0,1]$.*

(i) *For every $f \in \mathcal{M}^1$, for every $h \geq 0$, one has $d_f(h) \leq \min(h,1)$.*

(ii) *There exists a residual set $\mathcal{R}_1$ in $\mathcal{M}^1$ such that for every $f \in \mathcal{R}_1$,*

$$d_f(h) = h \ \text{ for every } h \in [0,1],$$

and $E^h_f = \emptyset$ if $h > 1$.

(iii) $\mu_f(E^0_f) = f(1) - f(0)$, *that is $\mu_f([0,1] \setminus E^{0,\leq}_f) = 0$, where μ_f is the Borel integral of f: $f(x) = \int_0^x d\mu_f$.*

By (i) of the above theorem, $0 = \dim_H E^0_f = \dim_H E^{0,\leq}_f$ and (iii) shows that all the "increasing" of f takes place on this set E^0_f of zero Hausdorff dimension. Since a typical monotone function is strictly monotone increasing (see [3]), its level sets are points. We deduce that heuristically, for "most" levels in the range of f, the corresponding points belong to the zero dimensional set E^0_f.

The new higher dimensional results are gathered in the following three theorems. First we obtain an upper estimate of the singularity spectrum which is valid for arbitrary functions in $\mathcal{M}^d$.

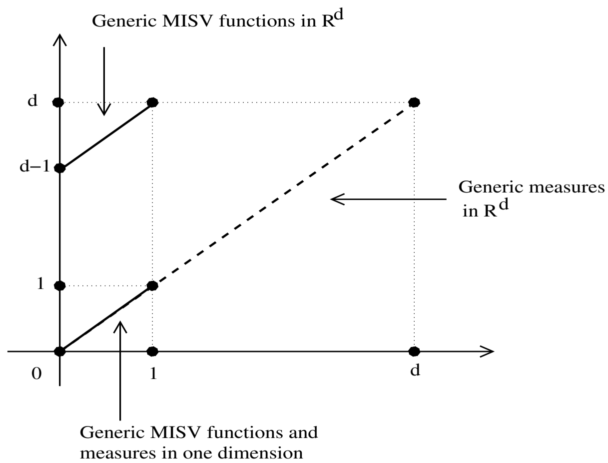

FIGURE 1. Typical spectra for measures and for MISV functions

Theorem 5. *For all $f \in \mathcal{M}^d$ and $h \geq 0$, we have*

$$\dim_H E_f^{h,\leq} \leq \min(d-1+h, d). \tag{4}$$

In particular, $d_f(h) = \dim_H(E_f^h) \leq \min(d-1+h, d)$.

One recognizes item (i) of Theorem 4 in the case $d = 1$. The next theorem shows that for $h \in [0,1]$ the generic functions can be as bad as possible from the multifractal standpoint, in the sense that the level sets of the Hölder exponent function corresponding to the exponents less than 1 are as large as possible.

Theorem 6. *There exists a dense G_δ set $\mathcal{R} \subset \mathcal{M}^d$ such that for all $f \in \mathcal{R}$ we have $d_f(h) = d-1+h$ for all $h \in [0,1]$. For these functions, for every $h > 1$ the set E_f^h is empty.*

Theorems 5 and 6 must be compared to the results obtained for measures supported by $[0,1]^d$, which can be viewed as a special case of functions monotone increasing in several variables (see Figure 1). Indeed, if μ is supported in $[0,1]^d$, then the function $f : [0,1]^d \to \mathbb{R}$ defined as $f(x_1, x_2, ..., x_d) = \mu([0,x_1] \times [0,x_2] \times ... \times [0,x_d])$ is increasing in several variables. We proved in [4] that for every measure, one has the upper bound

$$d_\mu(h) \leq \min(h, d) \quad \text{for every } h \geq 0,$$

and for typical measures supported in $[0,1]^d$,

$$d_\mu(h) = h \quad \text{for every } 0 \leq h \leq d, \quad \text{and} \quad E_\mu^h = \emptyset \text{ if } h > d.$$

This is in sharp contrast with typical continuous MISV functions when $d \geq 2$, whose local behavior is "worse" (the level sets of the Hölder exponents smaller than 1 may have a bigger value).

Next, we turn to the level sets of MISV functions. We define for every $a \in \mathbb{R}$ the level set $L_f(a)$ by

$$L_f(a) = \{x \in [0,1]^d : f(x) = a\}. \tag{5}$$

The following simple argument, valid for any continuous function, shows that the Hausdorff dimension of the level sets $L_f(a)$ in the interior of the range of an $f \in \mathcal{M}^d$ is at least $d-1$, for every a.

Indeed, if $\min_{x\in[0,1]^d} f(x) = f(0,...,0) < a < f(1,...,1) = \max_{x\in[0,1]^d} f(x)$ then we can choose $0 < t_1 < t_2 < t_3 < 1$ such that

$$f(t_1,...,t_1) < a = f(t_2,...,t_2) < f(t_3,...,t_3)$$

and there exists $\varepsilon > 0$ such that the three balls $B((t_i,...,t_i),\varepsilon)$, $i = 1,2,3$, are included in $[0,1]^d$, $f(x) < a$ for all $x \in B((t_1,...,t_1),\varepsilon)$ and $f(x) > a$ for all $x \in B((t_3,...,t_3),\varepsilon)$. Therefore, for any $(0,u_2,...,u_d) \in B((0,...,0),\varepsilon)$, on any line segment connecting points of the form $(t_1, t_1+u_2,...,t_1+u_d)$ and $(t_3,t_3+u_2,...,t_3+u_d)$, there exists a point $y(u_2,...,u_d)$ such that $f(y(u_2,...,u_d)) = a$. Using a projection argument, this implies that the level set $L_f(a)$ is of dimension at least $d-1$ for any $f \in \mathcal{M}^d$.

Theorem 7. *There exists a dense G_δ subset $\mathcal{L}$ in $\mathcal{M}^d$ such that for all $f \in \mathcal{L}$ the following holds.*

There exist a set $X_f \subset [0,1]^d$ and a set $A_f \subset (f(0,...,0), f(1,...,1)) = (m_f, M_f)$ satisfying:

(i) $\dim_H X_f = d-1$, $\dim_H A_f = 0$,
(ii) *for every $a \in (m_f, M_f)$, there is at most one point of $L_f(a)$ which does not belong to X_f (in other words, $L_f(a) \cap ([0,1]^d \setminus X_f)$ contains at most one point).*
(iii) *for every $a \in (m_f, M_f) \setminus A_f$, $L_f(a) \subset X_f$.*

In other words, X_f contains Lebesgue-almost every level sets $L_f(a)$, and for those level sets $L_f(a)$ which are not entirely contained in X_f (this occurs for a set of values of a of Hausdorff dimension 0), exactly one point of $L_f(a)$ does not belong to A_f. This entails that our function f is "increasing" only on the small $d-1$ dimensional set X_f which has the minimum possible dimension to contain at least one level set. Most points in the domain of f belong to $[0,1]^d \setminus X_f$, which can intersect just "very few" level sets and in no more than one point. In particular, for all $x, x' \in [0,1]^d \setminus X_f$ (this set has full Lebesgue measure in $[0,1]$), $f(x) \neq f(x')$.

The level sets of generic continuous MISV functions are quite simple compared to the level sets of generic continuous functions (see for example [2]).

Set

$$\mathbb{R}^d_+ = \{(l_1,...,l_d) : \forall i,\ l_i \geq 0\} \text{ and } \mathbb{R}^d_- = \{(l_1,...,l_d) : \forall i,\ l_i \leq 0\}.$$

It is well-known that generic continuous functions on $[0,1]$ are nowhere monotone (see for example [1], Chapter 10). MISV functions are obviously monotone increasing along lines $\underline{l}t + \underline{b} = (l_1 t + b_1, ..., l_d t + b_d)$, $(t \in \mathbb{R})$ if $\underline{l} \in \mathbb{R}^d_+$ and monotone decreasing if $\underline{l} \in \mathbb{R}^d_-$. For the generic functions in $\mathcal{M}^d$ one cannot say much more:

Theorem 8. *There exists a dense G_δ subset $\mathcal{G}$ in $\mathcal{M}^d$ such that for any $f \in \mathcal{G}$, if $\underline{l} = (l_1,...,l_d) \notin \mathbb{R}^d_+ \cup \mathbb{R}^d_-$ and $\underline{b} = (b_1,...,b_d) \in \mathbb{R}^d$, then the function $g_{\underline{l},\underline{b}}(t) = f(\underline{l}t + \underline{b})$, $t \in \mathbb{R}$ is monotone on no non-empty open subinterval on its domain.*

A two-dimensional variant of this result was Problem 6 of the annual Miklós Schweitzer competition of the János Bolyai Mathematical Society in 2010, proposed by the first listed author. Here we state this problem using the notation of our paper.

Problem for the Schweitzer competition. *Does there exist a function $f(x,y)$ which is continuous on $\mathbb{R}^2$ and such that for all $b \in \mathbb{R}$, the function $g_{l,b}(t) = f(t, lt+b)$ is strictly increasing on $\mathbb{R}$ for $l \geq 0$ and monotone on no nonempty open interval for $l < 0$.*

Generic functions in the compact-open topology of $C(\mathbb{R}^2)$ provide functions requested by the above problem, but there are not too difficult direct constructions as well.

2. Preliminary results

We use the metric in $\mathbb{R}^d$ coming from the sup norm, that is, $|x-y| = \max_i |x_i - y_i|$ for $x, y \in \mathbb{R}^d$. This way, the diameter of $[0,1]^d$ equals one.

The open ball centered at $x \in \mathbb{R}^d$ and of radius r is denoted by $B(x,r)$. By $|A|$ and $\lambda(A)$ we denote the diameter and Lebesgue measure of A, respectively.

