Convergence of ergodic averages for many group rotations

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November 3, 2014

Abstract

Suppose that $G$ is a compact Abelian topological group, $m$ is the Haar measure on $G$ and $f : G \to \mathbb{R}$ is a measurable function. Given $(n_k)$, a strictly monotone increasing sequence of integers we consider the nonconventional ergodic/Birkhoff averages

$$M_N^\alpha f(x) = \frac{1}{N+1} \sum_{k=0}^{N} f(x + n_k \alpha).$$

The $f$-rotation set is

$$\Gamma_f = \{ \alpha \in G : M_N^\alpha f(x) \text{ converges for } m \text{ a.e. } x \text{ as } N \to \infty. \}$$

∗This author was supported by the Hungarian National Foundation for Scientific Research K075242.
†This author was supported by the Hungarian National Foundation for Scientific Research K104178.

2010 Mathematics Subject Classification: Primary 22D40; Secondary 37A30, 28D99, 43A40.

Keywords: Birkhoff average, locally compact Abelian group, torsion, $p$-adic integers
We prove that if $G$ is a compact locally connected Abelian group and $f : G \to \mathbb{R}$ is a measurable function then from $m(\Gamma_f) > 0$ it follows that $f \in L^1(G)$.

A similar result is established for ordinary Birkhoff averages if $G = \mathbb{Z}_p$, the group of $p$-adic integers.

However, if the dual group, $\hat{G}$ contains “infinitely many multiple torsion” then such results do not hold if one considers non-conventional Birkhoff averages along ergodic sequences.

What really matters in our results is the boundedness of the tail, $f(x + nk\alpha)/k, k = 1, \ldots$ for a.e. $x$ for many $\alpha$, hence some of our theorems are stated by using instead of $\Gamma_f$ slightly larger sets, denoted by $\Gamma_{f,b}$.

1 Introduction

The starting point of this paper is a result of the first listed author in [3] which states that if $f$ is a (Lebesgue) measurable function on the unit circle $\mathbb{T}$ and $\Gamma_f$ denotes the set of those $\alpha$’s for which the Birkhoff averages

$$M_\alpha^n f(x) = \frac{1}{n+1} \sum_{k=0}^{n} f(x + k\alpha)$$

converge for almost every $x$ then from $m(\Gamma_f) > 0$ it follows that $f \in L^1(\mathbb{T})$. Hence $M_\alpha^n f$ converges for all $\alpha \in \mathbb{T}$.

In this paper we consider generalizations of this result to compact Abelian groups equipped with their Haar measure $m$. Theorem 1 implies that an analogous result is true even for non-conventional ergodic averages considered on a compact, locally connected Abelian group $G$.

On the other hand, if there is “sufficiently many multiple torsion” in the dual group $\hat{G}$ then Theorem 6 implies that there are non-$L^1$ measurable functions $f$ for which $m(\Gamma_f) = 1$ (in fact, $\Gamma_f = G$) if one considers non-conventional Birkhoff averages along ergodic sequences. Having lots of torsion in $\hat{G}$ means that $G$ is highly disconnected. In our opinion the most surprising result of this paper is Theorem 7 which states that if $G = \mathbb{Z}_p$, the group of $p$-adic integers and one considers the ordinary ergodic averages of a measurable function $f$ then from $m(\Gamma_f) > 0$ it follows that $f \in L^1(G)$. The group $\mathbb{Z}_p$ is zero-dimensional and all elements of its dual group, $\mathbb{Z}(p^\infty)$, are of finite order. If one considers a group $G$ which is a countable product of $\mathbb{Z}_p$’s then there is enough “multiple torsion” (see Definition 3) in $\Gamma_f$ and Theorem 6 implies that the result of Theorem 7 does not hold in these groups. If $M_\alpha^n f(x)$ converges then the tail $\frac{f(x + n\alpha)}{n} \to 0$. In our proofs the sets $\Gamma_{f,0}$
(and $\Gamma_{f,b}$), the sets of those $\alpha$’s where $f(x + n\alpha) \to 0$, (or $|f(x + n\alpha)|$ is bounded) for a.e. $x$ play an important role. Since $\Gamma_f \subset \Gamma_{f,0} \subset \Gamma_{f,b}$ from $m(\Gamma_f) > 0$ it follows that the other sets are also of positive measure and hence in the statements of Theorems 1 and 7 these sets are used. Again the tail of the ergodic averages plays an important role, like in [1], where we showed that for $L^1$ functions and ordinary ergodic averages the return time property for the tail may might fail and hence Bourgain’s return time property [2] does not hold in these situations.

