The (L^p, L^q) bilinear Hardy-Littlewood function for the tail

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Abstract

Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving dynamical system on a finite measure space. Consider the maximal function $R^* : (f, g) \in L^p \times L^q \to R^*(f, g)(x) = \sup_n \dfrac{f(T^n x)g(T^{2n}x)}{n}$. We prove that if p and q are greater or equal than one and $\frac{1}{p} + \frac{1}{q} < 2$ then R^* maps $L^p \times L^q$ into any L^r as long as $0 < r < 1/2$. This implies that $R^*(f, g)$ is finite almost everywhere and $\frac{f(T^n x)g(T^{2n}x)}{n} \to 0$ for a.e. x as $n \to \infty$.

1 Introduction

It is a well known fact that the Hardy–Littlewood maximal function

$$H^* : f \in L^1 \to H^* f(x) = \sup_t \frac{1}{2t} \int_{-t}^{t} f(x+u)du$$

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maps L^1 functions into weak L^1. In other words H^* satisfies a weak type $(1,1)$ inequality. The bilinear Hardy–Littlewood maximal function was introduced by Alberto Calderón in the 1960's. It is defined for f, g measurable functions as

$$M^*(f,g)(x) = \sup_t \frac{1}{2t} \int_{-t}^{t} f(x+s)g(x+2s)ds.$$

A simple application of Hölder's inequality combined with the weak type $(1,1)$ property of H^* shows that $M^*(f,g)$ is almost everywhere finite if $f \in L^p$, $g \in L^q$ and $\frac{1}{p}+\frac{1}{q} \leq 1$. Calderón made a famous conjecture by stating that M^* is integrable as soon as f and g are in L^2. In [6], M. Lacey built his work with C. Thiele [7] about the celebrated Carleson–Hunt theorem on the almost everywhere convergence of Fourier series to solve Calderón's conjecture. He showed that M^* maps actually $L^p \times L^q$ into L^r as long as $p, q \geq 1$, $\frac{1}{p}+\frac{1}{q} = \frac{1}{r}$ and $r > 2/3$. The intriguing aspect of this deep result is the bound $2/3$. What could happen if $3/2 < \frac{1}{p} + \frac{1}{q} \leq 2$? Unfortunately, Lacey's method fails for $r \leq 2/3$ as he indicated in his paper. One can understand why the bilinear maximal function is more difficult to control when one looks at the trilinear Hardy–Littlewood maximal function

$$R^*(f,g,h)(x) = \sup_t \frac{1}{2t} \int_{-t}^{t} f(x+s)g(x+2s)h(x+3s)ds.$$

The dependence of the monomials $x+s$, $x+2s$, and $x+3s$ allows to obtain in a relatively simple way negative results on the range of the functions f, g and h (see [3]). Positive results have been obtained by Demeter, Thiele and Tao in [4] beyond the usual bounds given by Hölder's inequality by extending Lacey's method. However, in the case of the bilinear Hardy–Littlewood M^*, primarily because of the independence of the monomials $x+s$ and $x+2s$ there was no negative result known close to L^1. Our purpose in this paper is to bring some new ideas and results to these problems by using the ergodic setting.

One considers an ergodic measure preserving transformation on a non-atomic probability measure space and one looks at the maximal function

$$\mathcal{M}(f,g)(x) = \sup_N \frac{1}{2N+1} \sum_{n=-N}^{N} f(T^n x) g(T^{2n} x)$$

for functions $f \in L^p$ and $g \in L^q$. The equivalent problem in this setting is to find the range of values $p, q \geq 1$ for which $\mathcal{M}(f,g)(x) < \infty$ a.e. A transference argument can bring us back to the bilinear maximal function. In Ergodic Theory a good indicator of the behavior of the averages is generally given by the tail of the averages, i.e. the term $\frac{f(T^N x)g(T^{2N}x)}{2N+1}$, when one averages along all natural numbers. For instance, the maximal function associated with the tail of the ergodic averages $\sup_n \frac{f(T^n x)}{n}$, satisfies similar weak type inequalities as the maximal function for the ergodic averages. There is also another obvious reason to consider $\sup_n \frac{f(T^n x)g(T^{2n}x)}{n}$ since we have

$$\mathcal{M}(f,g)(x) \geq \sup_N \frac{f(T^N x)g(T^{2N}x)}{2N+1}.$$

It is normal then to try to find out first what happens to the maximal function

$$R^*(f,g)(x) = \sup_n \frac{f(T^n x)g(T^{2n}x)}{n}.$$

The main result of this paper is the following theorem.

