The \((L^p, L^q)\) bilinear Hardy-Littlewood function for the tail

Idris Assani\(^*\) and Zoltán Buczolich\(^†\)

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Abstract

Let \((X, \mathcal{B}, \mu, T)\) be a measure preserving dynamical system on a finite measure space. Consider the maximal function \(R^* : (f, g) \in L^p \times L^q \rightarrow R^*(f, g)(x) = \sup_n \frac{f(T^n x)g(T^{2n} x)}{n}\). We prove that if \(p\) and \(q\) are greater or equal than one and \(\frac{1}{p} + \frac{1}{q} < 2\) then \(R^*\) maps \(L^p \times L^q\) into any \(L^r\) as long as \(0 < r < 1/2\). This implies that \(R^*(f, g)\) is finite almost everywhere and \(\frac{f(T^n x)g(T^{2n} x)}{n} \rightarrow 0\) for a.e. \(x\) as \(n \rightarrow \infty\).

1 Introduction

It is a well known fact that the Hardy–Littlewood maximal function

\[H^* : f \in L^1 \rightarrow H^*(f)(x) = \sup_{t} \frac{1}{2t} \int_{-t}^{t} f(x + u)du\]

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maps $L^1$ functions into weak $L^1$. In other words $H^*$ satisfies a weak type $(1, 1)$ inequality. The bilinear Hardy–Littlewood maximal function was introduced by Alberto Calderón in the 1960’s. It is defined for $f, g$ measurable functions as

$$M^*(f, g)(x) = \sup_t \frac{1}{2t} \int_{-t}^t f(x+s)g(x+2s)ds.$$ 

A simple application of Hölder’s inequality combined with the weak type $(1, 1)$ property of $H^*$ shows that $M^*(f, g)$ is almost everywhere finite if $f \in L^p$, $g \in L^q$ and $\frac{1}{p} + \frac{1}{q} \leq 1$. Calderón made a famous conjecture by stating that $M^*$ is integrable as soon as $f$ and $g$ are in $L^2$. In [6], M. Lacey built his work with C. Thiele [7] about the celebrated Carleson–Hunt theorem on the almost everywhere convergence of Fourier series to solve Calderón’s conjecture. He showed that $M^*$ maps actually $L^p \times L^q$ into $L^r$ as long as $p, q \geq 1$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ and $r > 2/3$. The intriguing aspect of this deep result is the bound $2/3$. What could happen if $3/2 < \frac{1}{p} + \frac{1}{q} \leq 2$? Unfortunately, Lacey’s method fails for $r \leq 2/3$ as he indicated in his paper. One can understand why the bilinear maximal function is more difficult to control when one looks at the trilinear Hardy–Littlewood maximal function

$$R^*(f, g, h)(x) = \sup_t \frac{1}{2t} \int_{-t}^t f(x+s)g(x+2s)h(x+3s)ds.$$ 

The dependence of the monomials $x+s$, $x+2s$, and $x+3s$ allows to obtain in a relatively simple way negative results on the range of the functions $f, g$ and $h$ (see [3]). Positive results have been obtained by Demeter, Thiele and Tao in [4] beyond the usual bounds given by Hölder’s inequality by extending Lacey’s method. However, in the case of the bilinear Hardy–Littlewood $M^*$, primarily because of the independence of the monomials $x+s$ and $x+2s$ there was no negative result known close to $L^1$. Our purpose in this paper is to bring some new ideas and results to these problems by using the ergodic setting.

One considers an ergodic measure preserving transformation on a non-atomic probability measure space and one looks at the maximal function

$$\mathcal{M}(f, g)(x) = \sup_N \frac{1}{2N+1} \sum_{n=-N}^N f(T^nx)g(T^{2n}x)$$ 

for functions $f \in L^p$ and $g \in L^q$. The equivalent problem in this setting is to find the range of values $p, q \geq 1$ for which $\mathcal{M}(f, g)(x) < \infty$ a.e. A transference
argument can bring us back to the bilinear maximal function. In Ergodic
Theory a good indicator of the behavior of the averages is generally given by
the tail of the averages, i.e. the term \( \frac{f(T^n x)g(T^{2n} x)}{2N + 1} \), when one averages
along all natural numbers. For instance, the maximal function associated
with the tail of the ergodic averages \( \sup_n \frac{f(T^n x)g(T^{2n} x)}{n} \), satisfies similar weak type
inequalities as the maximal function for the ergodic averages. There is also
another obvious reason to consider \( \sup_n \frac{f(T^n x)g(T^{2n} x)}{n} \) since we have

\[
\mathcal{M}(f, g)(x) \geq \sup_N \frac{f(T^n x)g(T^{2n} x)}{2N + 1}.
\]

It is normal then to try to find out first what happens to the maximal function

\[
\mathcal{R}^*(f, g)(x) = \sup_n \frac{f(T^n x)g(T^{2n} x)}{n}.
\]

The main result of this paper is the following theorem.