The proof of Theorem 1 is a rather straightforward generalization of Theorem 1 in [3]. We provide its details, since they are also used with some non-trivial modifications in the proof of Theorem 7.

Next we say a few words about the background history and related questions to this paper. Answering a question raised by the first listed author of this paper P. Major in [9] constructed two ergodic transformations $S, T : X \to X$ on a probability space $(X, \mu)$ and a measurable function $f : X \to \mathbb{R}$ such that for $\mu$ a.e. $x$

$$\lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} f(S^k x) = 0,$$

and

$$\lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} f(T^k x) = a \neq 0.$$ 

M. Laczkovich raised the question whether $S$ and $T$ can be irrational rotations of $\mathbb{T}$. In Major’s example $S$ and $T$ are conjugate. Therefore, his method did not provide an answer to Laczkovich’s question.

The results of Z. Buczolich in [4] imply that for any two independent irrationals $\alpha$ and $\beta$ one can find a measurable $f : \mathbb{T} \to \mathbb{R}$ such that $M_n^\alpha f(x) \to c_1$ and $M_n^\beta f(x) \to c_2$ for a.e. $x$ with $c_1 \neq c_2$. In this case by Birkhoff’s ergodic theorem $f \not\in L^1(\mathbb{T})$. It is shown in [3] that for any sequence $(\alpha_j)$ of independent irrationals one can find a measurable $f : \mathbb{T} \to \mathbb{R}$ such that $f \not\in L^1(\mathbb{T})$, but $\alpha_j \in \Gamma_f$ for all $j = 1,\ldots$. By Theorem 1 of [3] from $f \not\in L^1(\mathbb{T})$ it follows that $m(\Gamma_f) = 0$. It was a natural question to see how large $\Gamma_f$ could be for an $f \not\in L^1(\mathbb{T})$. In [14] R. Svetic showed that $\Gamma_f$ can be $c$-dense for an $f \not\in L^1(\mathbb{T})$.

The question about the possible largest Hausdorff dimension of $\Gamma_f$ for an $f \not\in L^1(\mathbb{T})$ remained open for a while until in [5] it was shown that there are $f \not\in L^1(\mathbb{T})$ such that $\dim_H(\Gamma_f) = 1$ (of course with $m(\Gamma_f) = 0$.)

For us motivation to consider non-conventional ergodic averages in this paper came from the project in [6] concerning almost everywhere convergence questions of Birkhoff averages along the squares.

It is also worth mentioning that ergodic averages of non-$L^1$ functions and rotations on $\mathbb{T}$ were also considered in [13] and [12].
2 Preliminaries

We suppose that $G$ is a compact Abelian topological group, the group operation will be addition. The dual group of the compact Abelian topological group $G$ is denoted by $\widehat{G}$. By Pontryagin duality $\widehat{G}$ is a discrete Abelian group. For $\gamma \in \widehat{G}$ the corresponding Fourier coefficient is

$$\widehat{f}(\gamma) = \int_{G} g(x) \gamma(-x)dm(x),$$

where $m$ denotes the Haar measure on $G$. By the Parseval formula

$$\int_{G} f(x)\overline{g(x)}dm(x) = \sum_{\gamma \in \widehat{G}} \widehat{f}(\gamma)\overline{\widehat{g}(\gamma)} \text{ for } f, g \in L^2(G).$$

By [8, 24.25] or [11, 2.5.6 Theorem] if $G$ is a compact Abelian group then $G$ is connected if and only if $\widehat{G}$ is torsion-free.

Suppose that $p_1, p_2, \ldots$ is a sequence of prime numbers. Recall that the direct product $G = (Z/p_1) \times (Z/p_2) \times \ldots$ is compact and its dual group $\widehat{G} = (Z/p_1) \bigoplus (Z/p_2) \bigoplus \ldots$ is the direct sum with the discrete topology see [11, 2.2 p.36] or [8].

We denote by $Z_p$ the group of $p$-adic integers and its dual group, the Prüfer $p$-group with the discrete topology will be denoted by $Z(p^\infty)$.

For other properties of topological groups we refer to standard textbooks like [7], [8] or [11].

