Convergence of ergodic averages for many group rotations

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Abstract

Suppose that G is a compact Abelian topological group, m is the Haar measure on G and $f : G \to \mathbb{R}$ is a measurable function. Given (n_k), a strictly monotone increasing sequence of integers we consider the nonconventional ergodic/Birkhoff averages
$M_N^\alpha f(x) = \frac{1}{N+1} \sum_{k=0}^{N} f(x + n_k\alpha)$.
The f-rotation set is
$\Gamma_f = \{\alpha \in G : M_N^\alpha f(x)$ converges for m a.e. x as $N \to \infty.\}$

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We prove that if G is a compact locally connected Abelian group and $f : G \to \mathbb{R}$ is a measurable function then from $m(\Gamma_f) > 0$ it follows that $f \in L^1(G)$.

A similar result is established for ordinary Birkhoff averages if $G = Z_p$, the group of p-adic integers.

However, if the dual group, $\widehat{G}$ contains "infinitely many multiple torsion" then such results do not hold if one considers non-conventional Birkhoff averages along ergodic sequences.

What really matters in our results is the boundedness of the tail, $f(x + n_k\alpha)/k$, $k = 1, \ldots$ for a.e. x for many α, hence some of our theorems are stated by using instead of Γ_f slightly larger sets, denoted by $\Gamma_{f,b}$.

1 Introduction

The starting point of this paper is a result of the first listed author in [3] which states that if f is a (Lebesgue) measurable function on the unit circle $\mathbb{T}$ and Γ_f denotes the set of those α's for which the Birkhoff averages

$$M_n^\alpha f(x) = \frac{1}{n+1} \sum_{k=0}^{n} f(x + k\alpha)$$

converge for almost every x then from $m(\Gamma_f) > 0$ it follows that $f \in L^1(\mathbb{T})$. Hence $M_n^\alpha f$ converges for all $\alpha \in \mathbb{T}$.

In this paper we consider generalizations of this result to compact Abelian groups equipped with their Haar measure m. Theorem 1 implies that an analogous result is true even for non-convential ergodic averages considered on a compact, locally connected Abelian group G.

On the other hand, if there is "sufficiently many multiple torsion" in the dual group $\widehat{G}$ then Theorem 6 implies that there are non-L^1 measurable functions f for which $m(\Gamma_f) = 1$ (in fact, $\Gamma_f = G$) if one considers non-conventional Birkhoff averages along ergodic sequences. Having lots of torsion in $\widehat{G}$ means that G is highly disconnected. In our opinion the most surprizing result of this paper is Theorem 7 which states that if $G = Z_p$, the group of p-adic integers and one considers the ordinary ergodic averages of a measurable function f then from $m(\Gamma_f) > 0$ it follows that $f \in L^1(G)$. The group Z_p is zero-dimensional and all elements of its dual group, $Z(p^\infty)$, are of finite order. If one considers a group G which is a countable product of Z_p's then there is enough "multiple torsion" (see Definition 3) in Γ_f and Theorem 6 implies that the result of Theorem 7 does not hold in these groups. If $M_n^\alpha f(x)$ converges then the tail $\dfrac{f(x + n\alpha)}{n} \to 0$. In our proofs the sets $\Gamma_{f,0}$

(and $\Gamma_{f,b}$), the sets of those α's where $\frac{f(x+n\alpha)}{n} \to 0$, (or $\frac{|f(x+n\alpha)|}{n}$ is bounded) for a.e. x play an important role. Since $\Gamma_f \subset \Gamma_{f,0} \subset \Gamma_{f,b}$ from $m(\Gamma_f) > 0$ it follows that the other sets are also of positive measure and hence in the statements of Theorems 1 and 7 these sets are used. Again the tail of the ergodic averages plays an important role, like in [1], where we showed that for L^1 functions and ordinary ergodic averages the return time property for the tail may might fail and hence Bourgain's return time property [2] does not hold in these situations.

The proof of Theorem 1 is a rather straightforward generalization of Theorem 1 in [3]. We provide its details, since they are also used with some non-trivial modifications in the proof of Theorem 7.

Next we say a few words about the background history and related questions to this paper. Answering a question raised by the first listed author of this paper P. Major in [9] constructed two ergodic transformations $S, T : X \to X$ on a probability space (X, μ) and a measurable function $f : X \to \mathbb{R}$ such that for μ a.e. x

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} f(S^k x) = 0, \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} f(T^k x) = a \neq 0.$$

M. Laczkovich raised the question whether S and T can be irrational rotations of $\mathbb{T}$. In Major's example S and T are conjugate. Therefore, his method did not provide an answer to Laczkovich's question.

The results of Z. Buczolich in [4] imply that for any two independent irrationals α and β one can find a measurable $f : \mathbb{T} \to \mathbb{R}$ such that $M_n^\alpha f(x) \to c_1$ and $M_n^\beta f(x) \to c_2$ for a.e. x with $c_1 \neq c_2$. In this case by Birkhoff's ergodic theorem $f \notin L^1(\mathbb{T})$. It is shown in [3] that for any sequence (α_j) of independent irrationals one can find a measurable $f : \mathbb{T} \to \mathbb{R}$ such that $f \notin L^1(\mathbb{T})$, but $\alpha_j \in \Gamma_f$ for all $j = 1, \ldots$. By Theorem 1 of [3] from $f \notin L^1(\mathbb{T})$ it follows that $m(\Gamma_f) = 0$. It was a natural question to see how large Γ_f could be for an $f \notin L^1(\mathbb{T})$. In [14] R. Svetic showed that Γ_f can be c-dense for an $f \notin L^1(\mathbb{T})$.

The question about the possible largest Hausdorff dimension of Γ_f for an $f \notin L^1(\mathbb{T})$ remained open for a while until in [5] it was shown that there are $f \notin L^1(\mathbb{T})$ such that $\dim_H(\Gamma_f) = 1$ (of course with $m(\Gamma_f) = 0$.)