**Theorem 1.** Let \((X, \mathcal{B}, \mu, T)\) be a measure preserving transformation on a
probability measure space. Then, for all \(p, q \geq 1\) such that \(\frac{1}{p} + \frac{1}{q} < 2\), \(\mathcal{R}^*\) maps
\(L^p \times L^q\) into \(L^r\) as soon as \(0 < r < 1/2\).

In [2] we show that there are functions \((f, g) \in L^1 \times L^1\) for which \(\mathcal{R}^*(f, g)\)
is not finite almost everywhere. Since Theorem 1 implies that \(\mathcal{R}^*(f, g)\) is
finite a.e. for \((f, g) \in L^p \times L^q\) when \(p, q \geq 1\) and \(\frac{1}{p} + \frac{1}{q} < 2\) we have a
complete characterization of the range of values \((p, q)\) for which \(\mathcal{R}^*(f, g)\) is
finite a.e.

Theorem 1 is an immediate consequence of the following maximal in-
equality whose proof is given in Section 3:

**Theorem 2.** Given \(p > 1\) there exists a universal finite constant \(C_p^*\) such that
if \((X, \mathcal{B}, \mu, T)\) is any invertible dynamical system on a probability measure
space \((X, \mathcal{B}, \mu)\) then the following holds. For every function \(f \in L^p\), for
every \(g \in L^1\), and for each \(s > 0\) we have

\[
\mu\left\{ x : \sup_{0 < l} \frac{f(T^l x)g(T^{2l} x)}{l} \geq s \right\} \leq C_p^* \sqrt{\frac{\|f\|_p \|g\|_1}{s}}.
\]
Therefore, for such functions $f$ and $g$ we have

$$\frac{f(T^lx)g(T^{2l}x)}{l} \to 0 \text{ as } l \to \infty. \quad (2)$$

Furthermore, for $1 < p < 2$ there exists a universal constant $C$ such that

$$C_p \leq \frac{C}{p-1}.$$

A similar maximal inequality can be obtained if one considers instead $f \in L^1$ and $g \in L^p$.

Theorem 2 implies easily Theorem 1. Because of the finiteness of the measure $\mu$, we have $\int |R^*(f,g)|^r d\mu \sim \sum_{k=1}^\infty \mu\{x : |R^*(f,g)(x)| > k^{1/r}\}$ which is finite by (1) as soon as $1/2r > 1$, or equivalently if $0 < r < 1/2$. □

Theorem 2 shows that for the maximal function $R^*$ one can go beyond $2/3$ and up to $1/2$. In fact, $R^*$ will map functions in $L^p \times L^q$ into any of the $L^r$ spaces as long as $1 \leq \frac{1}{p} + \frac{1}{q} < 2$ and $0 < r < 1/2$. For $1 \leq \frac{1}{p} + \frac{1}{q} < 3/2$ it does not recover the full strength of Lacey’s result as $r$ is just between 0 and 1/2. But it provides with a different approach the finiteness of $R^*$ for all cases of $p$ and $q$ including those not covered by Lacey’s result.

Theorem 2 follows by transference from the following result on the integers.

**Lemma 3.** Given $p > 1$ there exists a universal constant $C_p$ such that for each $s > 0$, for every $L \in \mathbb{N}$ and for all $K \in \mathbb{N}$, $K > 6L$ we have

$$\#\left\{j \in \mathbb{N} : 2L \leq j \leq K-1-4L, \sup_{0 \leq l \leq L} \frac{a_l+jb_{2l+j}}{l} \geq s\right\} \leq C_p \sqrt{\frac{\|a\|_p \|b\|_1}{s}}$$

for all $a = (a_i) \in l_p$ and $b = (b_i) \in l_1$ satisfying

$$1 \leq |a_i|, |b_i| < \sqrt{sL/8} \text{ for all } i = 0, ..., K-1. \quad (3)$$

Furthermore, for $1 < p < 2$ there exists a universal constant $C'$ such that

$$C_p \leq \frac{C'}{p-1}.$$

(4)
Remark 1. In a previous version of this paper the maximal inequality in Theorem 2 was established for functions $f$ and $g$ with the additional assumptions $|f| > 1$, and $|g| > 1$. We eliminated these restrictions in [1]. Our current treatment of this question is a mixture of the approach suggested by the referee of this paper and the one presented in [1].

We explain at the end of the paper (see Remarks 2 and 3) why the maximal inequality in Theorem 2 does not extend to measure preserving systems on $\sigma$-finite measure spaces.

2 Proof of Lemma 3

In this section we prove Lemma 3. The proof is divided into several steps.