For ease of notation we keep the open ball notation in metric spaces of functions as well, that is, for $f \in \mathcal{M}^d$ and $\delta > 0$ we denote by $B(f,\delta)$ the open ball in $\mathcal{M}^d$ centered at f and of radius δ (here the supremum norm is used).

We refer to [6, 7] for the standard definition of Hausdorff measures $\mathcal{H}^s(E)$ and Hausdorff dimension $\dim_H(E)$ of a set E.

Recall that the lower local dimension of a Borel measure μ at x is defined as (see [7])

$$\underline{\dim}_{\rm loc}\mu(x) = \liminf_{r\to 0+} \frac{\log \mu(B(x,r))}{\log r} \tag{6}$$

and $\dim \mu \stackrel{\rm def}{=} \sup\{s : \underline{\dim}_{\rm loc}\mu(x) \geq s \text{ for } \mu \text{ a.e. } x\}$. By Proposition 10.2 of [7], we have

$$\dim_H(\mu) = \inf\{\dim_H(E) : E \subset [0,1] \text{ Borel and } \mu(E) > 0\}.$$

The following property will be useful:

$$\begin{array}{c}\text{if } \dim_H(\mu) \geq h, \text{ then } \mu(E) = 0 \text{ for every Borel set} \\ E \subset [0,1] \text{ of dimension strictly less than } h.\end{array} \tag{7}$$

We recall the Mass Distribution Principle, see for example [6], Chapter 4.

Theorem 9. *Let μ be a finite mass distribution (measure) on $E \subset \mathbb{R}$. If for every $x \in E$,*

$$\liminf_{r\to 0+} \frac{\log \mu(B(x,r))}{\log r} \geq s, \tag{8}$$

then $\dim_H E \geq \dim_H \mu \geq s$.

In particular, $\mu(U) = 0$ for all sets U with $\dim_H U < s$.

3. Upper bound for the spectrum of all functions in $\mathcal{M}^d$

For every integer $j \geq 1$, we denote by $\mathcal{U}_j$ the system of cubes

$$U_{j,(k_1,...,k_d)} = [k_1 2^{-j}, (k_1+1)2^{-j}] \times ... \times [k_d 2^{-j}, (k_d+1)2^{-j}], \ (k_1,...,k_d) \in \mathbb{Z}^d.$$

We have a trivial property implied by the monotonicity property, that will be repeatedly used in the sequel: Suppose that $x, y \in U_{j,(k_1,...,k_d)}$ and $f \in \mathcal{M}^d$. Then

$$|f(x) - f(y)| \leq f((k_1+1)2^{-j}, ..., (k_d+1)2^{-j}) - f(k_1 2^{-j}, ..., k_d 2^{-j}). \tag{9}$$

Auxiliary cubes and parallelepipeds will be used during the proofs.

Definition 10. Denote by w the vector $(2,...,2,1) \in \mathbb{R}^d$. Given $j \in \mathbb{N}$ and $(k_1,...,k_d) \in \mathbb{Z}^d$, by $\widehat{U}_{j,(k_1,...,k_{d-1})}$ we denote the projected cube

$$\widehat{U}_{j,(k_1,...,k_{d-1})} = [k_1 2^{-j}, (k_1+1)2^{-j}] \times ... \times [k_{d-1} 2^{-j}, (k_{d-1}+1)2^{-j}] \times \{0\},$$

and we also consider the parallelepipeds

$$P_{j,(k_1,...,k_d)} = \{x' + tw : x' \in \widehat{U}_{j,(k_1,...,k_{d-1})}, \ t \in [k_d 2^{-j}, (k_d+1)2^{-j}]\}.$$

The system of these parallelepipeds of generation j is denoted by $\mathcal{P}_j$ (see Figure 2 for a schema).

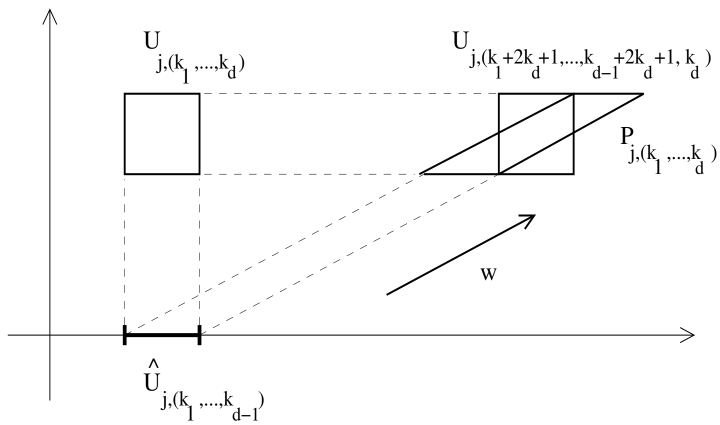

FIGURE 2. Illustration of the relative location of $U_{j,(k_1,\dots,k_d)}$, $\widehat{U}_{j,(k_1,\dots,k_{d-1})}$, $P_{j,(k_1,\dots,k_d)}$ and $U_{j,(k_1+2k_d+1,\dots,k_{d-1}+2k_d+1,k_d)}$

The advantage of using the parallelepipeds $\mathcal{P}_j$ is the following: If f is MISV and is defined on $P_{j,(k_1,\dots,k_d)}$ then for any $x' \in \widehat{U}_{j,(k_1,\dots,k_{d-1})}$ we have

$$\begin{aligned} & f(x' + (k_d+1)2^{-j}w) - f(x' + (k_d)2^{-j}w) \\ \text{(10)} \quad \geq\ & f\Big((k_1 2^{-j}, \dots, k_{d-1}2^{-j}, 0) + (k_d+1)2^{-j}w\Big) \\ & -f\Big(((k_1+1)2^{-j}, \dots, (k_{d-1}+1)2^{-j}, 0) + k_d 2^{-j}w\Big) \\ \text{(11)} \quad =\ & f\Big((k_1+2k_d+2)2^{-j}, \dots, (k_{d-1}+2k_d+2)2^{-j}, (k_d+1)2^{-j}\Big) \\ & -f\Big((k_1+2k_d+1)2^{-j}, \dots, (k_{d-1}+2k_d+1)2^{-j}, k_d 2^{-j}\Big), \end{aligned}$$

which is nothing but the increment of f on the main diagonal of the cube

$$U_{j,(k_1+2k_d+1,\dots,k_{d-1}+2k_d+1,k_d)}.$$

We denote by $\mathcal{N}(U_{j,(k_1,\dots,k_d)})$ the neighbors of $U_{j,(k_1,\dots,k_d)}$, that is the set of those $U_{j,(k'_1,\dots,k'_d)}$ for which $|k'_i - k_i| \leq 1$ for all $i = 1, \dots, d$, allowing $U_{j,(k_1,\dots,k_d)}$ itself being its own "neighbor".

Lemma 11. *Suppose $x, y \in [0,1]^d$, $x \in U_{j,(k_1,\dots,k_d)}$, and $|x-y| \leq 2^{-j}$. Then there exists $U_{j,(k'_1,\dots,k'_d)} \in \mathcal{N}(U_{j,(k_1,\dots,k_d)})$ such that the main diagonal difference satisfies*

$$f((k'_1+1)2^{-j}, \dots, (k'_d+1)2^{-j}) - f(k'_1 2^{-j}, \dots, k'_d 2^{-j}) \geq |f(x) - f(y)|/(d+1). \tag{12}$$

Proof. One only needs to observe that the segment xy can be covered by no more than $d+1$ cubes from $\mathcal{N}(j, (k_1, \dots, k_d))$. Suppose that $U_{j,(k'_1,\dots,k'_d)}$ is chosen among these squares so that if $x'y'$ is the subsegment of xy covered by $U_{j,(k'_1,\dots,k'_d)}$ then $|f(x') - f(y')| \geq |f(x) - f(y)|/(d+1)$. By the partial monotonicity (9) of the MISV function f, the left-hand side of (12) is larger than $|f(x') - f(y')|$. □

We are now ready to prove Theorem 5.

For every $x \in \mathbb{R}^d \setminus [0,1]^d$, denote by x_Q the point of $[0,1]^d$ which is closest to x. Extend the definition of f onto $\mathbb{R}^d$ by letting $f(x) = f(x_Q)$, for every $x \in \mathbb{R}^d \setminus [0,1]^d$. It is easy to see that this way we obtain a continuous MISV function defined on $\mathbb{R}^d$. We will still consider only points of E_f^h and $E_f^{h,\le}$ in $[0,1]^d$, but this extension will make our calculations easier in case of some parallelepipeds are not entirely in $[0,1]^d$.

Without limiting generality we can also assume that

$$0 \le f \le 1. \tag{13}$$

Let $h \in (0,1)$. We would like to verify that $\dim_H E_f^{h,\le} \le d-1+h+\varepsilon$ for all $\varepsilon > 0$. Suppose that ε and $\delta > 0$ are given.

Consider now $x \in E_f^{h,\le}$. By Remark 3 we can select a sequence $y(n,x) \to x$ (where $y(n,x) \in [0,1]^d$) such that

$$|f(x) - f(y(n,x))| > |x - y(n,x)|^{h+\varepsilon}. \tag{14}$$

We choose $j(n,x) \in \mathbb{N} \cup \{0\}$ as the unique integer such that

$$2^{-j(n,x)-1} < |x - y(n,x)| \le 2^{-j(n,x)}, \tag{15}$$

and we denote by $U_{n,x} := U_{j(n,x),(k_1(n,x),\dots,k_d(n,x))}$ the unique cube of generation $j(n,x)$ such that $x \in U_{j(n,x),(k_1(n,x),\dots,k_d(n,x))}$. By applying Lemma 11, we select the cube $U'_{n,x}$ in the neighborhood of $U_{n,x}$ defined as

$$U'_{n,x} = U_{j(n,x),(k'_1(n,x),\dots,k'_d(n,x))} \in \mathcal{N}(U_{n,x})$$

such that (12) holds with the integer parameters equal to $j(n,x), k'_1(n,x), \dots, k'_d(n,x)$. If we denote by $V_{n,x}$ the cube concentric with $U'_{n,x}$ but of side length $3 \cdot 2^{-j(n,x)}$, then it is clear that $x \in V_{n,x}$. Therefore, $E_f^{h,\le} \subset \cup_{x \in E_f^{h,\le}} V_{n,x}$.