For us motivation to consider non-conventional ergodic averages in this paper came from the project in [6] concerning almost everywhere convergence questions of Birkhoff averages along the squares.

It is also worth mentioning that ergodic averages of non-L^1 functions and rotations on $\mathbb{T}$ were also considered in [13] and [12].

2 Preliminaries

We suppose that G is a compact Abelian topological group, the group operation will be addition. The dual group of the compact Abelian topological group G is denoted by $\widehat{G}$. By Pontryagin duality $\widehat{G}$ is a discrete Abelian group. For $\gamma \in \widehat{G}$ the corresponding Fourier coefficient is

$$\widehat{f}(\gamma) = \int_G g(x)\gamma(-x)dm(x),$$

where m denotes the Haar mesure on G. By the Parseval formula

$$\int_G f(x)\bar{g}(x)dm(x) = \sum_{\gamma \in \widehat{G}} \widehat{f}(\gamma)\overline{\widehat{g}(\gamma)} \quad \text{for } f, g \in L^2(G).$$

By [8, 24.25] or [11, 2.5.6 Theorem] if G is a compact Abelian group then G is connected if and only if $\widehat{G}$ is torsion-free.

Suppose that $p_1, p_2, ...$ is a sequence of prime numbers. Recall that the direct product $G = (Z/p_1) \times (Z/p_2) \times \dots$ is compact and its dual group $\widehat{G} = (Z/p_1) \bigoplus (Z/p_2) \bigoplus \dots$ is the direct sum with the discrete topology see [11, 2.2 p.36] or [8].

We denote by Z_p the group of p-adic integers and its dual group, the Prüfer p-group with the discrete topology will be denoted by $Z(p^\infty)$.

For other properties of topological groups we refer to standard textbooks like [7], [8] or [11].

Suppose that $f : G \to \mathbb{R}$ is a measurable function. We suppose that the group rotation $T_\alpha = x + \alpha, \ \ \alpha \in G$ is fixed.

Given a strictly monotone increasing sequence of integers (n_k) we consider the nonconventional ergodic averages

$$M_N^\alpha f(x) = \frac{1}{N+1} \sum_{k=0}^{N} f(x + n_k \alpha).$$

Of course, if $n_k = k$ we have the usual Birkhoff averages.

The f-rotation set is

$$\Gamma_f = \{\alpha \in G : M_N^\alpha f(x) \text{ converges for } m \text{ a.e. } x \text{ as } N \to \infty\}.$$

As we mentioned in the introduction it was proved in [3] that if $G = \mathbb{T}, m = \lambda$, the Lebesgue measure on $\mathbb{T}$, and $n_k = k$ then for any measurable $f : \mathbb{T} \to \mathbb{R}$ from $m(\Gamma_f) > 0$ it follows that $f \in L^1(\mathbb{T})$.

Scrutinizing the proof of this result one can see that the set

$$\Gamma_{f,0} = \left\{ \alpha \in G : \frac{f(x+n_k\alpha)}{k} \to 0 \text{ for } m \text{ a.e. } x \right\}$$

played an important role. It is obvious that $\Gamma_f \subset \Gamma_{f,0}$.

In [3] it was shown that from $m(\Gamma_{f,0}) > 0$ it follows that $f \in L^1(\mathbb{T})$, when the sequence $n_k = k$ is considered. In this paper we will also use the slightly larger set

$$\Gamma_{f,b} = \left\{ \alpha \in G : \limsup_{k\to\infty} \frac{|f(x+n_k\alpha)|}{k} < \infty \text{ for } m \text{ a.e. } x \right\}. \tag{1}$$

3 Main results

First we generalize Theorem 1 of [3] for compact, locally connected Abelian groups.

Theorem 1. *If (n_k) is a strictly monotone increasing sequence of integers and G is a compact, locally connected Abelian group and $f : G \to \mathbb{R}$ is a measurable function then from $m(\Gamma_{f,b}) > 0$ it follows that $f \in L^1(G)$.*

Remark 2. Since $\Gamma_{f,b} \supset \Gamma_{f,0} \supset \Gamma_f$ Theorem 1 implies that if one considers the non-conventional ergodic averages $M_N^\alpha f$ on a locally compact Abelian group for group rotations and $m(\Gamma_f) > 0$ then $f \in L^1(G)$.

Proof. Set $n_0 = 0$. First we suppose that G is connected. Given an integer K put

$$G_{\alpha,K} = \{x : |f(x+n_k\alpha)| < K \cdot k \text{ for every } k > K \tag{2}$$
$$\text{and } |f(x+n_k\alpha)| < K^2 \text{ for } k = 0, \dots, K\}.$$

If $\alpha \in \Gamma_{f,b}$ then $m(G_{\alpha,K}) \to 1$ as $K \to \infty$.

Choose and fix K and $\varepsilon > 0$ such that the set

$$B = \{\alpha : m(G_{\alpha,K}) > \varepsilon\} \tag{3}$$

is of positive m-measure. From the measurability of f it follows that B and the sets $g_{\alpha,K}$ are also measurable.

Set

$$L_k(f) = \{x \in G : |f(x)| > k\}. \tag{4}$$

From $k > K$ and $x \in G_{\alpha,K} + n_k\alpha$ it follows that

$$|f(x)| = |f(x - n_k\alpha + n_k\alpha)| < k \cdot K.$$

Set $H_\alpha = G \backslash G_{\alpha,K}$, (keep in mind that K is fixed). From $k > K$ and $x \in L_{k \cdot K}(f)$ it follows that $x \notin G_{\alpha,K} + n_k\alpha$, that is, $x \in H_\alpha + n_k\alpha$.

For $\alpha \in B$ we set $a(\alpha) = m(H_\alpha) < 1 - \varepsilon$, by (3). This implies $1/(1 - a(\alpha)) < 1/\varepsilon$.

For $\alpha \in B$ put

$$h(x, \alpha) = \begin{cases} 1 \text{ if } x \in H_\alpha, \\ -\left(\frac{a(\alpha)}{1-a(\alpha)}\right) \text{ if } x \notin H_\alpha. \end{cases} \tag{5}$$

For $\alpha \notin B$ set $h(x, \alpha) = 0$ for any $x \in G$.