2.1 Reduction to sequences taking finite values being powers of 2

It is sufficient to consider only non-negative sequences $a$ and $b$ supported on the interval $[0, K - 1]$. This means that $a_i = b_i = 0$ when $i$ does not belong to the interval $[0, K - 1]$. In this first step we reduce the proof of Lemma 3 to sequences taking finite values in $\{2^j : j \in \mathbb{N}\}$. More precisely, we will assume first that $a_i \in \{2^n : n \in \mathbb{N}\}$ and $b_i \in \{2^k : k \in \mathbb{N}\}$. Moreover, we also make the assumption that $s = 2^{\gamma'}$ with $\gamma' \in \mathbb{Z}$.

Given $L \in \mathbb{N}$ and $K > 6L$ we will show that for the above specific sequences

$$\# \left\{ j \in \mathbb{N} : 2L \leq j \leq K - 1 - 4L; \sup_{0 < l \leq L} \frac{a_{l+j}b_{2l+j}}{l} \geq s \right\} \leq \frac{C_p}{2^{(p+2)/2}} \left( \frac{\|a\|_p \|b\|_1}{s} \right)$$

holds with $C_p$ independent of $L$ and $s$ provided that $a$ and $b$ satisfy (9), which is a technical consequence of (4).

We can observe that if (6) holds for sequences $a$ and $b$ taking values which are powers of 2 then (3) will also hold for sequences with finite support taking values greater than one. Indeed, if $a_i \geq 1$ and $2^{m_i} \leq a_i < 2^{m_{i}+1}$ for some $m_i \in \mathbb{N} \cup \{0\}$ then the sequence $\tau$ defined at $i$ as $2^{m_i+1}$ is taking values which are powers of 2 and $a_i < \tau_i \leq 2a_i$. Hence $\|\tau\|_p \leq 2^p \|a\|_p$ and with similar
consideration we would have $\|\hat{b}\|_1 \leq 2\|b\|_1$. We also choose $\gamma' \in \mathbb{Z}$ such that $2^{\gamma'+1} > s \geq 2^{\gamma'}$ and set $\overline{s} = 2^{\gamma'}$.

It is also clear that

$$\#\left\{ j \in \mathbb{N} : 2L \leq j \leq K - 1 - 4L, \sup_{0 < l \leq L} \frac{a_{l+j}b_{2l+j}}{l} \geq s \right\} \leq C_p \frac{\sqrt{\|a\|_p\|b\|_1}}{s L} \leq C_p \frac{\sqrt{\|a\|_p\|b\|_1}}{s} .$$

(8)

For ease of notation in the sequel we denote by $a$ and $b$ the modified sequences $\overline{a}$ and $\overline{b}$ consisting of powers of 2 or zero. Similarly we use $s = 2^{\gamma'}$ instead of $\overline{s}$.

After these adjustments we can suppose based on (4) that (for the modified $a$, $b$ and $s$) we have

$$1 < a_i, b_i < N_L = \sqrt{sL} .$$

(9)

Recall that we also assumed that

$$6L < K .$$

(10)

Suppose $i \in \{0, \ldots, K - 1 - 2L\}$ and there exists $l$ such that

$$\frac{a_{l+i}b_{2l+i}}{l} \geq s .$$

(11)

Then by (9)

$$\frac{sL}{l} \geq \frac{N_L^2}{l} \geq \frac{a_{l+i}b_{2l+i}}{l} \geq s ,$$

that is, $L \geq l$. Therefore, $l + i, 2l + i \in \{0, \ldots, K - 1\}$.

Choose $(a_{n,i})$ and $(b_{k,i})$ such that $a_i = \sum_{n \leq N_L} a_{n,i}$, $b_i = \sum_{k \leq N_L} b_{k,i}$, $a_{n,i} \in \{0, 2^n\}$, and $b_{k,i} \in \{0, 2^k\}$. Set

$$I_{n,k} = \left\{ i \in \{2L, \ldots, K - 1 - 4L\} : \sup_{0 < l \leq L} \frac{a_{n,l+i}b_{k,2l+i}}{l} \geq s \right\} .$$

(12)
If \[ \frac{a_{n,l+i}b_{k,2l+i}}{l} \geq s > 0 \] (13)
then \( a_{n,l+i} = 2^n \) and \( b_{k,2l+i} = 2^k \). Therefore,

\[ \frac{2^{n+k}}{s} \geq l, \quad \text{that is,} \quad 2^{n+k-\gamma'} \geq l \geq 1 > 0, \] (14)
which implies \( n + k - \gamma' \geq 0 \). Moreover, for these \( a_{n,l+i} \) and \( b_{k,2l+i} \) from (9) it also follows that

\[ L > \frac{a_{n,l+i}b_{k,2l+i}}{s} = 2^{n+k-\gamma'}. \] (15)

This “length”, \( 2^{n+k-\gamma'} \) will serve as a natural “unit” in our construction.