For each $x \in E_f^{h,\le}$, we fix an integer $n(x)$ such that

$$\text{if } j(x) \overset{\text{def}}{=} j(n(x),x), \quad \text{then } 2^{-j(x)} < \delta/(4M), \tag{16}$$

where M is some positive constant that will be fixed later.

Given $U'_{n(x),x}$, choose a parallelepiped $P_x = P_{j(x),(\overline{k}_1(x),\dots,\overline{k}_d(x))}$ such that

$$(\overline{k}_i(x) + 2\overline{k}_d(x) + 1)2^{-j(x)} = k'_i(n(x),x)2^{-j(x)} \text{ for } i = 1,\dots,d-1$$

and

$$\overline{k}_d(x)2^{-j(x)} = k'_d(n(x),x)2^{-j(x)}.$$

This implies that $U'_{n(x),x}$ and P_x share a diagonal with endpoints (see Figures 2 and 3 for the illustrations)

$$\begin{aligned} A_x &= \Big((\overline{k}_1(x) + 2\overline{k}_d(x) + 1), \dots, (\overline{k}_{d-1}(x) + 2\overline{k}_d(x) + 1), \overline{k}_d(x)\Big) \cdot 2^{-j(x)} \\ &= (k'_1(n(x),x), \dots, k'_d(n(x),x)) \cdot 2^{-j(x)}, \\ \text{and } B_x &= \Big((\overline{k}_1(x) + 2\overline{k}_d(x) + 2), \dots, (\overline{k}_{d-1}(x) + 2\overline{k}_d(x) + 2), (\overline{k}_d(x) + 1)\Big) \cdot 2^{-j(x)} \\ &= ((k'_1(n(x),x) + 1), \dots, (k'_d(n(x),x) + 1)) \cdot 2^{-j(x)}. \end{aligned}$$

By (14), (15) and the choice of $U'_{n(x),x}$ (especially by (12)), we have

$$f(B_x) - f(A_x) \ge \frac{|f(x) - f(y(n(x),x))|}{d+1} > \frac{(2^{-j(x)-1})^{h+\varepsilon}}{d+1}. \tag{17}$$

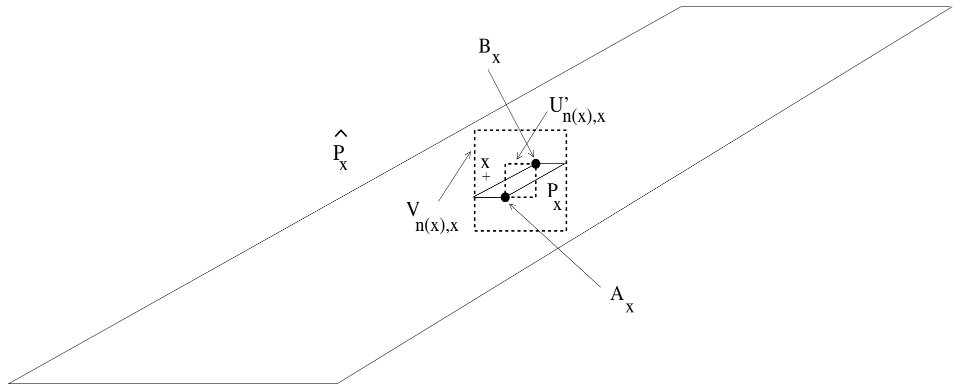

FIGURE 3. Illustration of the relative location of x, $U'_{n(x),x}$, $V_{n(x),x}$, P_x and $\widehat{P}_x$

By the partial monotonicity of the MISV function f, analogously to (12), for any $x' \in \widehat{U}_{j(x),(\overline{k}_1(x),...,\overline{k}_{d-1}(x))}$ we have

$$\begin{aligned} f(x' + (\overline{k}_d(x)+1)2^{-j(x)}w) - f(x' + \overline{k}_d(x)2^{-j(x)}w) &\geq f(B_x) - f(A_x) \\ &> \frac{(2^{-j(x)-1})^{h+\varepsilon}}{d+1}. \end{aligned} \tag{18}$$

Observe that there exists a constant M (independent of the dimension d and the integers j and k_i) such that if $\widehat{P}_x$ denotes the parallelepiped concentric with P_x but of sidelengths M times that of P_x, then $x \in V_{n(x),x} \subset \widehat{P}_x$ (see Figure 3). This constant M is the constant that we use in equation (16), and M is fixed now once for all. It is also clear that for the diameter of $\widehat{P}_x$ we have

$$|\widehat{P}_x| \leq M \cdot |P_x| < 4 \cdot M \cdot 2^{-j(x)} < \delta. \tag{19}$$

Let us introduce the projection $\pi_w : \mathbb{R}^d \mapsto \mathbb{R}^d$ in the direction of w onto the hyperplane

$$\{(x_1, ..., x_{d-1}, 0) : (x_1, ..., x_{d-1}) \in \mathbb{R}^{d-1}\},$$

that is, if a point $y \in \mathbb{R}^d$ is written in the form $y = (x_1, ..., x_{d-1}, 0) + tw$ for some $t \in \mathbb{R}$, then $\pi_w(y) = (x_1, ..., x_{d-1}, 0)$. Remark that we have $\pi_w([0,1]^d) = [-2,1]^{d-1} \times \{0\}$.

Since the parallelepipeds $\{P_x\}_{x \in E_f^{h,\leq}}$ form a nested family, we can select a non-overlapping system of them, that we denote by $\{P^{(n)}\}_{n\geq 1}$. By construction, the $\{P^{(n)}\}_{n\geq 1}$ form a δ-cover of $E_f^{h,\leq}$.

For each $x \in E_f^{h,\leq}$, the parallelepiped P_x is a subset of a $P^{(n)}$ for a suitable (uniquely determined) n. This property also implies that for every n, if $\widehat{P}^{(n)}$ denotes the M times enlarged copy of $P^{(n)}$, then $E_f^{h,\leq} \subset \cup_n \widehat{P}^{(n)}$.

Now, set for every integer $n \geq 1$

$$\chi_n(x_1, ..., x_{d-1}) = \begin{cases} 1 & \text{if } (x_1, ..., x_{d-1}, 0) \in \pi_w(P^{(n)}), \\ 0 & \text{otherwise.} \end{cases}$$

Each $P^{(n)}$ equals a suitable $P_{x(n)}$, for some $x(n) \in E_f^{h,\leq}$. Denote the integer $j(x(n))$ by $j^{(n)}$. It is clear that the diameter of $P^{(n)}$ satisfies

$$|P^{(n)}| < 4 \cdot 2^{-j^{(n)}}. \tag{20}$$

We introduce the following auxiliary function $F:\mathbb{R}^{d-1}\to\mathbb{R}_+$ defined by

$$F(x_1,...,x_{d-1})=\sum_{n\geq 1}\chi_n(x_1,...,x_{d-1})\cdot(2^{-j^{(n)}})^{h+\varepsilon}.$$

Lemma 12. *For every* $(x_1,...,x_{d-1})\in\mathbb{R}^{d-1}$, $0\leq F(x_1,...,x_{d-1})\leq(d+1)2^{h+\varepsilon}$.

Proof. Fix $(x_1,...,x_{d-1})\in\mathbb{R}^{d-1}$. Recall that for every integer n, $P^{(n)}=P_{x(n)}$. By (17) we have

$$f(B_{x(n)})-f(A_{x(n)})>\frac{(2^{-j^{(n)}-1})^{h+\varepsilon}}{d+1}. \tag{21}$$

It is also clear that if $\chi_n(x_1,...,x_{d-1})=1$, then the straight line $\{(x_1,...,x_{d-1},0)+tw: t\in\mathbb{R}\}$ intersects $P^{(n)}$ in a segment of the form

$$S_n(x_1,...,x_{d-1})=\{(x_1,...,x_{d-1},0)+tw:k^{(n)}2^{-j^{(n)}}\leq t\leq(k^{(n)}+1)2^{-j^{(n)}}\}$$

for some integer $k^{(n)}$. Then, from (18) and (21) we deduce that

$$\begin{aligned}&f\Big((x_1,...,x_{d-1},0)+(k^{(n)}+1)2^{-j^{(n)}}w\Big)-f\Big((x_1,...,x_{d-1},0)+k^{(n)}2^{-j^{(n)}}w\Big)\\&\geq f(B_{x(n)})-f(A_{x(n)})>\frac{(2^{-j^{(n)}-1})^{h+\varepsilon}}{d+1}.\end{aligned} \tag{22}$$

Since the $P^{(n)}$ parallelepipeds are non-overlapping, the segments $S_n(x_1,...,x_{d-1})$ are also non-overlapping. Hence, by the partial monotonicity of f, by (13) and by (22) we have

$$\begin{aligned}F(x_1,...,x_{d-1}) &\leq \sum_{n:S_n(x_1,...,x_{d-1})\text{ exists}}(2^{-j^{(n)}})^{h+\varepsilon}\\
&\leq (d+1)2^{h+\varepsilon}\sum_{n:S_n(x_1,...,x_{d-1})\text{ exists}}\Big(f(B_{x(n)})-f(A_{x(n)})\Big)\\
&\leq (d+1)2^{h+\varepsilon}\sum_{n:S_n(x_1,...,x_{d-1})\text{ exists}}\Big[f\Big((x_1,...,x_{d-1},0)+(k^{(n)}+1)2^{-j^{(n)}}w\Big)\\
&\qquad -f\Big((x_1,...,x_{d-1},0)+k^{(n)}2^{-j^{(n)}}w\Big)\Big]\\
&\leq (d+1)2^{h+\varepsilon}\Big(f((x_1,...,x_{d-1},0)+1\cdot w)-f((x_1,...,x_{d-1},0)+0\cdot w)\Big)\\
&\leq (d+1)2^{h+\varepsilon}\big(f(1,1,...,1)-f(0,0,...,0)\big)=(d+1)2^{h+\varepsilon}.\end{aligned}$$