Theorem 1. *Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving transformation on a probability measure space. Then, for all $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} < 2$, R^* maps $L^p \times L^q$ into L^r as soon as $0 < r < 1/2$.*

In [2] we show that there are functions $(f,g) \in L^1 \times L^1$ for which $R^*(f,g)$ is not finite almost everywhere. Since Theorem 1 implies that $R^*(f,g)$ is finite a.e. for $(f,g) \in L^p \times L^q$ when $p, q \geq 1$ and $\frac{1}{p} + \frac{1}{q} < 2$ we have a complete characterization of the range of values (p,q) for which $R^*(f,g)$ is finite a.e.

Theorem 1 is an immediate consequence of the following maximal inequality whose proof is given in Section 3:

Theorem 2. *Given $p > 1$ there exists a universal finite constant C_p^* such that if $(X, \mathcal{B}, \mu, T)$ is any invertible dynamical system on a probability measure space $(X, \mathcal{B}, \mu)$ then the following holds. For every function $f \in L^p$, for every $g \in L^1$, and for each $s > 0$ we have*

$$\mu\Big\{x : \sup_{0<l} \frac{f(T^l x)g(T^{2l}x)}{l} \geq s\Big\} \leq C_p^* \sqrt{\frac{||f||_p ||g||_1}{s}}. \tag{1}$$

Therefore, for such functions f and g we have

$$\frac{f(T^l x)g(T^{2l}x)}{l} \to 0 \text{ as } l \to \infty. \tag{2}$$

Furthermore, for $1 < p < 2$ there exists a universal constant C such that $C_p^* \leq \dfrac{C}{p-1}$.

A similar maximal inequality can be obtained if one considers instead $f \in L^1$ and $g \in L^p$.

Theorem 2 implies easily Theorem 1. Because of the finiteness of the measure μ, we have $\int |R^*(f,g)|^r d\mu \sim \sum_{k=1}^{\infty} \mu\{x : |R^*(f,g)(x)| > k^{1/r}\}$ which is finite by (1) as soon as $1/2r > 1$, or equivalently if $0 < r < 1/2$. □

Theorem 2 shows that for the maximal function R^* one can go beyond $2/3$ and up to $1/2$. In fact, R^* will map functions in $L^p \times L^q$ into any of the L^r spaces as long as $1 \leq \frac{1}{p} + \frac{1}{q} < 2$ and $0 < r < 1/2$. For $1 \leq \frac{1}{p} + \frac{1}{q} < 3/2$ it does not recover the full strength of Lacey's result as r is just between 0 and $1/2$. But it provides with a different approach the finiteness of R^* for all cases of p and q including those not covered by Lacey's result.

Theorem 2 follows by transference from the following result on the integers.

Lemma 3. *Given $p > 1$ there exists a universal constant C_p such that for each $s > 0$, for every $L \in \mathbb{N}$ and for all $K \in \mathbb{N}$, $K > 6L$ we have*

$$\#\Big\{ j \in \mathbb{N} : 2L \leq j \leq K-1-4L, \sup_{0<l\leq L} \frac{a_{l+j} b_{2l+j}}{l} \geq s \Big\} \leq C_p \sqrt{\frac{\|a\|_p^p \|b\|_1}{s}} \tag{3}$$

for all $a = (a_i) \in l_p$ and $b = (b_i) \in l_1$ satisfying

$$1 \leq |a_i|, |b_i| < \sqrt{sL/8} \text{ for all } i = 0, ..., K-1. \tag{4}$$

Furthermore, for $1 < p < 2$ there exists a universal constant C' such that

$$C_p \leq \frac{C'}{p-1}. \tag{5}$$

Remark 1. In a previous version of this paper the maximal inequality in Theorem 2 was established for functions f and g with the additional assumptions $|f| > 1$, and $|g| > 1$. We eliminated these restrictions in [1]. Our current treatment of this question is a mixture of the approach suggested by the referee of this paper and the one presented in [1].

We explain at the end of the paper (see Remarks 2 and 3) why the maximal inequality in Theorem 2 does not extend to measure preserving systems on σ-finite measure spaces.

2 Proof of Lemma 3

In this section we prove Lemma 3. The proof is divided into several steps.

2.1 Reduction to sequences taking finite values being powers of 2

It is sufficient to consider only non-negative sequences a and b supported on the interval $[0, K-1]$. This means that $a_i = b_i = 0$ when i does not belong to the interval $[0, K-1]$. In this first step we reduce the proof of Lemma 3 to sequences taking finite values in $\{2^j : j \in \mathbb{N}\}$. More precisely, we will assume first that $a_i \in \{2^n : n \in \mathbb{N}\}$ and $b_i \in \{2^k : k \in \mathbb{N}\}$. Moreover, we also make the assumption that $s = 2^{\gamma'}$ with a $\gamma' \in \mathbb{Z}$.