Then $h(x, \alpha)$ is a bounded measurable function defined on $G \times G$ and

$$\int_G h(x, \alpha) dm(x) = 0 \text{ for any } \alpha \in G. \tag{6}$$

From $k > K$ and $x \in L_{k \cdot K}(f)$ it follows that $x \in H_\alpha + n_k\alpha$ for any $\alpha \in B$. This implies

$$h(x - n_k\alpha, \alpha) = 1 \text{ for any } x \in L_{k \cdot K}(f) \text{ and } \alpha \in B. \tag{7}$$

Taking average

$$\frac{1}{m(B)} \int_B h(x - n_k\alpha, \alpha) dm(\alpha) = 1 \text{ for } k > K \text{ and } x \in L_{k \cdot K}(f). \tag{8}$$

Keep α fixed and select a character $\gamma \in \widehat{G}$. Consider in the Fourier-series of $h(x, \alpha)$ the coefficient $c_\gamma(\alpha)$ corresponding to this character, that is,

$$c_\gamma(\alpha) = \int_G h(x, \alpha)\gamma(-x) dm(x). \tag{9}$$

Since $h(x, \alpha)$ is a bounded measurable function, the function $c_\gamma(\alpha)$ is also bounded and measurable. Then

$$h(x, \alpha) \sim \sum_{\gamma \in \widehat{G}} c_\gamma(\alpha)\gamma(x). \tag{10}$$

If $\gamma_0(x) \equiv 1$ then by (6) we have

$$c_{\gamma_0}(\alpha) = 0 \text{ for any } \alpha \in G. \tag{11}$$

For a fixed $\alpha \in B$ we have

$$h(x - n_k\alpha, \alpha) \sim \sum_{\gamma \in \widehat{G}} c_\gamma(\alpha)\gamma(-n_k\alpha)\gamma(x). \tag{12}$$

By (8)

$$m(L_{k\cdot K}(f)) \le \int_G \left| \frac{1}{m(B)} \int_B h(x - n_k\alpha, \alpha) dm(\alpha) \right|^2 dm(x) \tag{13}$$
$$= \int_G |\varphi_k(x)|^2 dm(x) = \circledast,$$

where $\varphi_k(x) = \dfrac{1}{m(B)} \displaystyle\int_B h(x - n_k\alpha, \alpha) dm(\alpha)$ is a bounded measurable function. If γ is a given character then using that h is bounded and recalling (9) we obtain

$$\begin{aligned}
\widehat{\varphi}_k(\gamma) &= \int_G \frac{1}{m(B)} \int_B h(x - n_k\alpha, \alpha) dm(\alpha)\gamma(-x) dm(x) \\
&= \frac{1}{m(B)} \int_B \int_G h(x - n_k\alpha, \alpha)\gamma(-x) dm(x) dm(\alpha) \\
&= \frac{1}{m(B)} \int_G \chi_B(\alpha) \int_G h(u, \alpha)\gamma(-u - n_k\alpha) dm(u) dm(\alpha) \\
&= \frac{1}{m(B)} \int_G \chi_B(\alpha)\gamma(-n_k\alpha) \int_G h(u, \alpha)\gamma(-u) dm(u) dm(\alpha) \\
&= \frac{1}{m(B)} \int_G \chi_B(\alpha)\gamma(-n_k\alpha) c_\gamma(\alpha) dm(\alpha).
\end{aligned} \tag{14}$$

By using the Parseval formula we can continue $\circledast$ in (13) to obtain

$$\begin{aligned}
m(L_{k\cdot K}(f)) &\le \sum_{\gamma \in \widehat{G}} |\widehat{\varphi}_k(\gamma)|^2 \\
&= \sum_{\gamma \in \widehat{G}} \frac{1}{(m(B))^2} \left| \int_G \chi_B(\alpha)\gamma(-n_k\alpha) c_\gamma(\alpha) dm(\alpha) \right|^2 \\
&= \frac{1}{(m(B))^2} \sum_{\gamma \in \widehat{G}} \left| \int_G \chi_B(\alpha) c_\gamma(\alpha)\gamma^{n_k}(-\alpha) dm(\alpha) \right|^2 .
\end{aligned} \tag{15}$$

Since $\chi_B(\alpha)c_\gamma(\alpha)$ is a bounded measurable function and $\gamma^{n_k} \in \widehat{G}$, the expression $\int_G \chi_B(\alpha)c_\gamma(\alpha)\gamma^{n_k}(-\alpha)dm(\alpha)$ is a Fourier coefficient of this function.

Now we use that G is connected and hence $\widehat{G}$ is torsion-free. If $\gamma^{n_k} = \gamma^{n_{k'}}$ then $\gamma^{n_k - n_{k'}} = \gamma_0 \equiv 1$, but γ is of infinite order and hence it is only possible if $n_k - n_{k'} = 0$, that is $k = k'$. Hence for $k \ne k'$ the characters γ^{n_k} and $\gamma^{n_{k'}}$ are different. By Parseval's formula for a fixed $\gamma \in \widehat{G}$

$$\sum_{k=K}^{\infty} \left| \int_G \chi_B(\alpha) c_\gamma(\alpha)\gamma^{n_k}(-\alpha) dm(\alpha) \right|^2 \le \int_G |\chi_B(\alpha) c_\gamma(\alpha)|^2 dm(\alpha). \tag{16}$$

This, Parseval's formula, (5), (9) and (15) yield

$$\begin{aligned}\sum_{k=K+1}^{\infty} m(L_{k\cdot K}(f)) &\le \frac{1}{(m(B))^2}\sum_{\gamma\in\widehat{G}}\int_G |\chi_B(\alpha)c_\gamma(\alpha)|^2\,dm(\alpha)\\ &= \frac{1}{(m(B))^2}\int_G \chi_B(\alpha)\sum_{\gamma\in\widehat{G}}|c_\gamma(\alpha)|^2\,dm(\alpha) \qquad (17)\\ &= \frac{1}{(m(B))^2}\int_G \chi_B(\alpha)\int_G |h(x,\alpha)|^2 dm(x)dm(\alpha) < \infty.\end{aligned}$$

Since $\int_G |f| \le K\cdot \sum_{k=0}^{\infty} m(L_{k\cdot K}(f))$ from (17) and $m(G)=1$ it follows that $f\in L^1(G)$.