Set for \( t \in \mathbb{N} \)

\[ I_{n,k,t} = I_{n,k} \cap \{ t2^{n+k-\gamma'}, \ldots, (t+1)2^{n+k-\gamma'} - 1 \}, \quad \text{and} \quad \] (16)
\[ A_{n,k,t} = \{ t2^{n+k-\gamma'}, \ldots, (t+3)2^{n+k-\gamma'} - 1 \}. \]

By the introduction of the sequences \( a_{n,i} \) and \( b_{k,j} \) we want to split the ranges of \( a \) and \( b \) into some standardized “subsequences” and then use the sets \( I_{n,k,t} \) and a counting argument in (17-18) to obtain an upper estimate of the number of those indices where (13) holds. The “t-range” of a block \( I_{n,k,t} \) is greater or equal than \( l \), see (14).

Observe that if \( i \in I_{n,k,t} \) then there exists \( l \leq 2^{n+k-\gamma'} \) such that \( a_{n,l+i} = 2^n \), \( b_{k,2l+i} = 2^k \) and \( l + i, 2l + i \in A_{n,k,t} \). We want to give an upper estimate of \( \#I_{n,k,t} \). If \( a_{n,i} = 2^n \) for some \( i \) and \( b_{k,j} = 2^k \) for some \( j \)
with \( i, j \in \{ t2^{n+k-\gamma'}, \ldots, (t+3)2^{n+k-\gamma'} - 1 \} \) then there can be at most one \( i' \in \{ t2^{n+k-\gamma'}, \ldots, (t+3)2^{n+k-\gamma'} - 1 \} \) such that for \( i' \) there exists \( l \) such that \( i'+l = i \) and \( 2l+i' = j \). Indeed, in this case \( l = j-i \) and \( i' = i-(j-i) = 2i-j \).

### 2.2 Finding an upper bound of the cardinality of the sets \( I_{n,k,t} \), the counting argument

Denote by \( \mathcal{N}(n,k,t,a) \) the number of those \( i \in \{ t2^{n+k-\gamma'}, \ldots, (t+3)2^{n+k-\gamma'} - 1 \} \cap \mathbb{Z} \) for which \( a_{n,i} = 2^n \). Similarly, \( \mathcal{N}(n,k,t,b) \) denotes the number of those \( j \in \{ t2^{n+k-\gamma'}, \ldots, (t+3)2^{n+k-\gamma'} - 1 \} \cap \mathbb{Z} \) for which \( b_{k,j} = 2^k \). Now,

\[ \#I_{n,k,t} \leq \mathcal{N}(n,k,t,a)\mathcal{N}(n,k,t,b). \] (17)
On the other hand,

\[ \mathcal{N}(n, k, t, a) = \frac{\sum_{i \in A_{n,k,t}} a_{n,i}^p}{2^{np}}, \quad \text{and} \]

\[ \mathcal{N}(n, k, t, b) = \frac{\sum_{i \in A_{n,k,t}} b_{k,i}}{2^k}. \]

Therefore, by (17)

\[ \#I_{n,k,t} \leq \frac{1}{2^{np+k}} \sum_{i \in A_{n,k,t}} a_{n,i}^p \cdot \sum_{i \in A_{n,k,t}} b_{k,i}. \]

### 2.3 Refining the sets \( A_{n,k,t} \) with disjoint subsets \( A'_{n,k,t} \)

Set

\[ A'_{n,k,t} = \{t2^{n+k-\gamma}, \ldots, (t+1)2^{n+k-\gamma} - 1\}. \]

While for \( n, k \) fixed the sets \( A_{n,k,t} \) overlap, the sets \( A'_{n,k,t} \) are disjoint and

\[ A_{n,k,t} = A'_{n,k,t} \cup (A'_{n,k,t} + 2^{n+k-\gamma}) \cup (A'_{n,k,t} + 2 \cdot 2^{n+k-\gamma}). \]

Set

\[ T(n, k) = \left\{ 2, 3, \ldots, \left\lfloor \frac{K - 1 - 3L}{2^{n+k-\gamma}} \right\rfloor \right\}, \]

and \( T'(n, k) \) will consist of those \( t' \) for which \( A'_{n,k,t'} \subset \bigcup_{t \in T(n,k)} A_{n,k,t} \). It is useful to keep in mind that by (14) and (15) we have \( L > 2^{n+k-\gamma} \geq 1 \) and

\[ \{2L, 2L + 1, \ldots, K - 1 - 4L\} \subset \bigcup_{t \in T(n,k)} A_{n,k,t} \subset \{0, \ldots, K - 1\}. \]

### 2.4 Separation of two estimate cases, the set \( I^{**} \)

The constant \( \tilde{C}_{a,b} \) will be specified later in (34).