Given $L \in \mathbb{N}$ and $K > 6L$ we will show that for the above specific sequences

$$\#\Big\{j \in \mathbb{N} : \ 2L \le j \le K-1-4L; \sup_{0<l\le L} \frac{a_{l+j}b_{2l+j}}{l} \ge s\Big\} \le \tag{6}$$

$$\frac{C_p}{2^{(p+2)/2}}\sqrt{\frac{\|a\|_p^p\|b\|_1}{s}}$$

holds with C_p independent of L and s provided that a and b satisfy (9), which is a technical consequence of (4).

We can observe that if (6) holds for sequences a and b taking values which are powers of 2 then (3) will also hold for sequences with finite support taking values greater than one. Indeed, if $a_i \ge 1$ and $2^{n_i} \le a_i < 2^{n_i+1}$ for some $n_i \in \mathbb{N} \cup \{0\}$ then the sequence $\overline{a}$ defined at i as 2^{n_i+1} is taking values which are powers of 2 and $a_i < \overline{a}_i \le 2a_i$. Hence $\|\overline{a}\|_p^p \le 2^p\|a\|_p^p$ and with similar consideration we would have $\|\overline{b}\|_1 \leq 2\|b\|_1$. We also choose $\gamma' \in \mathbb{Z}$ such that $2^{\gamma'+1} > s \geq 2^{\gamma'}$ and set $\overline{s} = 2^{\gamma'}$.

It is also clear that

$$\#\Big\{j \in \mathbb{N} : 2L \leq j \leq K-1-4L, \sup_{0<l\leq L} \frac{a_{l+j}b_{2l+j}}{l} \geq s\Big\} \leq \tag{7}$$

$$\#\Big\{j \in \mathbb{N} : 2L \leq j \leq K-1-4L, \sup_{0<l\leq L} \frac{\overline{a}_{l+j}\overline{b}_{2l+j}}{l} \geq \overline{s}\Big\} \leq$$

(using (6))

$$\frac{C_p}{2^{(p+2)/2}}\sqrt{\frac{\|\overline{a}\|_p^p\|\overline{b}\|_1}{\overline{s}}} \leq C_p\sqrt{\frac{\|a\|_p^p\|b\|_1}{s}}. \tag{8}$$

For ease of notation in the sequel we denote by a and b the modified sequences $\overline{a}$ and $\overline{b}$ consisting of powers of 2 or zero. Similarly we use $s = 2^{\gamma'}$ instead of $\overline{s}$.

After these adjustments we can suppose based on (4) that (for the modified a, b and s) we have

$$1 < a_i, b_i < N_L = \sqrt{sL}. \tag{9}$$

Recall that we also assumed that

$$6L < K. \tag{10}$$

Suppose $i \in \{0, ..., K-1-2L\}$ and there exists l such that

$$\frac{a_{l+i}b_{2l+i}}{l} \geq s. \tag{11}$$

Then by (9)

$$\frac{sL}{l} \geq \frac{N_L^2}{l} \geq \frac{a_{l+i}b_{2l+i}}{l} \geq s,$$

that is, $L \geq l$. Therefore, $l+i$, $2l+i \in \{0, ..., K-1\}$.

Choose $(a_{n,i})$ and $(b_{k,i})$ such that $a_i = \sum_{n\leq N_L} a_{n,i}$, $b_i = \sum_{k\leq N_L} b_{k,i}$, $a_{n,i} \in \{0, 2^n\}$, and $b_{k,i} \in \{0, 2^k\}$. Set

$$I_{n,k} = \Big\{i \in \{2L, ..., K-1-4L\} : \sup_{0<l\leq L} \frac{a_{n,l+i}b_{k,2l+i}}{l} \geq s\Big\}. \tag{12}$$

If

$$\frac{a_{n,l+i}b_{k,2l+i}}{l} \geq s > 0 \tag{13}$$

then $a_{n,l+i} = 2^n$ and $b_{k,2l+i} = 2^k$. Therefore,

$$\frac{2^{n+k}}{s} \geq l, \text{ that is, } 2^{n+k-\gamma'} \geq l \geq 1 > 0, \tag{14}$$

which implies $n + k - \gamma' \geq 0$. Moreover, for these $a_{n,l+i}$ and $b_{k,2l+i}$ from (9) it also follows that

$$L > \frac{a_{n,l+i}b_{k,2l+i}}{s} = 2^{n+k-\gamma'}. \tag{15}$$

This "length", $2^{n+k-\gamma'}$ will serve as a natural "unit" in our construction.