□

Since by Lemma 12, F is bounded, we have

$$\int_{[-2,1]^{d-1}}F(x_1,...,x_{d-1})dx_1\cdots dx_{d-1}\leq C:=3^{d-1}(d+1)2^{h+\varepsilon}.$$

Moreover, by the definition of the parallelepipeds, for every integer n, $\chi_n(x_1, ..., x_{d-1}) = 1$ on a $(d-1)$-dimensional cube of sidelength $2^{-j(n)}$. Hence

$$\begin{aligned} C &\geq \int_{[-2,1]^{d-1}} F(x_1, ..., x_{d-1})\, dx_1 \cdots dx_{d-1} \\ &= \sum_{n\geq 1} \int_{[-2,1]^{d-1}} \chi_n(x_1, ..., x_{d-1}) (2^{-j(n)})^{h+\varepsilon}\, dx_1 \cdots dx_{d-1} \\ &= \sum_{n\geq 1} (2^{-j(n)})^{d-1+h+\varepsilon}. \end{aligned}$$

Then, (20) implies that $\sum_{n\geq 1} |P^{(n)}|^{d-1+h+\varepsilon}$ converges. Subsequently, using the equivalence between the diameters of $P^{(n)}$ and $\widehat{P}^{(n)}$, we obtain that $\sum_{n\geq 1} |\widehat{P}^{(n)}|^{d-1+h+\varepsilon}$ converges. Since $\{\widehat{P}^{(n)}\}_{n\geq 1}$ forms a δ-cover of $E_f^{h,\leq}$, we deduce that the $(d-1+h+\varepsilon)$-dimensional Hausdorff measure $\mathcal{H}^{d-1+h+\varepsilon}$ of $E_f^{h,\leq}$ is finite.

Therefore, $\dim_H(E_f^{h,\leq}) \leq d-1+h+\varepsilon$ for all $\varepsilon > 0$. Letting ε tend to zero we obtain Theorem 5.

4. Lower bound for the spectrum of generic functions in $\mathcal{M}^d$

We now prove Theorem 6. The argument is reduced to the one-dimensional case. One can choose, say, the first coordinate, and use perturbation functions which are constant in the direction of the other coordinates.

For every $l \in \mathbb{N}$, we will use the function $\gamma_l : [0,1] \to [0,1]$ defined as follows (see Figure 4):

- For every integer $j = 0, ..., 2^{l^2} - 1$, if $x_1 \in [j2^{-l^2}, (j+1)2^{-l^2} - 2^{-l^4})$, we set $\gamma_l(x_1) = j2^{-l^2-l}$,
- $\gamma_l(1) = 2^{-l}$,
- On the intervals $((j+1)2^{-l^2} - 2^{-l^4}, (j+1)2^{-l^2})$, we define γ_l so that it is linear on these intervals.

These functions γ_l are continuous, changing from 0 to 2^{-l}, but are strictly increasing only on the 2^{l^2} many very small, uniformly distributed, intervals of length 2^{-l^4}.

We select a countable dense set of functions $\{f_m : m = 1, 2, ...\}$ in $\mathcal{M}^d$. We will perturb these functions f_m using the "perturbation functions"

$$\overline{g}_l(x_1, ..., x_d) = \gamma_l(x_1). \tag{23}$$

Definition 13. For every integer n, we define the dense open sets

$$\mathcal{R}_n = \bigcup_{m=1}^{\infty} B(f_m + \overline{g}_{m+n}, \rho_{m+n}),$$

where $\rho_n = 2^{-n^8}$, and finally the dense G_δ set

$$\mathcal{R} = \bigcap_{n=1}^{\infty} \mathcal{R}_n.$$

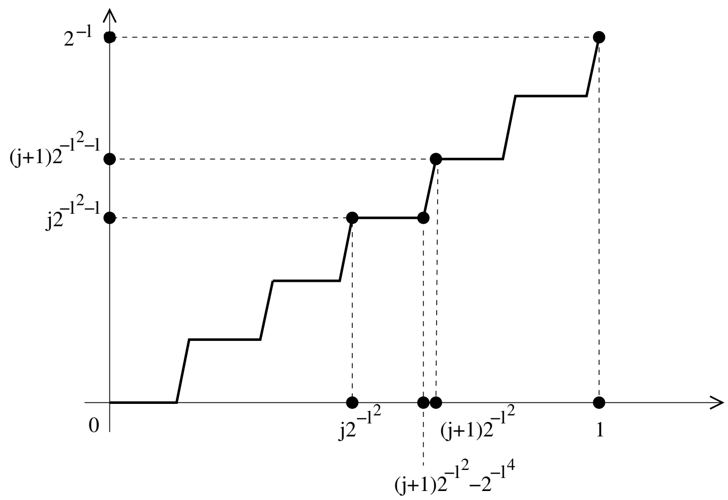

FIGURE 4. Illustration of the graph of the functions γ_l

We need to verify that for all $f \in \mathcal{R}$ the conclusion of Theorem 6 holds. We will construct, for every $h \in [0, 1)$, a Cantor set $\widetilde{F}_h$ of dimension $d - 1 + h$ which consists of points where the pointwise Hölder exponent for f equals h.

Suppose $f \in \mathcal{R}$. Then there exists a sequence of integers $(m_n)_{n\geq 1}$ such that f belongs to $B(f_{m_n} + \overline{g}_{m_n+n}, \rho_{m_n+n})$ for every integer n.

We will select a subsequence $l_k = m_{n_k} + n_k$ and will use the notation

$$\phi_k = f_{m_{n_k}}, \quad g_{l_k} = \overline{g}_{m_{n_k}+n_k}, \tag{24}$$

and the fact that

$$f \in B(\phi_k + g_{l_k}, \rho_{l_k}) \text{ for all } k = 1, \dots . \tag{25}$$

For every integer $k \geq 1$, we set the constants

$$D_k = 2^{(l_k)^2} \cdot 2^{-((l_k)^2+l_k)k-2} < 1 \quad \text{and for all } 0 < h < 1, \ D_{h,k} = 2^{(l_k)^2} \cdot 2^{-((l_k)^2+l_k)\frac{1}{h}-2} < 1. \tag{26}$$

We suppose that the subsequence n_k is chosen so that for all integers $k \geq 2$ with $l_k = m_{n_k} + n_k$ we have

$$l_k > 2^k, \quad ((l_k)^2 + l_k)k + 1 < (l_k)^4, \quad 2^{-((l_{k-1})^2+l_{k-1})(k-1)-1} > 100 \cdot 2^{-(l_k)^2} \tag{27}$$

and

$$D_1 \cdots D_{k-1} > 2^{-l_k}. \tag{28}$$

Suppose $0 < h < 1$. We define the compact set $F_{h,k}$ as the union of the intervals

$$F_{h,k} = \bigcup_{j=0,\dots,2^{(l_k)^2}-1} \left[(j+1)2^{-(l_k)^2} - 2^{-((l_k)^2+l_k)\frac{1}{h}}, (j+1)2^{-(l_k)^2} - \frac{1}{2}2^{-((l_k)^2+l_k)\frac{1}{h}}\right].$$

For $h = 0$ we use the set $F_{0,k}$

$$F_{0,k} = \bigcup_{j=0,\dots,2^{(l_k)^2}-1} \left[(j+1)2^{-(l_k)^2} - 2^{-((l_k)^2+l_k)k}, (j+1)2^{-(l_k)^2} - \frac{1}{2}2^{-((l_k)^2+l_k)k}\right].$$

Proposition 14. *Suppose* $0 < h < 1$ *is fixed. Set* $k_h = \lceil 1/h \rceil + 1$ *and*

$$F_h := \bigcap_{k \geq k_h} F_{h,k}.$$

For every $x = (x_1, ..., x_d) \in \widetilde{F}_h := F_h \times [0,1]^{d-1}$*, we have* $h_f(x) \leq h$.

Proof. Observe that if $k \geq k_h$, then $1/h < k$ and $D_k < D_{h,k}$.