This completes the proof of the case of connected G.

Next we show how one can reduce the case of a locally connected G to the connected case. If G is locally connected then by [8, 24.45] if C denotes the component of G containing O_G (the neutral element of G) then C is an open subgroup of G and G is topologically isomorphic to $C\times(G/C)$. Since G is compact G/C should be finite. Suppose that its order is n. Using that $G=C\times(G/C)$ we write the elements of G in the form $g=(g_1,g_2)$ with $g_1\in C,\ g_2\in G/C$.

Suppose that $f\notin L^1(G)$ is measurable and $m(\Gamma_{f,b})>0$. Set

$$X_{\alpha,f}=\Big\{x\in G : \limsup_{k\to+\infty}\frac{|f(x+n_k\alpha)|}{k}<+\infty\Big\}.$$

If $\alpha\in\Gamma_{f,b}$ then $m(X_{\alpha,f})=1$. Suppose that $g_j^*,\ j=1,\dots,n$ is a list of all elements of G/C.

For $x=(x_1,x_2)\in G$ define

$$f^*(x)=f^*(x_1,x_2)=\sum_{j=1}^{n}|f(x_1,x_2+g_j^*)|.$$

Set

$$X^*_{\alpha,f}=\bigcap_{j=1}^{n}\Big(X_{\alpha,f}+(0_C,g_j^*)).$$

Clearly $m(X_{\alpha,f})=1$ implies $m(X^*_{\alpha,f})=1$.

For $x\in X^*_{\alpha,f}$ we have $\limsup_{k\to\infty}\frac{|f^*(x+n_k\alpha)|}{k}<+\infty$. Since f^* is not depending on its second coordinate we have $f^*(x+n_k(\alpha_1,\alpha_2))=f^*(x+$

$n_k(\alpha_1, 0_{G/C}))$. Define $f^{**} : C \to \mathbb{R}$ such that $f^{**}(x_1) = f^*(x_1, 0_{G/C})$. Since we assumed that $f \notin L^1(G)$ we have $f^* \notin L^1(G)$ and this implies $f^{**} \notin L^1(C)$.

Set

$$\Gamma^*_{f,b} = \pi_C(\Gamma_{f,b}) = \{\alpha_1 : \exists \alpha_2 \in G/C \text{ such that } \alpha = (\alpha_1, \alpha_2) \in \Gamma_{f,b}\}.$$

Then for $\alpha_1 \in \Gamma^*_{f,b}$ we have

$$\limsup_{k\to\infty} \frac{|f^{**}(x_1 + n_k\alpha_1)|}{k} < +\infty. \tag{18}$$

Since the Haar measure on C is a positive constant multiple of the Haar measure on G restricted to C, on the compact connected Abelian group C we would obtain a measurable function $f^{**} \notin L^1(C)$ such that for a set of positive measure of rotations (18) holds. This would contradict the first part of this proof concerning connected groups. $\square$

Theorem 1 says that if we do not have "too much torsion" in $\widehat{G}$ then from $m(\Gamma_{f,b}) > 0$ it follows that $f \in L^1(G)$. In the next definition we define what we mean by "a lot of torsion" in a group.

Definition 3. We say that the group G contains infinitely many multiple torsion if

(i) either there is a prime number p such that G contains a subgroup algebraically isomorphic to the direct sum $(Z/p) \bigoplus (Z/p) \bigoplus \dots$ (countably many copies of Z/p),

(ii) or there are infinitely many different prime numbers $p_1, p_2, \dots$ such that G contains for any j subgroups of the form $(Z/p_j) \times (Z/p_j)$.

Theorem 4. *Suppose that (n_k) is a strictly monotone increasing sequence of integers and G is a compact Abelian group such that its dual group $\widehat{G}$ contains infinitely many multiple torsion. Then there exists a measurable $f \notin L^1(G)$ such that*

$$m(\Gamma_{f,0}) = m(\Gamma_{f,b}) = 1, \text{ where } m \text{ is the Haar-measure on } G. \tag{19}$$

In fact, we show that $\Gamma_{f,0} = \Gamma_{f,b} = G$.

Proof. First suppose that in Definition 3 property (i) holds for $\widehat{G}$. Then for any k we can select a subgroup $\widehat{G}_k$ in $\widehat{G}$ such that it is isomorphic to $\underbrace{(Z/p) \times (Z/p) \times \dots \times (Z/p)}_{k \text{ many times}}$. Suppose that the characters $\gamma_1, \dots, \gamma_k$ are the generators of $\widehat{G}_k$.

Put $H_k = \bigcap_{j=1}^{k} \gamma_j^{-1}(1)$. Then H_k is a closed subgroup of G. Since $y \in x + H_k$, that is $y - x \in H_k$ if and only if $\gamma_j(y) = \gamma_j(x)$ for $j = 1, \dots, k$, which means that $\gamma_j(y - x) = \gamma_j(y)/\gamma_j(x) = 1$ for $j = 1, \dots, k$ one can see that G is tiled with p^k many translated copies of H_k. The sets $x + H_k$ are all closed and therefore H_k is a closed-open subgroup of G.

We also have

$$m(H_k) = \frac{1}{p^k}. \tag{20}$$

Set $f_k(x) = p^k$ if $x \in H_k$ and $f_k(x) = 0$ otherwise.

Put $f = \sum_{k=1}^{\infty} f_k$. By the Borel-Cantelli lemma and (20) the function f is m a.e. finite. It is also clear that f is measurable and $f \notin L^1(G)$.