Set for $t \in \mathbb{N}$

$$\begin{aligned} I_{n,k,t} &= I_{n,k} \cap \{t2^{n+k-\gamma'}, ..., (t+1)2^{n+k-\gamma'} - 1\}, \text{ and} \\ A_{n,k,t} &= \{t2^{n+k-\gamma'}, ..., (t+3)2^{n+k-\gamma'} - 1\}. \end{aligned} \tag{16}$$

By the introduction of the sequences $a_{n,i}$ and $b_{k,i}$ we want to split the ranges of a and b into some standardized "subsequences" and then use the sets $I_{n,k,t}$ and a counting argument in (17-18) to obtain an upper estimate of the number of those indices where (13) holds. The "t-range" of a block $I_{n,k,t}$ is greater or equal than l, see (14).

Observe that if $i \in I_{n,k,t}$ then there exists $l \leq 2^{n+k-\gamma'}$ such that $a_{n,l+i} = 2^n$, $b_{k,2l+i} = 2^k$ and $l + i$, $2l + i \in A_{n,k,t}$. We want to give an upper estimate of $\#I_{n,k,t}$. If $a_{n,i} = 2^n$ for some i and $b_{k,j} = 2^k$ for some j with $i, j \in \{t2^{n+k-\gamma'}, ..., (t+3)2^{n+k-\gamma'} - 1\}$ then there can be at most one $i' \in \{t2^{n+k-\gamma'}, ..., (t+3)2^{n+k-\gamma'} - 1\}$ such that for i' there exists l such that $i'+l = i$ and $2l+i' = j$. Indeed, in this case $l = j-i$ and $i' = i-(j-i) = 2i-j$.

2.2 Finding an upper bound of the cardinality of the sets $I_{n,k,t}$, the counting argument

Denote by $\mathcal{N}(n, k, t, a)$ the number of those $i \in \{t2^{n+k-\gamma'}, ..., (t+3)2^{n+k-\gamma'} - 1\} \cap \mathbb{Z}$ for which $a_{n,i} = 2^n$. Similarly, $\mathcal{N}(n, k, t, b)$ denotes the number of those $j \in \{t2^{n+k-\gamma'}, ..., (t+3)2^{n+k-\gamma'} - 1\} \cap \mathbb{Z}$ for which $b_{k,j} = 2^k$. Now,

$$\#I_{n,k,t} \leq \mathcal{N}(n, k, t, a)\mathcal{N}(n, k, t, b). \tag{17}$$

On the other hand,

$$\mathcal{N}(n,k,t,a) = \frac{\sum_{i\in A_{n,k,t}} a^p_{n,i}}{2^{np}}, \text{ and} \tag{18}$$
$$\mathcal{N}(n,k,t,b) = \frac{\sum_{i\in A_{n,k,t}} b_{k,i}}{2^{k}}.$$

Therefore, by (17)

$$\# I_{n,k,t} \le \frac{1}{2^{np+k}} \sum_{i\in A_{n,k,t}} a^p_{n,i} \cdot \sum_{i\in A_{n,k,t}} b_{k,i}. \tag{19}$$

2.3 Refining the sets $A_{n,k,t}$ with disjoint subsets $A'_{n,k,t}$

Set

$$A'_{n,k,t} = \{t2^{n+k-\gamma'}, ..., (t+1)2^{n+k-\gamma'} - 1\}. \tag{20}$$

While for n,k fixed the sets $A_{n,k,t}$ overlap, the sets $A'_{n,k,t}$ are disjoint and

$$A_{n,k,t} = A'_{n,k,t} \cup (A'_{n,k,t} + 2^{n+k-\gamma'}) \cup (A'_{n,k,t} + 2\cdot 2^{n+k-\gamma'}).$$

Set

$$T(n,k) = \left\{2,3,..., \left\lfloor \frac{K-1-3L}{2^{n+k-\gamma'}} \right\rfloor \right\}, \tag{21}$$

and $T'(n,k)$ will consist of those t' for which $A'_{n,k,t'} \subset \cup_{t\in T(n,k)} A_{n,k,t}$. It is useful to keep in mind that by (14) and (15) we have $L > 2^{n+k-\gamma'} \ge 1$ and

$$\{2L, 2L+1, ..., K-1-4L\} \subset \bigcup_{t\in T(n,k)} A_{n,k,t} \subset \{0,...,K-1\}.$$

2.4 Separation of two estimate cases, the set I^{**}

The constant $\widetilde{C}_{a,b}$ will be specified later in (34).

Denote by I^{***}_n the set of those i for which there exists a k and $t \in T'(n,k)$ such that $i \in A'_{n,k,t}$ and

$$\sum_{i'\in A'_{n,k,t}} a^p_{n,i'} > (\# A'_{n,k,t}) \widetilde{C}_{a,b}. \tag{22}$$

Due to dyadic grid properties, for fixed n and k the sets $A'_{n,k,t}$ are disjoint for different t's, while - still keeping n fixed - if for different k's such sets intersect then one contains the other. Using this property one can choose a disjoint system of maximal intervals $\{A'_{n,k_j,t_j}\}$ such that $I_n^{***} = \cup_j A'_{n,k_j,t_j}$. The intervals A'_{n,k_j,t_j} are maximal in the sense that for each j if $A'_{n,k,t} \supsetneq A'_{n,k_j,t_j}$ for a $t \in T'(n,k)$ then (22) does not hold.