Suppose that $x = (x_1, ..., x_d) \in \widetilde{F}_h$. Then there exists $j \in \{0, ..., 2^{(l_k)^2} - 1\}$ such that

$$x_1 \in \Big[(j+1)2^{-(l_k)^2} - 2^{-((l_k)^2 + l_k)\frac{1}{h}}, (j+1)2^{-(l_k)^2} - \frac{1}{2} 2^{-((l_k)^2 + l_k)\frac{1}{h}} \Big]. \tag{29}$$

By (27), we also have

$$(j+1)2^{-(l_k)^2} - 2^{-(l_k)^4} > (j+1)2^{-(l_k)^2} - \frac{1}{2} 2^{-((l_k)^2 + l_k)\frac{1}{h}},$$

and hence $\gamma_{l_k}((j+1)2^{-(l_k)^2}) - \gamma_{l_k}(x_1) \geq 2^{-((l_k)^2 + l_k)}$. From (23) the monotonicity of ϕ_k (defined by (24)), we deduce that

$$(\phi_k + g_{l_k})((j+1)2^{-(l_k)^2}, x_2, ..., x_d) - (\phi_k + g_{l_k})(x_1, x_2, ..., x_d) \geq 2^{-(l_k)^2 - l_k}.$$

Combining (25) and the fact that $\rho_{l_k} = 2^{-(l_k)^8}$, it follows that

$$\begin{aligned} f((j+1)2^{-(l_k)^2}, x_2, ..., x_d) - f(x_1, x_2, ..., x_d) &\geq (\phi_k + g_{l_k})((j+1)2^{-(l_k)^2}, x_2, ..., x_d) \\ &\quad -(\phi_k + g_{l_k})(x_1, x_2, ..., x_d) - 2\rho_{l_k} \\ &\geq 2^{(-(l_k)^2 + l_k)} - 2^{-(l_k)^8 + 1} \\ &\geq 2^{(-(l_k)^2 + l_k) - 1}. \end{aligned}$$

By (29) we have

$$|((j+1)2^{-(l_k)^2}, x_2, ..., x_d) - (x_1, x_2, ..., x_d)| \leq 2^{-((l_k)^2 + l_k)\frac{1}{h}}$$

and hence

$$f((j+1)2^{-(l_k)^2}, x_2, ..., x_d) - f(x_1, x_2, ..., x_d) \geq \frac{1}{2} |((j+1)2^{-(l_k)^2}, x_2, ..., x_d) - (x_1, x_2, ..., x_d)|^h.$$

Since this holds for all k and $l_k \to \infty$, we deduce that $h_f(x_1, ..., x_d) \leq h$. □

Proposition 15. *Let*

$$F_0 := \bigcap_{k \geq 3} F_{0,k}.$$

For every $x = (x_1, ..., x_d) \in \widetilde{F}_0 := F_0 \times [0,1]^{d-1}$*, we have* $h_f(x) = 0$.

Subsequently, $\dim_H E_f^0 = \{x : h_f(x) = 0\} = d - 1$.

Proof. By (27), F_0 is non-empty and it is easy to see that it has the cardinality of the continuum (it has the structure of a Cantor set).

Fix $x = (x_1, ..., x_d) \in \widetilde{F}_0$, and $k \geq 3$. There exists $j \in \{0, ..., 2^{(l_k)^2} - 1\}$ such that

$$x_1 \in [(j+1)2^{-(l_k)^2} - 2^{-((l_k)^2 + l_k)k}, (j+1)2^{-(l_k)^2} - \frac{1}{2} 2^{-((l_k)^2 + l_k)k}], \tag{30}$$

and as above, by (27), we have

$$(j+1)2^{-(l_k)^2} - 2^{-(l_k)^4} > (j+1)2^{-(l_k)^2} - \frac{1}{2} 2^{-((l_k)^2 + l_k)k}.$$

Hence $\gamma_{l_k}((j+1)2^{-(l_k)^2}) - \gamma_{l_k}(x_1) \geq 2^{-((l_k)^2+l_k)}$. Similarly to the case $0 < h < 1$, we can deduce that

$$f((j+1)2^{-(l_k)^2}, x_2, ..., x_d) - f(x_1, x_2, ..., x_d) \geq 2^{(-(l_k)^2+l_k)-1}.$$

By (30), we have

$$|((j+1)2^{-(l_k)^2}, x_2, ..., x_d) - (x_1, x_2, ..., x_d)| \leq 2^{-((l_k)^2+l_k)k},$$

which implies that

$$f((j+1)2^{-(l_k)^2}, x_2, ..., x_d) - f(x_1, x_2, ..., x_d) \geq \frac{1}{2}|((j+1)2^{-(l_k)^2}, x_2, ..., x_d) - (x_1, x_2, ..., x_d)|^{1/k}.$$

Since this holds for all integers k and for the increasing sequence of integers l_k, we conclude that $h_f(x_1, ..., x_d) = 0$.

We just proved that F_0 is non-empty. Hence, using the fact that $\widetilde{F}_0$ is a product set, $\dim_H \widetilde{F}_0 \geq d - 1 + \dim_H F_0$. Obviously $\widetilde{F}_0 \subset E_f^0$, and we know by Theorem 5 that $\dim_H E_f^0 \leq d - 1$. We deduce that $\dim_H E_f^0 = d - 1$. □

Proposition 16. *Suppose* $0 < h < 1$ *is fixed. Then* $\dim_H F_h \geq h$.

This implies that $\dim_H \widetilde{F}_h \geq d - 1 + h$. *More precisely there exists a measure* $\widetilde{\mu}_h$ *supported by* $\widetilde{F}_h$ *which satisfies* (8) *with* $s = d - 1 + h$.

Proof. We want to apply the Mass Distribution Theorem 9. For this we are going to construct iteratively a measure μ_h supported by F_h, and then by taking the product of μ_h with the $d-1$-dimensional Lebesgue measure, we will obtain $\widetilde{\mu}_h$.

The set F_{h,k_h} consists of $2^{(l_{k_h})^2}$ many equally spaced intervals. We start by defining a measure μ_h^1 by assigning a weight $2^{-(l_{k_h})^2}$ to each such interval (uniformly distributed in each interval). These component intervals are of length

$$2^{-((l_{k_h})^2+l_{k_h})\frac{1}{h}-1} > 2^{-((l_{k_h})^2+l_{k_h})k_h-1} > 100 \cdot 2^{-(l_{k_h+1})^2}, \tag{31}$$

where the last inequality holds due to (27).

Then, using a very standard construction, the mass distribution μ_h is constructed iteratively as follows: assume for $k \geq k_h$ that a mass distribution μ_h^k is uniformly distributed on some of the intervals of $F_{h,k}$, namely, on those intervals which belong to the intervals in $F_{h,k-1}$, selected at the previous step. Then we define the measure μ_h^{k+1} as the measure supported by the intervals of $F_{h,k+1}$ included in the support of μ_h^k, and such that the weight of each interval I in $F_{h,k}$ included in the support of μ_h^k is divided equally on those component intervals of $F_{h,k+1}$ which are included in I. It is very classical that the sequence of measures $(\mu_h^k)_{k\geq k_h}$ weakly converges to a Borel probability measure μ_h.

As a particular property, this measure μ_h assigns to each interval $I \in F_{h,k}$ appearing along the construction the same mass as μ_h^k does: for every $I \in F_{h,k}$, $\mu_h(I) = \mu_h^k(I)$.

Using (31), the number of F_{h,k_h+1} intervals in a component interval of F_{h,k_h} is larger than $2^{(l_{k_h+1})^2}\frac{1}{2}2^{-((l_{k_h})^2+l_{k_h})\frac{1}{h}-1}$. Hence, if I_{k_h+1} is such an interval, then, using (28) and (26), we find that

$$\begin{aligned} \mu_h(I_{k_h+1}) &\leq 2^{-(l_{k_h+1})^2} 2^{((l_{k_h})^2+l_{k_h})\frac{1}{h}+2} 2^{-(l_{k_h})^2} \leq 2^{-(l_{k_h+1})^2} D_{h,k_h}^{-1} \\ &\leq 2^{-(l_{k_h+1})^2} D_{k_h}^{-1} \cdots D_1^{-1} < 2^{-(l_{k_h+1})^2+l_{k_h+1}}. \end{aligned}$$

By induction, one easily sees that for $k > k_h$, if I_k is a component interval of $F_{h,k}$ and I_{k-1} is the unique component interval in $F_{h,k-1}$ containing I_k, then the same properties as just above hold:

$$\begin{aligned}\mu_h(I_k) &\le 2^{-(l_k)^2} 2^{((l_{k-1})^2+l_{k-1})\frac{1}{h}+2} \mu_h(I_{k-1}) \\ &\le 2^{-(l_k)^2} 2^{((l_{k-1})^2+l_{k-1})\frac{1}{h}+2} 2^{-(l_{k-1})^2} D_{k-2}^{-1} \cdots D_1^{-1} \le 2^{-(l_k)^2} D_{k-1}^{-1} \cdots D_1^{-1} < 2^{-(l_k)^2+l_k}.\end{aligned}$$

Let us prove now formula (8) for μ with $s = h$.

Let $x_1 \in F_h$ and suppose that $r < 2^{-(l_{k_h}+1)^2}$. Consider the unique integer k such that

$$2^{-((l_k)^2+l_k)\frac{1}{h}-1} < r \le 2^{-((l_{k-1})^2+l_{k-1})\frac{1}{h}-1}. \tag{32}$$

We separate two cases.

- **If** $2^{-((l_k)^2+l_k)\frac{1}{h}-1} < r \le 2^{-(l_k)^2}$ then $B(x_1, r)$ can intersect no more than three component intervals of level k. Hence by (32),

$$\frac{\log \mu_h(B(x_1, r))}{\log r} \ge \frac{\log(3 \cdot 2^{-((l_k)^2+l_k)})}{\log(2^{-((l_k)^2+l_k)\frac{1}{h}-1})}, \tag{33}$$

  which converges to h when k tends to infinity.
- **If** $2^{-(l_k)^2} < r \le 2^{-((l_{k-1})^2+l_{k-1})\frac{1}{h}-1}$ then denote by $I_{k-1}(x_1)$ and $I_k(x_1)$ the (unique) component intervals of $F_{h,k-1}$ and $F_{h,k}$ containing x_1. Each interval of the form $[j2^{-(l_k)^2}, (j+1)2^{-(l_k)^2}] \subset I_{k-1}(x_1)$ contains exactly one component interval of $F_{h,k}$. The weight of $I_{k-1}(x_1)$ is uniformly distributed among the equally spaced intervals of $F_{h,k}$. Taking into account some possible extra intersection at the boundary of $B(x, r)$ the proportional μ_h measure of $B(x, r)$ is not exceeding $3 \cdot \frac{|B(x,r)|}{|I_{k-1}(x_1)|}$ times the measure of $I_{k-1}(x_1)$. Subsequently,

$$\mu_h(B(x, r)) < 3\mu_h(I_{k-1}(x_1)) \frac{2r}{|I_{k-1}(x_1)|} \le 6 \cdot 2^{-(l_{k-1})^2+l_{k-1}} 2^{((l_{k-1})^2+l_{k-1})\frac{1}{h}+1} r.$$

  Therefore,

$$\begin{aligned}\frac{\log \mu_h(B(x, r))}{\log r} &\ge \frac{\log r + \log 6 + (-(l_{k-1})^2 + l_{k-1} + ((l_{k-1})^2 + l_{k-1})\frac{1}{h} + 1)\log 2}{\log r} \\ &\ge 1 + \frac{\log 6}{\log r} + \frac{-(l_{k-1})^2 + l_{k-1} + ((l_{k-1})^2 + l_{k-1})\frac{1}{h} + 1}{-((l_{k-1})^2 + l_{k-1})\frac{1}{h} - 1} \\ &\ge 1 + \frac{\log 6}{(-((l_{k-1})^2 + l_{k-1})\frac{1}{h} - 1)\log 2} + \Big(-1 + \frac{-(l_{k-1})^2 + l_{k-1}}{-((l_{k-1})^2 + l_{k-1})\frac{1}{h} - 1}\Big),\end{aligned}$$

  which converges to h when k tends to infinity.