Suppose $\alpha \in G$ is arbitrary. Set $X_k = \bigcup_{j=0}^{p-1} H_k - j\alpha$. Then $m(X_k) = p^{-k+1}$ and by the Borel-Cantelli lemma m a.e. x belongs to only finitely many X_k. If $x \notin X_k$ then $\forall j \in \mathbb{N}$, $x + j\alpha \notin H_k$ and hence

$$f_k(x + j\alpha) = 0 \text{ for any } j \in \mathbb{N}. \tag{21}$$

Therefore, $\frac{f(x+n_k\alpha)}{k} \to 0$ for m a.e. $x \in G$ and $\Gamma_{f,0} = G$.

If in Definition 3 property (ii) holds for $\widehat{G}$ then for any k select $\widehat{G}_k$ in $\widehat{G}$ such that it is isomorphic to $(Z/p_k) \times (Z/p_k)$. We suppose that $\gamma_{1,k}$ and $\gamma_{2,k}$ are the generators of $\widehat{G}_k$. Put $H_k = \gamma_{1,k}^{-1}(1) \cap \gamma_{2,k}^{-1}(1)$. One can see, similary to the previous case, that G is tiled by p_k^2 many translated copies of H_k. Turning to a subsequence if necessary, we can suppose that

$$\sum_{k=1}^{\infty} \frac{1}{p_k} < +\infty. \tag{22}$$

We also have

$$m(H_k) = \frac{1}{p_k^2}. \tag{23}$$

Set $f_k(x) = p_k^2$ if $x \in H_k$ and $f_k(x) = 0$ otherwise.

Put $f = \sum_{k=1}^{\infty} f_k$. Again, it is clear that f is m a.e. finite, measurable and $f \notin L^1(G)$. For an arbitrary $\alpha \in G$ one can define $X_k = \cup_{j=0}^{p_k-1} H_k - j\alpha$. Then $m(X_k) = \frac{1}{p_k}$.

From (22) and from the Borel-Cantelli lemma it follows that m a.e. x belongs to only finitely many X_k. One can conclude the proof as we did it in the previous case. □

It is natural to ask for a version of Theorem 4 for the non-conventional ergodic averages with $m(\Gamma_f) = 1$ in (19). For convergence of the non-conventional ergodic averages some arithmetic assumptions about n_k are

needed.
We recall from [10] Definition 1.2 with some notational adjustment.

Definition 5. . The sequence (n_k) is ergodic mod q if for any $h \in \mathbb{Z}$

$$\lim_{N\to\infty} \frac{\sum_{k=0}^{N} \chi_{h,q}(n_k)}{N+1} = \frac{1}{q}, \tag{24}$$

Where $\chi_{h,q}(x) = 1$ if $x \equiv h \bmod q$ and $\chi_{h,q}(x) = 0$ otherwise.
A sequence (n_k) is ergodic for periodic systems if it is ergodic mod q for every $q \in \mathbb{N}$.

For ergodic sequences with essentially the same proof we can state the following version of Theorem 4:

Theorem 6. *Suppose that n_k is a strictly monotone, ergodic sequence for periodic systems and G is a compact Abelian group such that its dual group $\widehat{G}$ contains infinitely many multiple torsion. Then there exists a measurable $f \notin L^1(G)$such that $\Gamma_f = G$, and hence $m(\Gamma_f) = 1$.*

Proof. As we mentioned earlier the argument of the proof of Theorem 4 is applicable. One needs to add the observation that if $x \in X_k$ then the ergodicity of n_k for periodic systems implies that $M_N^\alpha f_k$ converges. If $x \notin X_k$ then (21) can be used. Hence $M_N^\alpha f$ converges for all $\alpha \in G$ for a.e. x. $\square$

In Theorem 4 we saw that if there is "lots of torsion" in $\widehat{G}$, that is, G is "highly disconnected" then there are measurable functions f not in L^1 for which $m(\Gamma_{f,0}) = 1$. Since the p-adic integers, Z_p are the building blocks of 0-dimensional compact Abelian groups ([8, Theorem 25.22]) it is natural to consider them. If we take a countable product of Z_p with p fixed then the dual group will be the direct sum of $Z(p^\infty)$'s and will contain a subgroup algebraically isomorphic to the direct sum $(Z/p)\bigoplus(Z/p)\bigoplus \dots$. Then Theorem 4 is applicable.

If one considers an individual Z_p then its dual group is $Z(p^\infty)$ with all elements of finite order, so still there seems to be "lots of torsion" in the dual group. It is also clear that arithmetic properties of n_k might matter if we consider Z_p. For us it was quite surprising that if one considers ordinary ergodic averages, that is, $n_k = k$ then Z_p behaves like a locally connected group and the following theorem is true.

Theorem 7. *Suppose that $n_k = k$, and p is a fixed prime number. We consider $G = Z_p$, the group of p-adic integers. Then for any measurable function $f : G \to \mathbb{R}$ from $m(\Gamma_{f,b}) > 0$ it follows that $f \in L^1(G)$.*

Before turning to the proof of Theorem 7 we need some notation and a Claim simplifying the proof of Theorem 7. Denote by $\Gamma^j_{f,b}$, $j=-1,0,1,\dots$ the set of those $\alpha=(\alpha_0,\alpha_1,\dots)\in\Gamma_{f,b}$ for which $\alpha_{j+1}\neq 0$ but $\alpha_0=\dots=\alpha_j=0$. From $m(\Gamma_{f,b})>0$ it follows that there exists j_0 such that $m(\Gamma^{j_0}_{f,b})>0$. Given a finite string $(x_0,\dots,x_j)$ we denote by $[x_0,\dots,x_j]$ the corresponding cylinder set in G, that is,

$$[x_0,\dots,x_j]=\{(x'_0,x'_1,\dots)\in G:(x'_0,\dots,x'_j)=(x_0,\dots,x_j)\}.$$

Claim 8. *If from $m(\Gamma^{-1}_{f,b})>0$ it follows that $f\in L^1(G)$, then Theorem 7 is also true.*

Proof. As mentioned above if $m(\Gamma_{f,b})>0$ then we can choose j_0 such that $m(\Gamma^{j_0}_{f,b})>0$. Then for $\alpha\in\Gamma^{j_0}_{f,b}$ for any cylinder $[x_0,\dots,x_{j_0}]$ we have $[x_0,\dots,x_{j_0}]+\alpha=[x_0,\dots,x_{j_0}]$. If σ is the one-sided shift on Z_p, that is, $\sigma(x_0,x_1,\dots)=(x_1,\dots)$ then for $\alpha\in\Gamma^{j_0}_{f,b}$ we have $\sigma^{j_0+1}(x+\alpha)=\sigma^{j_0+1}x+\sigma^{j_0+1}\alpha$.