Denote by \( I_{n}^{***} \) the set of those \( i \) for which there exists a \( k \) and \( t \in T'(n,k) \) such that \( i \in A'_{n,k,t} \) and

\[ \sum_{i' \in A'_{n,k,t}} a_{n,i'}^p > (\#A'_{n,k,t}) \tilde{C}_{a,b}. \]
Due to dyadic grid properties, for fixed $n$ and $k$ the sets $A'_{n,k,t}$ are disjoint for different $t$'s, while - still keeping $n$ fixed - if for different $k$'s such sets intersect then one contains the other. Using this property one can choose a disjoint system of maximal intervals $\{A'_{n,k_j,t_j}\}$ such that $I^{**}_n = \bigcup_j A'_{n,k_j,t_j}$. The intervals $A'_{n,k_j,t_j}$ are maximal in the sense that for each $j$ if $A'_{n,k,t} \supseteq A'_{n,k_j,t_j}$ for a $t \in T'(n,k)$ then (22) does not hold.

Now, (22) implies

$$\sum_{i=0}^{K-1} a_{n,i}^p \geq \sum_{i \in I^{**}_n} a_{n,i}^p > \#I^{**}_n \tilde{C}_{a,b}.$$  \hspace{1cm} (23)

If $A'_{n,k_j,t_j}$ is one of the intervals considered above then let

$$B'_{n,k_j,t_j} \overset{\text{def}}{=} A'_{n,k_j,t_j} \cup (A'_{n,k_j,t_j} - 2^{n+k_j-\gamma'}) \cup (A'_{n,k_j,t_j} - 2 \cdot 2^{n+k_j-\gamma'}).$$

We put

$$I^{*}_n = \bigcup_j B'_{n,k_j,t_j}.$$  

From (23) it follows that

$$3 \sum_{i=0}^{K-1} a_{n,i}^p > \#(I^{*}_n) \tilde{C}_{a,b}.$$  \hspace{1cm} (24)

Set $I^{**} = \cup_n I^{**}_n$. Then adding (24) for $n$'s we obtain

$$3 \sum_{i=0}^{K-1} a_{n,i}^p > \#I^{**} \tilde{C}_{a,b}.$$  \hspace{1cm} (25)

### 2.5 The estimate for the set $I^*$

If $A'_{n,k,t+j'} \subset I^{**}_n$ for a $j' = 0, 1, 2$, then using the definition of the sets $B'_{n,k_j,t_j}$ and the maximality of the sets $A'_{n,k_j,t_j}$ one would obtain $A'_{n,k,t} \subset I^{**}_n$.

Hence, if $t \in T'(n,k)$, $A'_{n,k,t} \not\subset I^{**}$ then $A'_{n,k,t+j'} \not\subset I^{**}_n$ holds for $j' = 0, 1, 2$. This means by (22) that

if $A'_{n,k,t} \not\subset I^{**}_n$ and $A'_{n,k,t'} \subset A_{n,k,t}$ then

$$\sum_{i \in A'_{n,k,t'}} a_{n,i}^p \leq \#(A'_{n,k,t'}) \tilde{C}_{a,b}.$$  \hspace{1cm} (26)
From (26) it follows that

if \( t \in T'(n, k) \), \( A'_{n, k, t} \not\subset I^{**} = \cup_n I'^*_{n, t} \) and \( A'_{n, k, t'} \subset A_{n, k, t} \) then

\[
\sum_{i \in A'_{n, k, t'}} a^p_{n, i} \leq (\# A'_{n, k, t'}) \tilde{C}_{a, b}.
\]

(27)

Denote by \( T^{**}(n, k) \) the set of those \( t \in T'(n, k) \) for which \( A'_{n, k, t} \not\subset I^{**} \). Set \( I^{**}_{n, k} = I_{n, k} \setminus I^{**} \). Clearly,

\[
I^{**}_{n, k} \subset \bigcup_{t \in T^{**}(n, k)} I_{n, k, t} \subset \bigcup_{t \in T^{**}(n, k)} A'_{n, k, t'}.
\]

(28)

Denote by \( T''(n, k) \) the set of those \( t' \in T'(n, k) \) for which there exists \( t \in T^{**}(n, k) \) satisfying \( A'_{n, k, t'} \subset A_{n, k, t} \). For \( t' \in T''(n, k) \) one can apply (26) and (27).

Set

\[
C_{n, k, t} = A_{n, k, t} \cup (A_{n, k, t} - 2^{n+k-\gamma'}) \cup (A_{n, k, t} - 2 \cdot 2^{n+k-\gamma'}).
\]

By (19) and (28) we have

\[
\# I^{**}_{n, k} \leq \sum_{t \in T^{**}(n, k)} \# I_{n, k, t} \leq \sum_{t \in T^{**}(n, k)} \frac{1}{2np+k} \sum_{i \in A_{n, k, t}} a^p_{n, i} \sum_{i \in A_{n, k, t}} b_{k, i} \leq \sum_{t \in T''(n, k)} \frac{3}{2np+k} \sum_{i \in A'_{n, k, t'}} a^p_{n, i} \sum_{i \in C_{n, k, t}} b_{k, i}.
\]

(29)

(30)

Recall that \( \#(A'_{n, k, t}) = 2^{n+k-\gamma'} = 2^{n+k}/s \) when \( t \in T'(n, k) \).