From the above two cases we deduce that necessarily $\liminf_{r\to 0+} \frac{\log \mu_h(B(x_1,r))}{\log r} \ge h$ at all points of F_h. Theorem 9 implies that $\dim_H F_h \ge h$. In addition, we obtain that for every point $x \in \widetilde{F}_h$, we have

$$\liminf_{r\to 0+} \frac{\log \widetilde{\mu}_h(B((x_1, ..., x_d), r))}{\log r} \ge d - 1 + h,$$

where $\widetilde{\mu}_h := \mu_h \times \lambda_{d-1}$ and λ_{d-1} stands for the $d-1$-dimensional Lebesgue measure on $[0, 1]^{d-1}$. Hence, using again Theorem 9,

$$\dim_H \widetilde{F}_h \ge d - 1 + h$$

and the local dimension of $\widetilde{\mu}_h$ at any point of its support is larger than $d-1+h$. □

Proposition 17. *If $f \in \mathcal{R}$, then for all $0 < h \leq 1$ we have $\dim_H E_f^h = d-1+h$.*

Proof. From Propositions 14 and 16, we deduce that

$$\widetilde{\mu}_h(E_f^{h,\leq}) = \mu_h(\widetilde{F}_h) = 1.$$

By definition of $E_f^{h,\leq}$, we have

$$E_f^h = E_f^{h,\leq} \setminus \bigcup_{n\geq 1} E_f^{h-1/n,\leq}.$$

Using Theorem 5 and the upper bound (4), for every $n \geq 1$, $\dim_H E_f^{h-1/n,\leq} \leq d-1+h-1/n < d-1+h$. By Theorem 9, one necessarily has $\widetilde{\mu}_h(E_f^{h-1/n,\leq}) = 0$. We deduce that

$$\widetilde{\mu}_h(E_f^h) \geq \widetilde{\mu}_h(E_f^{h,\leq}) - \sum_{n\geq 1} \widetilde{\mu}_h(E_f^{h-1/n,\leq}) \geq 1.$$

Since $\widetilde{\mu}_h$ has dimension $d-1+h$, we conclude that $\dim_H E_f^h \geq d-1+h$.

For $h=1$, the same argument using the d-dimensional Lebesgue measure instead of $\widetilde{\mu}_h$ applies, and we obtain that $\dim_H E_f^1 \geq d$ and $\lambda_d(E_f^1) = 1$. □

We finish the proof of Theorem 6 with the following proposition.

Proposition 18. *If $f \in \mathcal{R}$, then for all $h > 1$, the sets E_f^h are empty.*

Proof. It is sufficient to verify that, taking any $x = (x_1, ..., x_d) \in [0,1]^d$, if we keep $(x_2, ..., x_d)$ fixed and consider the function $f_1(t) = f(t, x_2, ..., x_d)$, then $E_{f_1}^h = \emptyset$.

Proceeding towards a contradiction suppose that $h = h_{f_1}(x_1) > 1$. Then there exist a real number m and a constant C such that, when t is close enough to x_1,

$$|f_1(t) - f_1(x_1) - m(t-x_1)| \leq C|t-x_1|^{\min(h,2)-\varepsilon}, \tag{34}$$

where ε is chosen so that $\min(h,2)-\varepsilon > 1$.

Recall (25), and choose the integer j such that $x_1 \in [j2^{-(l_k)^2}, (j+1)2^{-(l_k)^2}]$. Without limiting generality, one can assume that $j \leq 2^{(l_k)^2} - 2$.

Using the definition of γ_{l_k}, the piecewise monotonicity of ϕ_k and (25), one sees that

$$\begin{aligned} f_1((j+2)2^{-(l_k)^2}) - f_1((j+2)2^{-(l_k)^2} - 2^{-(l_k)^4}) &\geq 2^{-(l_k)^2-l_k} - 2\rho_{l_k} = 2^{-(l_k)^2-l_k} - 2\cdot 2^{-(l_k)^8} \\ &> 2^{-(l_k)^2-l_k-1}. \end{aligned} \tag{35}$$

Using twice (34) with $t = (j+2)2^{-(l_k)^2}$ and $t = (j+2)2^{-(l_k)^2} - 2^{-(l_k)^4}$, we should have

$$\begin{aligned} & \Big| f_1((j+2)2^{-(l_k)^2}) - f_1(((j+2)2^{-(l_k)^2} - 2^{-(l_k)^4}) \Big| \\ \leq\ & \Big| f_1((j+2)2^{-(l_k)^2}) - f_1(x_1) - m((j+2)2^{-(l_k)^2} - x_1) \Big| \\ & + \Big| f_1(x_1) - f_1((j+2)2^{-(l_k)^2} - 2^{-l_k^4}) - m(x_1 - ((j+2)2^{-(l_k)^2} - 2^{-l_k^4})) \Big| \\ & + m2^{-l_k^4} \\ \leq\ & C|(j+2)2^{-(l_k)^2} - x_1|^{\min(h,2)-\varepsilon} + C|x_1 - ((j+2)2^{-(l_k)^2} - 2^{-(l_k)^4}))|^{\min(h,2)-\varepsilon} + m2^{-(l_k)^4} \\ \leq\ & 2\cdot 2^{\min(h,2)-\varepsilon} C 2^{-(l_k)^2(\min(h,2)-\varepsilon)} + m2^{-(l_k)^4}. \end{aligned}$$

This contradicts (35) when l_k is large. □

5. Study of the level sets of generic functions in $\mathcal{M}^d$

The following extension lemma is quite simple, but a little caution is needed since in general it is not true that if $K \subset [0,1]^d$ is compact and f is continuous MISV on K, then we can extend f to obtain an MISV function in $\mathcal{M}^d$. For instance, one can take K as the union of four points in $[0,1]^2$ such that all of their coordinates are different. Then any function defined on them is continuous MISV, since the functions $f^{(i)}$ (defined by (1)) are defined at at most one point, and one can easily find examples when there is no extension. The existence of a continuous MISV extension is easily proved when K is a cartesian product.

Lemma 19. *Suppose $K_i \subset [0,1]$ is compact for all $i = 1, ..., d$, $K = K_1 \times ... \times K_d$ and $f : K \to \mathbb{R}$ is MISV and continuous. Then there exists an extension of f onto $[0,1]^d$, still denoted by f which is also continuous MISV, and such that*

$$f([0,1]^d) = [\min_{x\in K} f(x), \max_{x\in K} f(x)]. \tag{36}$$

Proof. First we show that for all $i = 1, ..., d$, one can extend f onto

$$K(i) = K_1 \times ... \times K_{i-1} \times [0,1] \times K_{i+1} \times ... \times K_d$$

so that the extension is continuous MISV, and

$$f(K(i)) \subset [\min_{x\in K} f(x), \max_{x\in K} f(x)]. \tag{37}$$

By induction this clearly implies Lemma 19.

Without limiting generality, it is sufficient to show that one can extend f onto $[0,1] \times K_2 \times ... \times K_d$. Let $m_1 := \min\{t : t \in K_1\})$ and $M_1 := \max\{t : t \in K_1\})$. Observe that the complement of K_1 in $[m_1, M_1]$ can be written as an at most countable disjoint union of open intervals, that is, $[m_1, M_1] \setminus K_1 = \bigcup_n (a_n, b_n)$, where for every n, the $a_n, b_n \in K_1$ and $(a_n, b_n) \cap K_1 = \emptyset$. Now, for any fixed $(x_2, ..., x_d) \in K_2 \times ... \times K_d$, we set

$$f(t, x_2, ..., x_d) = \begin{cases} f(m_1, x_2, ..., x_d) & \text{if } 0 \le t \le m_1, \\ f(t, x_2, ..., x_d) & \text{if } t \in K_1, \\ \gamma f(a_n, x_2, ..., x_d) + (1-\gamma) f(b_n, x_2, ..., x_d) & \text{if } t = \gamma a_n + (1-\gamma) b_n \\ & \quad \text{for some } \gamma \in (0,1), \\ f(M_1, x_2, ..., x_d) & \text{if } M_1 \le t \le 1. \end{cases}$$

Essentially, f is extended outside K_1 by using affine interpolation. Clearly, f is monotone in the first variable. Let us prove the monotonicity property in the other variables.

Let $i > 1$.