For an $x'\in G$ we define the function $f_{x_0,\dots,x_{j_0}}(x')=f(x_0,\dots,x_{j_0},x')$, where $(x_0,\dots,x_{j_0},x')$ is the concatenation of the finite string $(x_0,\dots,x_{j_0})$ and $x'\in G=Z_p$. Then $\Gamma^{-1}_{f_{x_0,\dots,x_{j_0}},b}\supset\sigma^{j_0+1}(\Gamma^{j_0}_{f,b})$ and we can apply the Claim for $f_{x_0,\dots,x_{j_0}}$ to obtain that $f_{x_0,\dots,x_{j_0}}\in L^1(G)$, that is, $f\in L^1([x_0,\dots,x_{j_0}])$. Since this holds for any cylinder set $[x_0,\dots,x_{j_0}]$ we obtain that $f\in L^1(G)$. □

Proof of Theorem 7. By Claim 8 we can assume that $m(\Gamma^{-1}_{f,b})>0$. We need to adjust the proof of Theorem 1 for the case of $G=Z_p$. The key difficulty is the torsion in $\widehat{G}=Z(p^\infty)$ which makes it impossible to use a direct argument which lead to (16). Anyway, we start to argue as in the proof of Theorem 1, keeping in mind that now $n_k=k$. We introduce the sets $G_{\alpha,K}$, $B\subset\Gamma^{-1}_{f,b}$, $L_k(f)$ as in (2), (3), and (4), respectively. We fix K and define the set H_α and the auxiliary function $h(x,\alpha)$ as in (5). We have (6) again.

Our aim is to establish that for a suitable κ_0

$$\sum_{\kappa\ge\kappa_0}p^\kappa m(L_{p^{\kappa+2}\cdot K}(f))<\infty. \tag{25}$$

Suppose that the function φ equals $p^{\kappa+3}\cdot K$ on $L_{p^{\kappa+2}\cdot K}(f)\backslash L_{p^{\kappa+3}\cdot K}(f)$, $\kappa=\kappa_0,\kappa_0+1,\dots$ and equals $K\cdot p^{\kappa_0+2}$ on $G\backslash L_{p^{\kappa_0+2}\cdot K}(f)$. Then $\varphi\ge|f|$ and by (25)

$$\int_G\varphi dm\le K\cdot p^{\kappa_0+2}m(G)+\sum_{\kappa=\kappa_0}^{\infty}p^{\kappa+3}\cdot Km(L_{p^{\kappa+2}\cdot K}(f))<+\infty. \tag{26}$$

This implies that $f\in L^1(G)$.

Hence we need to establish (25). Choose and fix $\kappa_0 \in \mathbb{N}$ such that $p^{\kappa_0} > K$ and suppose that $\kappa \geq \kappa_0$.

Then, keeping in mind that $L_{k\cdot K}(f) \supset L_{p^{\kappa+2}.K}(f)$ for $k \leq p^{\kappa+2}$ we have instead of (7)

$$h(x - k\alpha, \alpha) = 1 \text{ for any } \alpha \in B,\ K < k < p^{\kappa+2} \text{ and } x \in L_{p^{\kappa+2}.K}(f). \tag{27}$$

Let

$$h_\kappa(x, \alpha) = \frac{1}{p^\kappa} \sum_{k=p^\kappa}^{2p^\kappa - 1} h(x - k\alpha, \alpha). \tag{28}$$

Then by (27)

$$h_\kappa(x - k\alpha, \alpha) = 1 \text{ for any } \alpha \in B,\ 0 \leq k < p^{\kappa+2} - 2p^\kappa \text{ and } x \in L_{p^{\kappa+2}.K}(f) \tag{29}$$

Taking average on B

$$\frac{1}{m(B)} \int_B h_\kappa(x - k\alpha, \alpha) dm(\alpha) = 1 \tag{30}$$

$$\text{for } \kappa \geq \kappa_0,\ 0 \leq k < p^{\kappa+2} - 2p^\kappa \text{ and } x \in L_{p^{\kappa+2}.K}(f).$$

Now we return to $h(x, \alpha)$ and we define $c_\gamma(\alpha)$ as in (9). Again, $c_\gamma(\alpha)$ is a bounded, measurable function and (10) holds.
Denoting again by $\gamma_0(x)$ the identically 1 character, the neutral element of $\widehat{G}$ we also have (11) satisfied. For $h_\kappa(x, \alpha)$ we have

$$h_\kappa(x, \alpha) \sim \sum_{\gamma \in \widehat{G}} c_{\gamma,\kappa}(\alpha)\gamma(x) = \sum_{\gamma \in \widehat{G}} c_\gamma(\alpha) \left(\frac{1}{p^\kappa} \sum_{k=p^\kappa}^{2p^\kappa - 1} \gamma(-k\alpha) \right) \gamma(x). \tag{31}$$

Since $\widehat{G} = Z(p^\infty)$, the order of γ is a power of p. We denote it by $\mathrm{ord}(\gamma)$. A $\gamma \in \widehat{G}$ of order p^r, $r > 0$ is of the form

$$\begin{gathered} \gamma(x) = \exp\left(\frac{2\pi i l}{p^r} (x_0 + px_1 + \cdots + p^{r-1}x_{r-1}) \right) \\ \text{for } x = (x_0, x_1, \dots) \in G = Z_p \text{ with } l \text{ not divisible by } p. \end{gathered} \tag{32}$$