By (27), (29), and (30) we have

\[
\# I^{**}_{n, k} \leq \sum_{t \in T''(n, k)} \frac{3}{2^{np+1}} \sum_{i \in A'_{n, k, t'}} \frac{a^p_{n, i}}{\# A'_{n, k, t'}} \sum_{i \in C_{n, k, t}} b_{k, i} \leq \sum_{t \in T''(n, k)} \frac{3}{2^{np+1}} \sum_{i \in C_{n, k, t}} b_{k, i} \leq \frac{27}{2^{np+1}} \tilde{C}_{a, b} \sum_{i=0}^{K-1} b_{k, i}.
\]

(31)
Set
\[ I^* = \bigcup_{n,k} I_{n,k} \setminus I^{**} = \bigcup_{n,k} I^{**}_{n,k}. \]

By (31) we have
\[ \# I^* \leq \sum_{n} \frac{27}{s^{2n(p-1)}} \tilde{C}_{a,b} \sum_{k} \sum_{i=0}^{K-1} b_{k,i} \leq \frac{27}{s(1 - 2^{-(p-1)})} \tilde{C}_{a,b} \sum_{i=0}^{K-1} b_{i}. \] (32)

### 2.6 Conclusion of the proof of Lemma 3

Now, by using (25) and (32) we have
\[ \# \left( \bigcup_{n,k} I_{n,k} \right) \leq \# \left( \bigcup_{n,k} (I_{n,k} \setminus I^{**}) \right) + \# I^{**} = \# I^* + \# I^{**} \leq \] (33)
\[ \frac{27}{s(1 - 2^{-(p-1)})} \tilde{C}_{a,b} \sum_{i=0}^{K-1} b_{i} + \frac{3 \sum_{i=0}^{K-1} d_{i}^{p}}{C_{a,b}}. \]

Choose
\[ \tilde{C}_{a,b} = \sqrt{s \sum_{i=0}^{K-1} d_{i}^{p} / \| b \|_{1}}. \] (34)

Then we obtained
\[ \# \left( \bigcup_{n,k} I_{n,k} \right) \leq \left( \frac{27}{1 - 2^{-(p-1)} + 3} \right) \sqrt{\frac{\| a \|_{p} \| b \|_{1}}{s}} = \frac{C_{p}}{2^{(p+2)/2}} \sqrt{\frac{\| a \|_{p} \| b \|_{1}}{s}}. \] (35)

Using (6), (7-8) and (12) we can infer (3). By elementary calculus from (35) one can also deduce the existence of a constant \( C' \) for which (5) holds when \( 1 < p < 2 \). \( \square \)

### 3 Proof of Theorem 2 by a transference argument

The transference argument is first applied for bounded functions \( f \) and \( g \). Moreover, we also make the auxiliary assumption that \( g \) is bounded away from zero. Later we remove these extra assumptions. For \( f \) some standard duality tricks are applicable, while for \( g \) with the \( L^1 \) norm, slightly modified arguments are needed.
### 3.1 The transference argument

Let $(X, B, \mu, T)$ be an invertible dynamical system on a probability measure space and let $f \in L^p$, $g \in L^1$. Usage of $|f|$ and $|g|$ instead of $f$ and $g$ increases the left hand side of (1) without changing its right hand side. Therefore, in the sequel we assume that $f$ and $g$ are nonnegative.

First we suppose that there exists $M > 1$ such that $0 \leq f, g \leq M$ and $g \geq 1/M$ everywhere. If $||f||_p = 0$ then $f = 0$ a.e. and we have nothing to prove.

Put

$$F = \frac{f}{||f||_p} + 1 \quad \text{and} \quad G = M \cdot g.$$  \hspace{1cm} (36)

Set

$$M^* = \max \left\{ \frac{M}{||f||_p} + 1, M^2 \right\}.$$  

Then $1 \leq F, G \leq M^*$ holds everywhere. Given $s' > 0$ we choose $L$ so large that $M^* < \sqrt{s'L}/8$ and choose $K > 6L$. In (2) we have $s > 0$ given, but we will see later that we need to start with a suitable $s'$ which will equal $sM/||f||_p$ in the end. For any given $x \in X$ consider the sequences $a_i = F(T_i x)$, $b_i = G(T_i x)$, for $i = 0, ..., K - 1$ and $a_i = b_i = 0$ for $i \in \mathbb{Z} \setminus \{0, ..., K - 1\}$.