Suppose that $f : G \to \mathbb{R}$ is a measurable function. We suppose that the group rotation $T_\alpha = x + \alpha, \; \alpha \in G$ is fixed.

Given a strictly monotone increasing sequence of integers $(n_k)$ we consider the nonconventional ergodic averages

$$M_N^\alpha f(x) = \frac{1}{N+1} \sum_{k=0}^{N} f(x + n_k \alpha).$$

Of course, if $n_k = k$ we have the usual Birkhoff averages.

The $f$-rotation set is

$$\Gamma_f = \{ \alpha \in G : M_N^\alpha f(x) \text{ converges for m a.e. } x \text{ as } N \to \infty \}.$$
Scrutinizing the proof of this result one can see that the set

$$
\Gamma_{f,0} = \left\{ \alpha \in G : \frac{f(x + n_k \alpha)}{k} \to 0 \text{ for m.a.e. } x \right\}
$$

played an important role. It is obvious that $\Gamma_f \subset \Gamma_{f,0}$.

In [3] it was shown that from $m(\Gamma_{f,0}) > 0$ it follows that $f \in L^1(\mathbb{T})$, when the sequence $n_k = k$ is considered. In this paper we will also use the slightly larger set

$$
\Gamma_{f,b} = \left\{ \alpha \in G : \limsup_{k \to \infty} \frac{|f(x + n_k \alpha)|}{k} < \infty \text{ for m.a.e. } x \right\}. \quad (1)
$$

### 3 Main results

First we generalize Theorem 1 of [3] for compact, locally connected Abelian groups.

**Theorem 1.** If $(n_k)$ is a strictly monotone increasing sequence of integers and $G$ is a compact, locally connected Abelian group and $f : G \to \mathbb{R}$ is a measurable function then from $m(\Gamma_{f,b}) > 0$ it follows that $f \in L^1(G)$.

**Remark 2.** Since $\Gamma_{f,b} \supset \Gamma_{f,0} \supset \Gamma_f$ Theorem 1 implies that if one considers the non-conventional ergodic averages $M_{N}^\alpha f$ on a locally compact Abelian group for group rotations and $m(\Gamma_f) > 0$ then $f \in L^1(G)$.

**Proof.** Set $n_0 = 0$. First we suppose that $G$ is connected. Given an integer $K$ put

$$
G_{\alpha,K} = \{ x : |f(x + n_k \alpha)| < K \cdot k \text{ for every } k > K \}
$$

and $|f(x + n_k \alpha)| < K^2$ for $k = 0, \ldots, K$.

If $\alpha \in \Gamma_{f,b}$ then $m(G_{\alpha,K}) \to 1$ as $K \to \infty$.

Choose and fix $K$ and $\varepsilon > 0$ such that the set

$$
B = \{ \alpha : m(G_{\alpha,K}) > \varepsilon \}
$$

is of positive $m$-measure. From the measurability of $f$ it follows that $B$ and the sets $g_{\alpha,K}$ are also measurable.

Set

$$
L_k(f) = \{ x \in G : |f(x)| > k \}. \quad (4)
$$

From $k > K$ and $x \in G_{\alpha,K} + n_k \alpha$ it follows that

$$
|f(x)| = |f(x - n_k \alpha + n_k \alpha)| < k \cdot K.
$$
Set $H_\alpha = G \setminus G_{\alpha,K}$, (keep in mind that $K$ is fixed). From $k > K$ and $x \in L_{k,K}(f)$ it follows that $x \notin G_{\alpha,K} + n_k \alpha$, that is, $x \in H_\alpha + n_k \alpha$.

For $\alpha \in B$ we set $a(\alpha) = m(H_\alpha) < 1 - \varepsilon$, by (3). This implies $1/(1 - a(\alpha)) < 1/\varepsilon$.

For $\alpha \in B$ put

$$h(x, \alpha) = \begin{cases} 1 & \text{if } x \in H_\alpha, \\ -\left(\frac{a(\alpha)}{1-a(\alpha)}\right) & \text{if } x \notin H_\alpha. \end{cases}$$

(5)

For $\alpha \notin B$ set $h(x, \alpha) = 0$ for any $x \in G$.