Since $B \subset \Gamma_{f,b}^{-1}$, for $\alpha \in B$ we have $\alpha_0 \neq 0$ which implies $\gamma(-\alpha) \neq 1$ and if γ is of order p^r, $r > 0$ then $\gamma(-\alpha) \in \mathbb{C}$ is also of order p^r, $r > 0$. Hence for $\mathrm{ord}(\gamma) = p^r \leq p^\kappa$ and $\alpha \in B$ we have

$$\sum_{k=p^\kappa}^{2p^\kappa - 1} \gamma(-k\alpha) = \sum_{k=p^\kappa}^{2p^\kappa - 1} \gamma^k(-\alpha) = \gamma(-p^\kappa\alpha) \frac{1 - \gamma^{p^\kappa}(-\alpha)}{1 - \gamma(-\alpha)} = 0. \tag{33}$$

This way we can get rid of some characters with small torsion in the Fourier-series of $h_\kappa(x,\alpha)$.

Recalling that $c_{\gamma_0}(\alpha) = \int_G h(x,\alpha)\cdot 1 dm(\alpha) = 0$ by (10) we have in (31)

$$c_{\gamma_0,\kappa}(\alpha) = 0 \text{ if } \alpha \in B. \tag{34}$$

Using (31) again we have

$$h_\kappa(x-k\alpha,\alpha) \sim \sum_{\gamma\in\widehat{G}} c_{\gamma,\kappa}(\alpha)\gamma(-k\alpha)\gamma(x) \tag{35}$$

and by (30) for any $0 \le k < p^{\kappa+2} - 2p^\kappa$

$$\begin{aligned} m(L_{p^{\kappa+2}\cdot K}(f)) &\le \int_G \left| \frac{1}{m(B)} \int_B h_\kappa(x-k\alpha,\alpha)dm(\alpha) \right|^2 dm(x) \\ &= \int_G |\varphi_{\kappa,k}(x)|^2 dm(x), \end{aligned} \tag{36}$$

where $\varphi_{\kappa,k}(x) = \dfrac{1}{m(B)} \displaystyle\int_B h_\kappa(x-k\alpha,\alpha)dm(\alpha)$ is a bounded measurable function.

Recall that by (31) we can express the Fourier-coefficients of h_κ by those of h, that is

$$c_{\gamma,\kappa}(\alpha) = \int_G h_\kappa(x,\alpha)\gamma(-x)dm(x) = c_\gamma(\alpha)\frac{1}{p^\kappa}\sum_{k=p^\kappa}^{2p^\kappa-1}\gamma(-k\alpha). \tag{37}$$

Therefore,

$$\begin{aligned} \widehat{\varphi}_{\kappa,k}(\gamma) &= \int_G \frac{1}{m(B)} \int_B h_\kappa(x-k\alpha,\alpha)dm(\alpha)\gamma(-x)dm(x) \\ &= \frac{1}{m(B)} \int_B \int_G h_\kappa(x-k\alpha,\alpha)\gamma(-x)dm(x)dm(\alpha) \\ &= \frac{1}{m(B)} \int_G \chi_B(\alpha) \cdot \int_G h_\kappa(u,\alpha)\gamma(-u-k\alpha)dm(u)dm(\alpha) \\ &= \frac{1}{m(B)} \int_G \chi_B(\alpha)\gamma(-k\alpha)c_{\gamma,\kappa}(\alpha)dm(\alpha). \end{aligned} \tag{38}$$

If $\gamma \ne \gamma_0$ and $\mathrm{ord}(\gamma) \le p^\kappa$ then by (33) and (37) we have $c_{\gamma,\kappa}(\alpha) = 0$ for any $\alpha \in B$, and hence $\widehat{\varphi}_{\kappa,k}(\gamma) = 0$.

Recall from (34) that if $\alpha \in B$ then $c_{\gamma_0,\kappa}(\alpha) = 0$. Hence $\widehat{\varphi}_{\kappa,k}(\gamma_0) = 0$ holds in this case as well.

Now suppose that $\gamma^{p^\kappa} \neq \gamma_0$. Then $\mathrm{ord}(\gamma) \geq p^{\kappa+1}$ and for $k = 0, \ldots, p^{\kappa+1} - 1$ the characters γ^k are different.

By using the Parseval-formula we can continue (36) to obtain for any $0 \leq k < p^{\kappa+2} - 2p^\kappa$ that

$$m(L_{p^{\kappa+2}\cdot K}(f)) \leq \sum_{\gamma \in \widehat{G}} |\widehat{\varphi}_{\kappa,k}(\gamma)|^2 \tag{39}$$

$$= \sum_{\gamma \in \widehat{G},\, \gamma^{p^\kappa} \neq \gamma_0} \frac{1}{(m(B))^2} \cdot \left| \int_G \chi_B(\alpha)\gamma(-k\alpha)c_{\gamma,\kappa}(\alpha)dm(\alpha) \right|^2 .$$

Since $p \geq 2$ implies $p^{\kappa+2} \geq 3p^\kappa$ we can use (29) and (39) for $k = 0, ..., p^\kappa - 1$. Adding equation (39) for all $\kappa \geq \kappa_0$ and $k = 0, ..., p^\kappa - 1$ we need to estimate

$$\sum_{\kappa \geq \kappa_0} p^\kappa m(L_{p^{\kappa+2}\cdot K}(f)) \tag{40}$$

$$\leq \sum_{\kappa \geq \kappa_0} \sum_{\gamma \in \widehat{G}, \gamma^{p^\kappa} \neq \gamma_0} \frac{1}{(m(B))^2} \sum_{k=0}^{p^\kappa - 1} \left| \int_G \chi_B(\alpha)c_{\gamma,\kappa}(\alpha)\gamma(-k\alpha)dm(\alpha) \right|^2 .$$