Since (4) is satisfied we can apply Lemma 3 to obtain the inequality

$$\# \left\{ j \in \mathbb{N} : 2L \leq j \leq K - 1 - 4L, \sup_{0 \leq l \leq L} \frac{F(T_{i+l} x)G(T_{i+l}^2 x)}{l} \geq s' \right\} \leq \frac{C_p}{s'} \left( \sum_{i=0}^{K-1} |F(T_i x)|^p \sum_{i=0}^{K-1} |G(T_i x)| \right)^{1/p}.$$  

By integrating this inequality with respect to $\mu$ and using that $T^j$ is measure preserving and hence the integral of the left hand side takes the same value for all $j$’s we infer

$$(K - 6L)\mu \left\{ x : \sup_{0 \leq l \leq L} \frac{F(T^i x)G(T^{2l} x)}{l} \geq s' \right\} \leq \frac{C_p}{s'} \int_X \left( \sum_{i=0}^{K-1} |F(T_i x)|^p \sum_{i=0}^{K-1} |G(T_i x)| \right) d\mu.$$
Dividing by $K - 6L$ one obtains

$$
\mu \left\{ x : \sup_{0 < t \leq L} \frac{F(T^t x) G(T^{2t} x)}{t} \geq s' \right\} \leq C_p \int_X \sqrt{\frac{\sum_{i=0}^{K-1} |F(T^i x)|^p \sum_{i=0}^{K-1} |G(T^i x)|}{(K - 6L)s'(K - 6L)}} \, d\mu. \tag{37}
$$

Now we can notice that by the pointwise ergodic theorem if we denote by $\mathcal{I}$ the $\sigma$-field of $T$-invariant sets then the averages $\frac{1}{K - 6L} \sum_{i=0}^{K-1} |G(T^i x)|$ converge a.e. to $E[|G|, \mathcal{I}](x)$, the conditional expectation of $|G|$ with respect to $\mathcal{I}$. Furthermore, the averages $\frac{1}{K - 6L} \sum_{i=0}^{K-1} |F(T^i x)|^p$ converge a.e. to $E[|F|^p, \mathcal{I}](x)$ as $K \to \infty$. Since $F$ and $G$ are bounded Lebesgue’s dominated convergence theorem applies and we have obtained the inequality

$$
\mu \left\{ x : \sup_{0 < t \leq L} \frac{F(T^t x) G(T^{2t} x)}{t} \geq s' \right\} \leq C_p \int_X \sqrt{\frac{E[|F|^p, \mathcal{I}](x)E[|G|, \mathcal{I}](x)}{s'}} \, d\mu. \tag{38}
$$

By using Hölder’s inequality the right handside of (38) is bounded above by

$$
C_p \sqrt{\frac{1}{s'}} \sqrt{\int_X E[|F|^p, \mathcal{I}](x) \, d\mu} \sqrt{\int_X E[|G|, \mathcal{I}](x) \, d\mu}.
$$

Using the integral preserving property of the conditional expectation this last term equals

$$
C_p \sqrt{\frac{1}{s'}} \sqrt{\int_X |F|^p \, d\mu} \sqrt{\int_X |G| \, d\mu} = C_p \sqrt{\frac{\|F\|^p \|G\|_q}{s'}}.
$$

We have reached then the following inequality

$$
\mu \left\{ x : \sup_{0 < t \leq L} \frac{F(T^t x) G(T^{2t} x)}{t} \geq s' \right\} \leq C_p \sqrt{\frac{\|F\|^p \|G\|_q}{s'}}. \tag{39}
$$

Next we see the consequences of (39) for our original functions $f$ and $g$. Since $C_p$ does not depend on $L$, first one can let $L \to \infty$. Recall that by Hölder’s inequality if $\frac{1}{p} + \frac{1}{q} = 1$ then $(\alpha + \beta) \leq 2^{1/q} (\alpha^p + \beta^p)^{1/p}$ and this yields

$$
\|F\|^p = \int_X \left( \frac{f}{\|f\|^p} + 1 \right)^p \, d\mu \leq 2^{p/q} \int_X \left( \frac{f^p}{\|f\|^p} + 1 \right) \, d\mu = 2^{\frac{p}{q} + 1}. \tag{40}
$$
Rewriting (39) and letting $L \to \infty$ we obtain

$$\mu \left\{ x : \sup_{0< \ell} \frac{f(T^l x) g(T^{2l} x) \cdot M}{l \cdot \|f\|_p} \geq s' \right\} \leq$$

$$\mu \left\{ x : \sup_{0< \ell} \frac{F(T^l x) G(T^{2l} x)}{l} \geq s' \right\} \leq$$

(using (40))

$$C_p \sqrt{\frac{\|F\|_p^p \|G\|_1}{s'}} \leq C_p \sqrt{\frac{2^{\frac{p+1}{p}} \|g\|_1 \cdot M}{s'}}. \quad (42)$$

Choosing $s' = \frac{s \cdot M}{\|f\|_p}$ we obtain from (41-42)

$$\mu \left\{ x : \sup_{0< \ell} \frac{f(T^l x) g(T^{2l} x)}{l} \geq s \right\} \leq C_p \sqrt{\frac{2^{\frac{p+1}{p}} \|f\|_p \|g\|_1}{s}} =$$

$$C_p 2^{\frac{p+1}{p}} \sqrt{\frac{\|f\|_p \|g\|_1}{s}}.$$

Hence we can choose $C_p^* = C_p 2^{\frac{p+1}{p}}$. This constant does not depend on $M$. The general case for arbitrary nonnegative functions $f \in L^p$ and $g \in L^1$ can be obtained by approximating $f$ by functions of the form $f_M = \min\{f, M\}$ and $g$ by $\min\{\max\{g, \frac{1}{M}\}, M\}$.