Now, (22) implies

$$\sum_{i=0}^{K-1} a_{n,i}^p \geq \sum_{i \in I_n^{***}} a_{n,i}^p > (\# I_n^{***}) \widetilde{C}_{a,b}. \tag{23}$$

If A'_{n,k_j,t_j} is one of the intervals considered above then let

$$B'_{n,k_j,t_j} \stackrel{\text{def}}{=} A'_{n,k_j,t_j} \cup (A'_{n,k_j,t_j} - 2^{n+k_j-\gamma'}) \cup (A'_{n,k_j,t_j} - 2 \cdot 2^{n+k_j-\gamma'}).$$

We put

$$I_n^{**} = \bigcup_j B'_{n,k_j,t_j}.$$

From (23) it follows that

$$3 \sum_{i=0}^{K-1} a_{n,i}^p > \#(I_n^{**}) \widetilde{C}_{a,b}. \tag{24}$$

Set $I^{**} = \cup_n I_n^{**}$. Then adding (24) for n's we obtain

$$3 \sum_{i=0}^{K-1} a_i^p > (\# I^{**}) \widetilde{C}_{a,b}. \tag{25}$$

2.5 The estimate for the set I^*

If $A'_{n,k,t+j'} \subset I_n^{***}$ for a $j' = 0,1,2$, then using the definition of the sets B'_{n,k_j,t_j} and the maximality of the sets A'_{n,k_j,t_j} one would obtain $A'_{n,k,t} \subset I_n^{**}$.

Hence, if $t \in T'(n,k)$, $A'_{n,k,t} \not\subset I_n^{**}$ then $A'_{n,k,t+j'} \not\subset I_n^{***}$ holds for $j' = 0,1,2$. This means by (22) that

$$\begin{gathered} \text{if } A'_{n,k,t} \not\subset I_n^{**} \text{ and } A'_{n,k,t'} \subset A_{n,k,t} \text{ then} \\ \sum_{i \in A'_{n,k,t'}} a_{n,i}^p \leq (\# A'_{n,k,t'}) \widetilde{C}_{a,b}. \end{gathered} \tag{26}$$

From (26) it follows that

$$\text{if } t \in T'(n,k),\ A'_{n,k,t} \not\subset I^{**} = \cup_{n'} I^{**}_{n'} \text{ and } A'_{n,k,t'} \subset A_{n,k,t} \text{ then} \quad (27)$$
$$\sum_{i \in A'_{n,k,t'}} a^p_{n,i} \le (\# A'_{n,k,t'}) \widetilde{C}_{a,b}.$$

Denote by $T^{**}(n,k)$ the set of those $t \in T'(n,k)$ for which $A'_{n,k,t} \not\subset I^{**}$. Set $I^{**}_{n,k} = I_{n,k} \setminus I^{**}$. Clearly,

$$I^{**}_{n,k} \subset \bigcup_{t \in T^{**}(n,k)} I_{n,k,t} \subset \bigcup_{t \in T^{**}(n,k)} A'_{n,k,t}. \quad (28)$$

Denote by $T''(n,k)$ the set of those $t' \in T'(n,k)$ for which there exists $t \in T^{**}(n,k)$ satisfying $A'_{n,k,t'} \subset A_{n,k,t}$. For $t' \in T''(n,k)$ one can apply (26) and (27).

Set

$$C_{n,k,t} = A_{n,k,t} \cup (A_{n,k,t} - 2^{n+k-\gamma'}) \cup (A_{n,k,t} - 2 \cdot 2^{n+k-\gamma'}).$$

By (19) and (28) we have

$$\# I^{**}_{n,k} \le \sum_{t \in T^{**}(n,k)} \# I_{n,k,t} \le \quad (29)$$

$$\sum_{t \in T^{**}(n,k)} \frac{1}{2^{np+k}} \sum_{i \in A_{n,k,t}} a^p_{n,i} \sum_{i \in A_{n,k,t}} b_{k,i} \le$$

$$\sum_{t \in T''(n,k)} \frac{3}{2^{np+k}} \sum_{i \in A'_{n,k,t}} a^p_{n,i} \sum_{i \in C_{n,k,t}} b_{k,i}. \quad (30)$$

Recall that $\#(A'_{n,k,t}) = 2^{n+k-\gamma'} = 2^{n+k}/s$ when $t \in T'(n,k)$.