Then $h(x, \alpha)$ is a bounded measurable function defined on $G \times G$ and

$$\int_G h(x, \alpha)dm(x) = 0 \text{ for any } \alpha \in G.$$ (6)

From $k > K$ and $x \in L_{k,K}(f)$ it follows that $x \in H_\alpha + n_k \alpha$ for any $\alpha \in B$. This implies

$$h(x - n_k \alpha, \alpha) = 1 \text{ for any } x \in L_{k,K}(f) \text{ and } \alpha \in B.$$ (7)

Taking average

$$\frac{1}{m(B)} \int_B h(x - n_k \alpha, \alpha)dm(\alpha) = 1 \text{ for } k > K \text{ and } x \in L_{k,K}(f).$$ (8)

Keep $\alpha$ fixed and select a character $\gamma \in \hat{G}$. Consider in the Fourier-series of $h(x, \alpha)$ the coefficient $c_\gamma(\alpha)$ corresponding to this character, that is,

$$c_\gamma(\alpha) = \int_G h(x, \alpha)\gamma(-x)dm(x).$$ (9)

Since $h(x, \alpha)$ is a bounded measurable function, the function $c_\gamma(\alpha)$ is also bounded and measurable. Then

$$h(x, \alpha) \sim \sum_{\gamma \in \hat{G}} c_\gamma(\alpha)\gamma(x).$$ (10)

If $\gamma_0(x) \equiv 1$ then by (6) we have

$$c_{\gamma_0}(\alpha) = 0 \text{ for any } \alpha \in G.$$ (11)

For a fixed $\alpha \in B$ we have

$$h(x - n_k \alpha, \alpha) \sim \sum_{\gamma \in \hat{G}} c_\gamma(\alpha)\gamma(-n_k \alpha)\gamma(x).$$ (12)
By (8)

\[ m(L_{k,K}(f)) \leq \int_G \left| \frac{1}{m(B)} \int_B h(x-n_k\alpha, \alpha)dm(\alpha) \right|^2 dm(x) \quad \text{(13)} \]

\[ = \int_G |\varphi_k(x)|^2 dm(x) = \Theta, \]

where \( \varphi_k(x) = \frac{1}{m(B)} \int_B h(x-n_k\alpha, \alpha)dm(\alpha) \) is a bounded measurable function. If \( \gamma \) is a given character then using that \( h \) is bounded and recalling (9) we obtain

\[
\hat{\varphi}_k(\gamma) = \int_G \frac{1}{m(B)} \int_B h(x-n_k\alpha, \alpha)dm(\alpha)\gamma(-x)dm(x) = \frac{1}{m(B)} \int_B \int_G h(x-n_k\alpha, \alpha)\gamma(-x)dm(x)dm(\alpha) = \frac{1}{m(B)} \int_G \chi_B(\alpha) \int_G h(u, \alpha)\gamma(-u-n_k\alpha)dm(u)dm(\alpha) = \frac{1}{m(B)} \int_G \chi_B(\alpha) \gamma(-n_k\alpha) \int_G h(u, \alpha)\gamma(-u)dm(u)dm(\alpha) = \frac{1}{m(B)} \int_G \chi_B(\alpha) \gamma(-n_k\alpha) c_\gamma(\alpha)dm(\alpha).
\]

By using the Parseval formula we can continue \( \Theta \) in (13) to obtain

\[ m(L_{k,K}(f)) \leq \sum_{\gamma \in \hat{G}} |\hat{\varphi}_k(\gamma)|^2 = \sum_{\gamma \in \hat{G}} \frac{1}{(m(B))^2} \left| \int_G \chi_B(\alpha) \gamma(-n_k\alpha)c_\gamma(\alpha)dm(\alpha) \right|^2 = \frac{1}{(m(B))^2} \sum_{\gamma \in \hat{G}} \left| \int_G \chi_B(\alpha)c_\gamma(\alpha)\gamma^{nk}(-\alpha)dm(\alpha) \right|^2 \quad \text{(15)} \]

Since \( \chi_B(\alpha)c_\gamma(\alpha) \) is a bounded measurable function and \( \gamma^{nk} \in \hat{G} \), the expression \( \int_G \chi_B(\alpha)c_\gamma(\alpha)\gamma^{nk}(-\alpha)dm(\alpha) \) is a Fourier coefficient of this function.