Using (31) and (37) first we estimate for $\kappa \geq \kappa_0$

$$\sum_{k=0}^{p^\kappa - 1} \left| \int_G \chi_B(\alpha)c_{\gamma,\kappa}(\alpha)\gamma(-k\alpha)dm(\alpha) \right|^2 \tag{41}$$

$$= \sum_{k=0}^{p^\kappa - 1} \left| \int_G \chi_B(\alpha)c_\gamma(\alpha)\frac{1}{p^\kappa} \sum_{k'=p^\kappa}^{2p^\kappa - 1} \gamma(-(k'+k)\alpha)dm(\alpha) \right|^2 = **$$

in the last expression $k' + k$ can take values between p^κ and $3p^\kappa - 2$. If $p \geq 3$ then $3p^\kappa - 2 \leq p^{\kappa+1} - 1$ so for the moment we suppose that $p \geq 3$. In the end of this proof we will point out the little adjustments which we need for the case $p = 2$.

For $p^\kappa \leq j \leq 3p^\kappa - 2 \leq p^{\kappa+1} - 1$ we denote by w'_j the number of those couples (k, k') for which $0 \leq k \leq p^\kappa - 1$, $p^\kappa \leq k' \leq 2p^\kappa - 1$ and $k + k' = j$. Obviously, $w'_j \leq p^\kappa$. Set $w_j = w'_j/p^\kappa \leq 1$. We select these w_j for all $\kappa_0 \leq \kappa \leq \mathrm{ord}(\gamma)$. For those values of j for which we have not defined w_j yet we set $w_j = 0$.

By using this notation we can continue $**$ from (41)

$$** \leq \sum_{j=p^\kappa}^{p^{\kappa+1}-1} w_j \left| \int_G \chi_B(\alpha)c_\gamma(\alpha) \cdot \gamma(-j\alpha)dm(\alpha) \right|^2 \tag{42}$$

$$\leq \sum_{j=p^\kappa}^{p^{\kappa+1}-1} \left| \int_G \chi_B(\alpha) c_\gamma(\alpha) \cdot \gamma(-j\alpha) dm(\alpha) \right|^2 .$$

Using (41) and (42) while continuing the estimation of (40) we obtain

$$\sum_{\kappa \geq \kappa_0} p^\kappa m(L_{p^{\kappa+2}\cdot K}(f)) \leq$$

$$\leq \sum_{\kappa \geq \kappa_0} \sum_{\gamma \in \widehat{G}, \gamma^{p^\kappa} \neq \gamma_0} \frac{1}{(m(B))^2} \sum_{j=p^\kappa}^{p^{\kappa+1}-1} \left| \int_G \chi_B(\alpha) c_\gamma(\alpha) \gamma(-j\alpha) dm(\alpha) \right|^2 \tag{43}$$

$$\leq \sum_{\gamma \in \widehat{G}} \sum_{j=1}^{\mathrm{ord}(\gamma)-1} \frac{1}{(m(B))^2} \cdot \left| \int_G \chi_B(\alpha) c_\gamma(\alpha) \gamma(-j\alpha) dm(\alpha) \right|^2 .$$

Since for a fixed γ the characters γ^{-j} are different, for different values $0 \leq j < \mathrm{ord}(\gamma)$ by Parseval's Theorem we infer

$$\sum_{j=1}^{\mathrm{ord}(\gamma)-1} \left| \int_G \chi_B(\alpha) c_\gamma(\alpha) \gamma(-j\alpha) dm(\alpha) \right|^2 \leq \int_G |\chi_B(\alpha) c_\gamma(\alpha)|^2 \, dm(\alpha). \tag{44}$$

Using this in (43) we obtain

$$\sum_{\kappa \geq \kappa_0} p^\kappa m(L_{p^{\kappa+2}\cdot K}(f)) \leq \frac{1}{(m(B))^2} \sum_{\gamma \in \widehat{G}} \int_G |\chi_B(\alpha) c_\gamma(\alpha)|^2 dm(\alpha)$$

$$= \frac{1}{(m(B))^2} \int_G \chi_B(\alpha) \sum_{\gamma \in \widehat{G}} |c_\gamma(\alpha)|^2 dm(\alpha) \tag{45}$$

$$= \frac{1}{(m(B))^2} \int_G \chi_B(\alpha) \int_G |h(x,\alpha)|^2 dm(x) dm(\alpha) < +\infty.$$

This completes the proof if $p \geq 3$.

In case of $p = 2$ the intervals $p^\kappa \leq j \leq 3p^\kappa - 2$ are not disjoint, but $3p^\kappa - 2 \leq p^{\kappa+2} - 1$. Instead of (43) we could obtain

$$\sum_{\kappa \geq \kappa_0} p^{\kappa+1} m(L_{p^{\kappa+1}\cdot K}(f)) \leq 2 \cdot \sum_{\gamma \in \widehat{G}} \sum_{j=1}^{2\mathrm{ord}(\gamma)-1} \frac{1}{(m(B))^2} \left| \int_G \chi_B(\alpha) c_\gamma(\alpha) \gamma(-j\alpha) dm(\alpha) \right|^2 .$$

For a fixed γ the characters $\gamma^{-j}(\alpha)$, $j \leq 2\mathrm{ord}(\gamma) - 1$ are not different but for each $j \leq 2\mathrm{ord}(\gamma) - 1$ there is at most one other $j' \leq 2\mathrm{ord}(\gamma) - 1$ such that $\gamma^{-j} = \gamma^{-j'}$, hence

$$\sum_{j=1}^{2\mathrm{ord}(\gamma)-1} \left| \int_G \chi_B(\alpha) c_\gamma(\alpha) \gamma(-j\alpha) dm(\alpha) \right|^2 \leq 2 \int_G |\chi_B(\alpha) c_\gamma(\alpha)|^2 dm(\alpha).$$

The conclusion of the proof is similar to the $p \geq 3$ case. $\square$

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