- If $x_1 = \gamma a_n + (1-\gamma) b_n$, where $\gamma \in (0,1)$. Assume that $x_i < x_i'$ and $(x_1, ..., x_d)$ and $(x_1, ..., x_{i-1}, x_i', x_{i+1}, ..., x_d)$ both belong to $[0,1] \times K_2 \times ... \times K_d$. Then

$$\begin{aligned} f(x_1, ..., x_{i-1}, x_i, x_{i+1}, ..., x_d) &= \gamma f(a_n, x_2, ..., x_{i-1}, x_i, x_{i+1}, ..., x_d) \\ &\quad + (1-\gamma) f(b_n, x_2, ..., x_{i-1}, x_i, x_{i+1}, ..., x_d) \\ &\le \gamma f(a_n, x_2, ..., x_{i-1}, x_i', x_{i+1}, ..., x_d) \\ &\quad + (1-\gamma) f(b_n, x_2, ..., x_{i-1}, x_i', x_{i+1}, ..., x_d) \\ &= f(x_1, ..., x_{i-1}, x_i', x_{i+1}, ..., x_d). \end{aligned}$$

- The cases $0 \le x_1 < \min\{t : t \in K_1\}$ and $\max\{t : t \in K_1\} < x_1 \le 1$ are similar and are left to the reader.

Hence, f is MISV on $[0,1]\times K_2\times...\times K_d$. The continuity of f on $[0,1]\times K_2\times...\times K_d$ can also be verified easily by using uniform continuity of f on K_1 and uniform continuity of the projections.

□

We are now able to prove Theorem 7.

For every fixed $n\geq 1$, we select a countable dense family in $\mathcal{M}^d$ consisting of functions $\{f_{m,n}\}_{m=1}^{\infty}$ satisfying the following:

- there exists a sequence $(l_{m,n})_{m\geq 1,n\geq 1}$ such that $l_{m,n}\geq m+n$ and the next properties hold:
- for every $j=(j_1,...,j_d)\in\{0,...,2^{l_{m,n}}-1\}^d$, for all

$$x\in Q_{m,n}(j):=\prod_{i=1}^{d}[j_i 2^{-l_{m,n}}+2^{-(l_{m,n})^2},(j_i+1)2^{-l_{m,n}}-2^{-(l_{m,n})^2}],$$

  we have $f_{m,n}(x)=y_{m,n}(j)$, that is, $f_{m,n}$ is constant and equal to the real number $y_{m,n}(j)$ on $Q_{m,n}(j)$.
- the real numbers $(y_{m,n}(j)))$, where $m,n\in\mathbb{N}^2$ and $j\in\{0,...,2^{l_{m,n}}-1\}^d$, are pairwise distinct. In other words,

$$0<d_{m,n}\ :=\min\{|y_{m,n}(j)-y_{m,n}(j')|:j,j'\in\{0,...,2^{l_{m,n}}-1\}^d,\ j\neq j'\}.$$

The existence of this family is an exercise, left to the reader (the density follows from the fact that we use the L^∞ norm for continuous functions).

Remark 20. Observe that, due to the definition of $Q_{m,n}(j)$, each function $f_{m,n}$ is constant on the largest part of the cubes $\prod_{i=1}^{d}[j_i 2^{-l_{m,n}},(j_i+1)2^{-l_{m,n}}]$ (here, the notion of "large" is understood with respect to the Lebesgue measure). More precisely, the complement of $Q_{m,n}(j)$ in the cube $\prod_{i=1}^{d}[j_i 2^{-l_{m,n}},(j_i+1)2^{-l_{m,n}}]$ consists of less than $2d\cdot\left(\frac{2^{-l_{m_n,n}}}{2^{-(l_{m_n,n})^2}}\right)^{d-1}$ many cubes of side length $2^{-(l_{m_n,n})^2}$.

Set

$$\rho_{m,n}=\min\{2^{-(l_{m,n})^2},d_{m,n}\},\quad \mathcal{L}_n=\bigcup_{m=1}^{\infty}B(f_{m,n},\rho_{m,n})\quad\text{and}\quad \mathcal{L}=\bigcap_{n=1}^{\infty}\mathcal{L}_n.$$

Each $\mathcal{L}_n$ is a dense open set, and $\mathcal{L}$ is a dense G_δ set. We will prove Theorem 7 for the functions in $\mathcal{L}$.

Suppose $f\in\mathcal{L}$. By definition, for all n, there exists m_n such that $f\in B(f_{m_n,n},\rho_{m_n,n})$. Set

$$\overline{X}_n=\bigcup_{j\in\{0,...,2^{l_{m_n,n}}-1\}^d}Q_{m_n,n}(j),\quad X_n=[0,1]^d\setminus\overline{X}_n\quad\text{and}\quad X_f=\bigcup_{N=1}^{\infty}\bigcap_{n\geq N}X_n.$$

The set X_f contains points x which belong to X_n for all n greater than some integer N_x, hence do not belong to any cube $Q_{m_n,n}(j)$, for any $n\geq N_x$ and any j.

We also put

$$A_n = \bigcup_{\{0,\dots,2^{l_{m_n,n}}-1\}^d} (y_{m_n,n}(j) - \rho_{m_n,n}, y_{m_n,n}(j) + \rho_{m_n,n}) \quad \text{and} \quad A_f = \bigcap_{N=1}^{\infty} \bigcup_{n\geq N} A_n.$$

The set A_f contains those real numbers a which satisfy $|a - y_{m_n,n}(j)| < \rho_{m_n,n}$ for an infinite number of integers n.

Lemma 21. $\dim_H X_f \leq d-1$ *and* $\dim_H A_f = 0$.

Proof. Using Remark 20, the set X_n is covered by $2^{d\, l_{m_n,n}} \cdot 2d \cdot \left(\frac{2^{-l_{m_n,n}}}{2^{-(l_{m_n,n})^2}}\right)^{d-1}$ cubes of side length $2^{-(l_{m,n})^2}$. Hence, for any $\varepsilon > 0$, the $(d-1+\varepsilon)$-dimensional Hausdorff measure of X_n is bounded from above by

$$C_d \sum_{n=1}^{\infty} \left(2^{d\, l_{m_n,n}} \cdot 2d \cdot \left(\frac{2^{-l_{m_n,n}}}{2^{-(l_{m_n,n})^2}}\right)^{d-1}\right) 2^{-(l_{m_n,n})^2(d-1+\varepsilon)} \leq C_d 2d \sum_{n=1}^{\infty} 2^{l_{m_n,n}} 2^{-\varepsilon (l_{m_n,n})^2}.$$

Obviously, this sum converges for every $\varepsilon > 0$, and tends to zero when n converges to infinity.

Remark that for every $N \geq 1$, $\bigcap_{n\geq N} X_n$ is obviously covered by any X_n, $n \geq N$. From this we deduce that for every $\varepsilon > 0$ and $N \geq 1$, $\dim_H \bigcap_{n\geq N} X_n \leq d-1+\varepsilon$. As a countable union of such sets, $\dim_H X_f \leq d-1+\varepsilon$, holds for every $\varepsilon > 0$. We conclude that $\dim_H(X_f) \leq d-1$.

Let us now bound from above the dimension of A_f. The sets A_n can be covered by $2^{l_{m_n,n}d}$ many intervals of length $2\rho_{m_n,n} \leq 2\cdot 2^{-(l_{m_n,n})^2}$. Remarking that for every $N \geq 1$, the union $\bigcup_{n\geq N} A_n$ is a cover of A_f, and using this union to cover A_f, for all $\varepsilon > 0$, the ε-dimensional Hausdorff measure of A_f is bounded from above by

$$2^{\varepsilon} \sum_{n\geq N}^{\infty} 2^{d l_{m_n,n}} 2^{-(l_{m_n,n})^2 \varepsilon},$$

which is the tail of a convergent series. This implies that $\dim_H A_f \leq 0$.

Obviously A_f is uncountable and hence non-empty. This implies that $\dim_H A_f = 0$. □

Lemma 22. *For all $a \in (f(0,...,0), f(1,...,1)) \setminus A_f$, the level set $L_f(a)$ (defined by (5)) is included in X_f. In particular,* $\dim_H X_f = d-1$.

Proof. Let $a \in (f(0,...,0), f(1,...,1)) \setminus A_f$. There exists N_a such that $a \notin A_n$ for all $n \geq N_a$. Since $f(\overline{X}_n) \subset A_n$ we have $L_f(a) \subset [0,1]^d \setminus \overline{X}_n = X_n$ for all $n \geq N_a$. Subsequently, $L_f(a) \subset X_f$. In the introduction, we explained that if $a \in (f(0,...,0), f(1,...,1))$, then $\dim_H L_f(a)$ is necessarily greater than $d-1$. We obtain finally that $\dim_H(X_f) = d-1$. □

Finally, we consider the level sets $L_f(a)$ when a belongs to A_f.

Lemma 23. *For all $a \in A_f$, the level set $L_f(a)$ contains at most one point in $[0,1]^d \setminus X_f$.*

Proof. We are interested in the intersection of $L_f(a)$ and the complement of X_f. We need to consider the possible intersection between $L_f(a)$ and the cubes $Q_{m_n,n}(j)$. By definition of A_f, there exists an infinite sequence of integers n_k such that $|a - y_{m_{n_k}(j^k),n_k}| \leq \rho_{m_{n_k},n_k}$ for some $j^k = (j_1^k, ..., j_d^k) \in \{0,1,...,2^{l_{m_{n_k},n_k}}-1\}^d$. This j^k is necessarily unique since the $y_{m_{n_k}(j),n_k}$, $j \in \{0,1,...,2^{l_{m_{n_k},n_k}}-1\}^d$ are distinct and $\rho_{m_{n_k},n_k}$ has been chosen small enough. Therefore, $L_f(a)$ can intersect at most one $Q_{m_{n_k},n_k}(j)$. This holds for an infinite sequence of integers n_k. Since the diameters of these sets $Q_{m_{n_k},n_k}(j)$ tend to zero as $n_k \to \infty$, the level set $L_f(a)$ can have at most one point in $[0,1]^d \setminus X_f$. □

6. Generic nowhere monotonicity in the directions not in $\mathbb{R}^d_+ \cup \mathbb{R}^d_-$

Finally, we turn to the proof of Theorem 8.