By (27), (29), and (30) we have

$$\# I^{**}_{n,k} \le \sum_{t \in T''(n,k)} \frac{3}{s 2^{n(p-1)}} \frac{\sum_{i \in A'_{n,k,t}} a^p_{n,i}}{\# A'_{n,k,t}} \sum_{i \in C_{n,k,t}} b_{k,i} \le \quad (31)$$

$$\sum_{t \in T''(n,k)} \frac{3}{s 2^{n(p-1)}} \widetilde{C}_{a,b} \sum_{i \in C_{n,k,t}} b_{k,i} \le \frac{27}{s 2^{n(p-1)}} \widetilde{C}_{a,b} \sum_{i=0}^{K-1} b_{k,i}.$$

Set

$$I^* = \bigcup_{n,k} I_{n,k} \setminus I^{**} = \bigcup_{n,k} I^{**}_{n,k}.$$

By (31) we have

$$\# I^* \leq \sum_n \frac{27}{s 2^{n(p-1)}} \widetilde{C}_{a,b} \sum_k \sum_{i=0}^{K-1} b_{k,i} \leq \frac{27}{s(1-2^{-(p-1)})} \widetilde{C}_{a,b} \sum_{i=0}^{K-1} b_i. \tag{32}$$

2.6 Conclusion of the proof of Lemma 3

Now, by using (25) and (32) we have

$$\#\Big(\bigcup_{n,k} I_{n,k}\Big) \leq \#\Big(\bigcup_{n,k} (I_{n,k} \setminus I^{**})\Big) + \# I^{**} = \# I^* + \# I^{**} \leq \tag{33}$$

$$\frac{27}{s(1-2^{-(p-1)})} \widetilde{C}_{a,b} \sum_{i=0}^{K-1} b_i + 3 \frac{\sum_{i=0}^{K-1} a_i^p}{\widetilde{C}_{a,b}}.$$

Choose

$$\widetilde{C}_{a,b} = \sqrt{\frac{s \sum_{i=0}^{K-1} a_i^p}{\sum_{i=0}^{K-1} b_i}} = \sqrt{\frac{s \|a\|_p^p}{\|b\|_1}}. \tag{34}$$

Then we obtained

$$\#\Big(\bigcup_{n,k} I_{n,k}\Big) \leq \Big(\frac{27}{1-2^{-(p-1)}} + 3\Big) \sqrt{\frac{\|a\|_p^p \|b\|_1}{s}} = \frac{C_p}{2^{(p+2)/2}} \sqrt{\frac{\|a\|_p^p \|b\|_1}{s}}. \tag{35}$$

Using (6), (7-8) and (12) we can infer (3). By elementary calculus from (35) one can also deduce the existence of a constant C' for which (5) holds when $1 < p < 2$. □

3 Proof of Theorem 2 by a transference argument

The transference argument is first applied for bounded functions f and g. Moreover, we also make the auxiliary assumption that g is bounded away from zero. Later we remove these extra assumptions. For f some standard duality tricks are applicable, while for g with the L^1 norm, slightly modified arguments are needed.

3.1 The transference argument

Let $(X, \mathcal{B}, \mu, T)$ be an invertible dynamical system on a probability measure space and let $f \in L^p$, $g \in L^1$. Usage of $|f|$ and $|g|$ instead of f and g increases the left handside of (1) without changing its right handside. Therefore, in the sequel we assume that f and g are nonnegative.

First we suppose that there exists $M > 1$ such that $0 \leq f, g \leq M$ and $g \geq 1/M$ everywhere. If $||f||_p = 0$ then $f = 0$ a. e. and we have nothing to prove.

Put

$$F = \frac{f}{||f||_p} + 1 \text{ and } G = M \cdot g. \tag{36}$$

Set

$$M^* = \max\Big\{\frac{M}{||f||_p} + 1, M^2\Big\}.$$

Then $1 \leq F, G \leq M^*$ holds everywhere. Given $s' > 0$ we choose L so large that $M^* < \sqrt{s'L/8}$ and choose $K > 6L$. In (2) we have $s > 0$ given, but we will see later that we need to start with a suitable s' which will equal $sM/||f||_p$ in the end. For any given $x \in X$ consider the sequences $a_i = F(T^i x)$, $b_i = G(T^i x)$, for $i = 0, ..., K-1$ and $a_i = b_i = 0$ for $i \in \mathbb{Z} \setminus \{0, ..., K-1\}$. Since (4) is satisfied we can apply Lemma 3 to obtain the inequality

$$\#\Big\{j \in \mathbb{N} : 2L \leq j \leq K - 1 - 4L, \sup_{0<l\leq L} \frac{F(T^{l+j}x)G(T^{2l+j}x)}{l} \geq s'\Big\} \leq$$

$$C_p \sqrt{\frac{\big(\sum_{i=0}^{K-1} |F(T^i x)|^p\big) \sum_{i=0}^{K-1} |G(T^i x)|}{s'}}.$$