### 3.2 Second part of Theorem 2

It remains to prove that

$$\frac{f(T^l x) g(T^{2l} x)}{l} \to 0 \text{ as } l \to \infty. \quad (43)$$

This follows by approximation. We give the details for sake of completeness. Since (43) is equivalent to $\frac{|f(T^l x)| \cdot |g(T^{2l} x)|}{l} \to 0$ we can assume that $f, g \geq 0$. Set $f_M = \min\{f, M\}$, $g_M = \min\{g, M\}$. For all $\epsilon > 0$ choose $M$ such that
\[ \int_X (f - f_M)^p \, d\mu < \epsilon, \int_X (g - g_M) \, d\mu < \epsilon. \] Then
\[
\begin{align*}
\left| \frac{f(T^i x)g(T^{2l}x)}{l} \right| &\leq \left| \frac{(f - f_M)(T^i x)(g - g_M)(T^{2l}x)}{l} \right| + \\
\left| \frac{f_M(T^i x)g(T^{2l}x)}{l} \right| &+ \left| \frac{f(T^i x)g_M(T^{2l}x)}{l} \right| + \left| \frac{f_M(T^i x)g_M(T^{2l}x)}{l} \right| \\
&\leq \left| (f - f_M)(T^i x)(g - g_M)(T^{2l}x) \right| + M \left| g(T^{2l}x) \right| + M \left| f(T^{2l}x) \right| + M^2 \frac{\epsilon}{l}.
\end{align*}
\]

The last three terms converge to zero almost everywhere as \( l \to \infty \). By (1) the measure of the set of those \( x \)'s where the limit superior of the first term is larger than \( s = \epsilon \) can be estimated from above by \( C_p^* \sqrt{\frac{\epsilon^2}{\epsilon}} = C_p^* \sqrt{\epsilon} \). Since \( \epsilon > 0 \) is arbitrary we obtain (43).

\[ \square \]

**Remark 2.** It is natural to ask whether \( \|a\|_p \) can be replaced by \( \|a\|_p \) in (3). The answer is no since this would imply the following version of (37)

\[ \mu \left\{ x : \sup_{0 < l \leq L} \frac{F(T^i x)G(T^{2l}x)}{l} \geq s' \right\} \leq \quad (44) \]

\[ C_p \int_X \left[ \left( \sum_{i=0}^{K-1} |F(T^i x)|^p \right)^{1/p} \sum_{i=0}^{K-1} |G(T^i x)| \right] d\mu. \]

By the pointwise ergodic theorem the averages \( \frac{1}{K-6L} \sum_{i=0}^{K-1} |G(T^i x)| \) converge a.e. to \( \mathbb{E}[|G|, T](x) \) as \( K \to \infty \). Furthermore, the averages

\[ \frac{1}{K-6L} \left( \sum_{i=0}^{K-1} |F(T^i x)|^p \right)^{1/p} \]

converge a.e. to \( 0 \cdot \left( \mathbb{E}[|F|^p, T](x) \right)^{1/p} = 0 \). Since \( F \) and \( G \) are bounded by Lebesgue's dominated convergence theorem we would obtain that the right hand side of (44) converges to zero as \( K \to \infty \) which is impossible for all possible choices of \( F, G \) and \( s' \).

**Remark 3.** One of the reasons why (1) does not hold on \( \mathbb{Z} \) is because it is not homogeneous with respect to \( \mu \). By this we mean that if one divides the measure \( \mu \) by \( N \) then the right hand side of (1) is not divided by \( N \) but by a power of \( N \), namely \( N^{1/2p+1/2} \).
Theorem 2 holds for finite measure spaces as this can be derived from probability measure spaces by simple computations. The constant $C_p$ depends then also on the total mass of the space $X$. However the failure of (1) on $\mathbb{Z}$ indicates that Theorem 2 does not hold in general for measure preserving systems on $\sigma$-finite measure spaces.

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References


Department of Mathematics, University of North Carolina at Chapel Hill, Chapel Hill, North Carolina 27599, USA
e-mail: assani@email.unc.edu
Assani and Buczolich, \((L^p, L^q)\) bilinear Hardy-Littlewood function

Department of Analysis, Eötvös Loránd University, Pázmány Péter Sétány 1/c, 1117 Budapest, Hungary
email: buczo@cs.elte.hu
www.cs.elte.hu/~buczo