Proof. Suppose that $\{f_n\}_{n=1}^\infty$ is a countable dense family in $\mathcal{M}^d$.

We select a sequence $(p_n)_{n\in\mathbb{N}} := ((p_{1,n},...,p_{d,n}))_{n\in\mathbb{N}}$ in $(0,1)^d$ such that every $(q_1,...,q_d) \in (0,1)^d$ with rational coordinates appears in the sequence $(p_n)_{n\in\mathbb{N}}$ infinitely often.

For every integer $n \geq 1$, we define $\widetilde{p}_n := \max_{i\in\{1,...,d\}} p_{i,n}$.

For every integers $m \geq 1$ and $n \geq 1$, we set $r_{m,n} = 1/(m+n) > 0$.

By uniform continuity of f_n, one can choose $\eta'_{m,n} > 0$ such that $\eta'_{m,n} < \min\{1/(m+n), 1-\widetilde{p}_n\}$ and if $x, x' \in [0,1]^d$, $|x-x'| < \eta'_{m,n}$ then

$$|f_n(x) - f_n(x')| < r_{m,n}/3. \tag{38}$$

Finally, for every integer $n \geq 1$, we set

$$\eta_{m,n} = \eta'_{m,n}/n > 0. \tag{39}$$

Definition 24. For $i \in \{1,...,d\}$, we introduce the function $h_{m,n,i} \in \mathcal{M}^d$ as follows:

- If $0 \leq x_i \leq p_{i,n}$ then set $h_{m,n,i}(x_1,...,x_d) = 0$,
- if $p_{i,n} + \eta_{m,n} \leq x_i \leq 1$ then set $h_{m,n,i}(x_1,...,x_d) = r_{m,n} = 1/(m+n)$,
- $h_{m,n,i}$ is linear on the set $\{(x_1,...,x_d) : p_{i,n} \leq x_i \leq p_{i,n} + \eta_{m,n}\}$.

For every integers m and n, we set $h_{m,n} = \sum_{i=1}^{d} h_{m,n,i}$ and $f_{m,n} = f_n + h_{m,n}$.

Obviously, for every m and n, for every $(x_1,x_2,...,x_d) \in (0,1)^d$, and for every $i \in \{1,...,d\}$, we have

$$\begin{aligned} h_{m,n,i}(x_1,...,x_{i-1},p_{i,n}+\eta_{m,n},x_{i+1},...,x_d) - h_{m,n,i}(x_1,...,x_{i-1},p_{i,n},x_{i+1},...,x_d) \\ \geq 1/(m+n) = r_{m,n}. \end{aligned} \tag{40}$$

We set $\rho_{m,n} = r_{m,n}/9$.

Definition 25. We introduce the sets $\mathcal{G}_n = \bigcup_{m=1}^{\infty} B(f_{m,n}, \rho_{m,n})$ and $\mathcal{G} = \bigcap_{n=1}^{\infty} \mathcal{G}_n$.

Obviously, $\mathcal{G}_n$ is dense open and hence $\mathcal{G}$ is residual in $\mathcal{M}^d$. We will prove Theorem 8 for the functions in $\mathcal{G}$.

Suppose that $f \in \mathcal{G}$, $\underline{l} = (l_1,...,l_d) \notin \mathbb{R}^d_+ \cup \mathbb{R}^d_-$, $\underline{b} = (b_1,...,b_d) \in \mathbb{R}^d$ are given, and assume that there is a non-empty open interval (α,β) such that $\underline{l}t + \underline{b} \in [0,1]^d$ for $t \in (\alpha,\beta)$. We are going to study the (non-)monotonicity properties of the function $g : t \in (\alpha,\beta) \longmapsto g(t) := f(\underline{l}t + \underline{b})$.

Since $\underline{l} \notin \mathbb{R}^d_+ \cup \mathbb{R}^d_-$ choose a coordinate, say i, such that $l_i < 0$ and another coordinate i' such that $l_{i'} > 0$.

Set $T_{\underline{l},i} = \max\{l_j/|l_i| : j = 1,...,d\}$. We select a real number $t_i \in (\alpha,\beta)$ such that $l_i t_i + b_i = q_i$, for some rational number $q_i \in \mathbb{Q} \cap [0,1]$. For $j \neq i$, we also select

$q_j \in \mathbb{Q} \cap (0,1)$ with $q_j \neq q_i$, $q_j \neq l_j t_i + b_j$. Keeping in mind that $l_i < 0$, we suppose that a sufficiently large integer $N_0 > T_{\underline{l},i}$ is selected such that

$$\underline{l}t + \underline{b} \in [0,1]^d \qquad \text{for every } t \in \Big[t_i + \frac{1}{N_0 l_i}, t_i\Big] \subset (\alpha,\beta), \tag{41}$$

$$l_j t + b_j \notin \Big[q_j, q_j + \tfrac{1}{N_0}\Big] \quad \text{for } j \neq i \text{ and for every } t \in \Big[t_i + \frac{1}{N_0 l_i}, t_i\Big]. \tag{42}$$

Observe that (42) implies that

$$h_{m,n,j} \text{ is constant on } \Big[t_i + \frac{1}{N_0 l_i}, t_i\Big], \text{ for } j \neq i. \tag{43}$$

From $N_0 > T_{\underline{l},i}$ and (39), it follows that for every integer $n \geq N_0$, we have

$$\left| \frac{l_j}{l_i} \cdot \eta_{m,n} \right| \leq T_{\underline{l},i} \eta_{m,n} < \eta'_{m,n}.$$

Hence

$$\left| \Big(\underline{l}\Big(t_i + \frac{\eta_{m,n}}{l_i}\Big) + \underline{b}\Big) - (\underline{l}t_i + \underline{b}) \right| < \eta'_{m,n} \tag{44}$$

since we use the sup norm in $\mathbb{R}^d$.

Since $f \in \mathcal{G}$, we can choose two integers m, n such that :

- $n \geq N_0$, which implies that $\eta'_{m,n} \leq r_{m,n} = 1/(m+n) < 1/N_0$,
- $p_{j,n} = q_j$ for every $j = 1, ..., d$,
- $f \in B(f_{m,n}, \rho_{m,n})$.

Since $f \in B(f_{m,n}, \rho_{m,n})$, we have

$$\begin{aligned} g\Big(t_i + \frac{\eta_{m,n}}{l_i}\Big) - g(t_i) &= f\Big(\underline{l}\Big(t_i + \frac{\eta_{m,n}}{l_i}\Big) + \underline{b}\Big) - f\Big(\underline{l}t_i + \underline{b}\Big) \\ &\geq f_{m,n}\Big(\underline{l}\Big(t_i + \frac{\eta_{m,n}}{l_i}\Big) + \underline{b}\Big) - f_{m,n}\Big(\underline{l}t_i + \underline{b}\Big) - 2\rho_{m,n}. \end{aligned}$$

Then, by (38) and (44), we have

$$\begin{aligned} g\Big(t_i + \frac{\eta_{m,n}}{l_i}\Big) - g(t_i) &\geq h_{m,n}\Big(\underline{l}\Big(t_i + \frac{\eta_{m,n}}{l_i}\Big) + \underline{b}\Big) - h_{m,n}\Big(\underline{l}t_i + \underline{b}\Big) - \frac{\eta_{m,n}}{3} - 2\rho_{m,n} \\ &= h_{m,n,i}\Big(\underline{l}\Big(t_i + \frac{\eta_{m,n}}{l_i}\Big) + \underline{b}\Big) - h_{m,n,i}\Big(\underline{l}t_i + \underline{b}\Big) - \frac{\eta_{m,n}}{3} - 2\rho_{m,n} \\ &\geq r_{m,n} - \frac{\eta_{m,n}}{3} - 2\rho_{m,n} > 5\rho_{m,n} > 0. \end{aligned}$$

where (43) and (40) have also been used to obtain the last two lines of the above displayed formula.

This implies that g cannot be monotone increasing on $(\alpha,\beta) \supset [t_i + \frac{\eta_{m,n}}{l_i}, t_i]$.

A similar argument used for the coordinate i' can show that g cannot be monotone decreasing on (α,β) either. □

References

[1] A. M. Bruckner, J. B. Bruckner and B. S. Thomson, *Real Analysis,* Prentice-Hall International, London, (2008).

[2] Z. Buczolich, U. B. Darji, *Pseudoarcs, pseudocircles, Lakes of Wada and generic maps on S^2,* Topology and its Applications 150 (1-3), 223-254, (2005).

[3] Z. Buczolich, J. Nagy, *Hölder spectrum of typical monotone continuous functions*, Real Anal. Exchange 26, 133–156, 2000/01.

[4] Z. Buczolich, S. Seuret, *Typical Borel measures on* $[0,1]^d$ *satisfy a multifractal formalism*, Nonlinearity, Vol. 23(11) 2010.

[5] Yves Chabrillac, J.-P. Crouzeix, *Continuity and differentiability properties of monotone real functions of several real variables*, Nonlinear Analysis and Optimization, Mathematical Programming Studies (1987) 30: 1-16.

[6] K. J. Falconer, *Fractal Geometry,* John Wiley & Sons, (1990).

[7] K. J. Falconer, *Techniques in Fractal Geometry,* Wiley, New York (1997).

[8] S. Jaffard, *On the Frisch-Parisi conjecture*, J. Math. Pures Appl. 79(6) 525–552, 2000.

[9] L. Olsen, *Fractal and multifractal dimensions of prevalent measures.* Indiana Univ. Math. J. 59 (2010), no. 2, 661690.

[10] L. Olsen, *Prevalent* L^q*-dimensions of measures.* Math. Proc. Cambridge Philos. Soc. 149 (2010), no. 3, 553571.

[11] W. H. Young, G.C. Young, *On the Discontinuties of Monotone Functions of Several Variables*, Proc. London Math. Soc. (1924) 2-22(1): 124–142.

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