By integrating this inequality with respect to μ and using that T^j is measure preserving and hence the integral of the left handside takes the same value for all j's we infer

$$(K - 6L)\mu\Big\{x : \sup_{0<l\leq L} \frac{F(T^l x)G(T^{2l}x)}{l} \geq s'\Big\} \leq$$

$$C_p \int_X \sqrt{\frac{\big(\sum_{i=0}^{K-1} |F(T^i x)|^p\big) \sum_{i=0}^{K-1} |G(T^i x)|}{s'}} d\mu.$$

Dividing by $K-6L$ one obtains

$$\mu\Big\{x : \sup_{0<l\leq L} \frac{F(T^l x)G(T^{2l}x)}{l} \geq s'\Big\} \leq \tag{37}$$

$$C_p \int_X \sqrt{\frac{\big(\sum_{i=0}^{K-1}|F(T^i x)|^p\big)\sum_{i=0}^{K-1}|G(T^i x)|}{(K-6L)s'(K-6L)}}d\mu.$$

Now we can notice that by the pointwise ergodic theorem if we denote by $\mathcal{I}$ the σ-field of T-invariant sets then the averages $\frac{1}{K-6L}\sum_{i=0}^{K-1}|G(T^i x)|$ converge a.e. to $\mathbb{E}[|G|, \mathcal{I}](x)$, the conditional expectation of $|G|$ with respect to $\mathcal{I}$. Furthermore, the averages $\frac{1}{K-6L}\sum_{i=0}^{K-1}|F(T^i x)|^p$ converge a.e. to $\mathbb{E}[|F|^p, \mathcal{I}](x)$ as $K \to \infty$. Since F and G are bounded Lebesgue's dominated convergence theorem applies and we have obtained the inequality

$$\mu\Big\{x : \sup_{0<l\leq L} \frac{F(T^l x)G(T^{2l}x)}{l} \geq s'\Big\} \leq C_p \int_X \sqrt{\frac{\mathbb{E}[|F|^p, \mathcal{I}](x)\mathbb{E}[|G|, \mathcal{I}](x)}{s'}}d\mu. \tag{38}$$

By using Hölder's inequality the right handside of (38) is bounded above by

$$C_p\sqrt{\frac{1}{s'}}\sqrt{\int_X \mathbb{E}[|F|^p, \mathcal{I}](x)d\mu}\sqrt{\int_X \mathbb{E}[|G|, \mathcal{I}](x)d\mu}.$$

Using the integral preserving property of the conditional expectation this last term equals

$$C_p\sqrt{\frac{1}{s'}}\sqrt{\int_X |F|^p d\mu}\sqrt{\int_X |G|d\mu} = C_p\sqrt{\frac{\|F\|_p^p\|G\|_1}{s'}}.$$

We have reached then the following inequality

$$\mu\Big\{x : \sup_{0<l\leq L} \frac{F(T^l x)G(T^{2l}x)}{l} \geq s'\Big\} \leq C_p\sqrt{\frac{\|F\|_p^p\|G\|_1}{s'}}. \tag{39}$$

Next we see the consequences of (39) for our original functions f and g. Since C_p does not depend on L, first one can let $L \to \infty$. Recall that by Hölder's inequality if $\frac{1}{p} + \frac{1}{q} = 1$ then $(\alpha + \beta) \leq 2^{1/q}(\alpha^p + \beta^p)^{1/p}$ and this yields

$$||F||_p^p = \int_X \Big(\frac{f}{||f||_p} + 1\Big)^p d\mu \leq 2^{p/q}\int_X \Big(\frac{f^p}{||f||_p^p} + 1\Big)d\mu = 2^{\frac{p}{q}+1}. \tag{40}$$

Rewriting (39) and letting $L \to \infty$ we obtain

$$\mu\Big\{x : \sup_{0<l} \frac{f(T^l x)g(T^{2l}x)\cdot M}{l\cdot ||f||_p} \geq s'\Big\} \leq \tag{41}$$

$$\mu\Big\{x : \sup_{0<l} \frac{F(T^l x)G(T^{2l}x)}{l} \geq s'\Big\} \leq$$

(using (40))

$$C_p\sqrt{\frac{||F||_p^p||G||_1}{s'}} \leq C_p\sqrt{\frac{2^{\frac{p}{q}+1}||g||_1\cdot M}{s'}}. \tag{42}$$

Choosing $s' = \frac{s\cdot M}{||f||_p}$ we obtain from (41-42)

$$\mu\Big\{x : \sup_{0<l} \frac{f(T^l x)g(T^{2l}x)}{l} \geq s\Big\} \leq C_p\sqrt{\frac{2^{\frac{p}{q}+1}||f||_p||g||_1}{s}} =$$

$$C_p 2^{\frac{p}{2q}+\frac{1}{2}}\sqrt{\frac{||f||_p||g||_1}{s}}.$$

Hence we can choose $C_p^* = C_p 2^{\frac{p}{2q}+\frac{1}{2}}$. This constant does not depend on M. The general case for arbitrary nonnegative functions $f \in L^p$ and $g \in L^1$ can be obtained by approximating f by functions of the form $f_M = \min\{f, M\}$ and g by $\min\{\max\{g, \frac{1}{M}\}, M\}$.

3.2 Second part of Theorem 2

It remains to prove that

$$\frac{f(T^l x)g(T^{2l}x)}{l} \to 0 \text{ as } l \to \infty. \tag{43}$$

This follows by approximation. We give the details for sake of completeness. Since (43) is equivalent to $\dfrac{|f(T^l x)|\cdot|g(T^{2l}x)|}{l} \to 0$ we can assume that $f, g \geq 0$. Set $f_M = \min\{f, M\}$, $g_M = \min\{g, M\}$. For all $\epsilon > 0$ choose M such that

$\int_X (f-f_M)^p d\mu < \epsilon$, $\int_X (g-g_M) d\mu < \epsilon$. Then

$$\left|\frac{f(T^l x)g(T^{2l}x)}{l}\right| \leq \left|\frac{(f-f_M)(T^l x)(g-g_M)(T^{2l}x)}{l}\right| +$$
$$\left|\frac{f_M(T^l x)g(T^{2l}x)}{l}\right| + \left|\frac{f(T^l x)g_M(T^{2l}x)}{l}\right| + \left|\frac{f_M(T^l x)g_M(T^{2l}x)}{l}\right| \leq$$
$$\left|\frac{(f-f_M)(T^l x)(g-g_M)(T^{2l}x)}{l}\right| + M\left|\frac{g(T^{2l}x)}{l}\right| + M\left|\frac{f(T^{2l}x)}{l}\right| + \frac{M^2}{l}.$$

The last three terms converge to zero almost everywhere as $l \to \infty$. By (1) the measure of the set of those x's where the limit superior of the first term is larger than $s = \epsilon$ can be estimated from above by $C_p^* \sqrt{\epsilon^2/\epsilon} = C_p^* \sqrt{\epsilon}$. Since $\epsilon > 0$ is arbitrary we obtain (43). $\square$

Remark 2. It is natural to ask whether $||a||_p^p$ can be replaced by $||a||_p$ in (3). The answer is no since this would imply the following version of (37)

$$\mu\Big\{x : \sup_{0<l\leq L} \frac{F(T^l x)G(T^{2l}x)}{l} \geq s'\Big\} \leq \tag{44}$$

$$C_p \int_X \sqrt{\frac{\left(\sum_{i=0}^{K-1} |F(T^i x)|^p\right)^{1/p} \sum_{i=0}^{K-1} |G(T^i x)|}{(K-6L)s'(K-6L)}} d\mu.$$

By the pointwise ergodic theorem the averages $\frac{1}{K-6L}\sum_{i=0}^{K-1} |G(T^i x)|$ converge a.e. to $\mathbb{E}[|G|, \mathcal{I}](x)$ as $K \to \infty$. Furthermore, the averages

$$\frac{1}{K-6L}\Big(\sum_{i=0}^{K-1} |F(T^i x)|^p\Big)^{1/p} = \frac{1}{(K-6L)^{1-\frac{1}{p}}}\Big(\frac{1}{K-6L}\sum_{i=0}^{K-1} |F(T^i x)|^p\Big)^{1/p}$$

converge a.e. to $0 \cdot (\mathbb{E}[|F|^p, \mathcal{I}](x))^{1/p} = 0$. Since F and G are bounded by Lebesgue's dominated convergence theorem we would obtain that the right hand side of (44) converges to zero as $K \to \infty$ which is impossible for all possible choices of F, G and s'.

Remark 3. One of the reasons why (1) does not hold on $\mathbb{Z}$ is because it is not homogeneous with respect to μ. By this we mean that if one divides the measure μ by N then the right hand side of (1) is not divided by N but by a power of N, namely $N^{1/2p+1/2}$.

Theorem 2 holds for finite measure spaces as this can be derived from probability measure spaces by simple computations. The constant C_p depends then also on the total mass of the space X. However the failure of (1) on $\mathbb{Z}$ indicates that Theorem 2 does not hold in general for measure preserving systems on σ-finite measure spaces.

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