Now we use that \( G \) is connected and hence \( \hat{G} \) is torsion-free. If \( \gamma^{nk} = \gamma^{nk'} \) then \( \gamma^{nk-k} = \gamma_0 \equiv 1 \), but \( \gamma \) is of infinite order and hence it is only possible if \( n_k - n_{k'} = 0 \), that is \( k = k' \). Hence for \( k \neq k' \) the characters \( \gamma^{nk} \) and \( \gamma^{nk'} \) are different. By Parseval’s formula for a fixed \( \gamma \in \hat{G} \)

\[ \sum_{k=K}^{\infty} \left| \int_G \chi_B(\alpha)c_\gamma(\alpha)\gamma^{nk}(-\alpha)dm(\alpha) \right|^2 \leq \int_G |\chi_B(\alpha)c_\gamma(\alpha)|^2 dm(\alpha). \quad \text{(16)} \]
This, Parseval’s formula, (5), (9) and (15) yield

\[
\sum_{k=K+1}^{\infty} m(L_{k,K}(f)) \leq \frac{1}{(m(B))^2} \sum_{\gamma \in \hat{G}} \int_G |\chi_B(\alpha) c_\gamma(\alpha)|^2 \, dm(\alpha)
\]

\[
= \frac{1}{(m(B))^2} \int_G \chi_B(\alpha) \sum_{\gamma \in \hat{G}} |c_\gamma(\alpha)|^2 \, dm(\alpha)
\]

(17)

\[
= \frac{1}{(m(B))^2} \int_G \chi_B(\alpha) \int_G |h(x,\alpha)|^2 dm(x) dm(\alpha) < \infty.
\]

Since \( \int_G |f| \leq K \cdot \sum_{k=0}^{\infty} m(L_{k,K}(f)) \) from (17) and \( m(G) = 1 \) it follows that \( f \in L^1(G) \).

This completes the proof of the case of connected \( G \).

Next we show how one can reduce the case of a locally connected \( G \) to the connected case. If \( G \) is locally connected then by [8, 24.45] if \( C \) denotes the component of \( G \) containing \( O_G \) (the neutral element of \( G \)) then \( C \) is an open subgroup of \( G \) and \( G \) is topologically isomorphic to \( C \times (G/C) \). Since \( G \) is compact \( G/C \) should be finite. Suppose that its order is \( n \). Using that \( G = C \times (G/C) \) we write the elements of \( G \) in the form \( g = (g_1, g_2) \) with \( g_1 \in C, g_2 \in G/C \).

Suppose that \( f \notin L^1(G) \) is measurable and \( m(G_{f,b}) > 0 \). Set

\[
X_{\alpha,f} = \left\{ x \in G : \limsup_{k \to +\infty} \frac{|f(x + n_k \alpha)|}{k} < +\infty \right\}.
\]

If \( \alpha \in G_{f,b} \) then \( m(X_{\alpha,f}) = 1 \). Suppose that \( g_j, j = 1,\ldots,n \) is a list of all elements of \( G/C \).

For \( x = (x_1, x_2) \in G \) define

\[
f^*(x) = f^*(x_1, x_2) = \sum_{j=1}^{n} |f(x_1, x_2 + g_j^*)|.
\]

Set

\[
X_{\alpha,f}^* = \bigcap_{j=1}^{n} \left( X_{\alpha,f} + (0_C, g_j^*) \right).
\]

Clearly \( m(X_{\alpha,f}) = 1 \) implies \( m(X_{\alpha,f}^*) = 1 \).

For \( x \in X_{\alpha,f}^* \) we have \( \limsup_{k \to +\infty} \frac{|f^*(x + n_k \alpha)|}{k} < +\infty \). Since \( f^* \) is not depending on its second coordinate we have \( f^*(x + n_k(\alpha_1, \alpha_2)) = f^*(x + \)
Define $f^{**} : C \to \mathbb{R}$ such that $f^{**}(x_1) = f^*(x_1, 0_{G/C})$. Since we assumed that $f \notin L^1(G)$ we have $f^* \notin L^1(G)$ and this implies $f^{**} \notin L^1(C)$.

Set

$$\Gamma^*_{f,b} = \pi^*_C(\Gamma_{f,b}) = \{ \alpha_1 : \exists \alpha_2 \in G/C \text{ such that } \alpha = (\alpha_1, \alpha_2) \in \Gamma_{f,b} \}.$$ 

Then for $\alpha_1 \in \Gamma^*_{f,b}$ we have

$$\limsup_{k \to \infty} \frac{|f^{**}(x_1 + n_k \alpha_1)|}{k} < +\infty. \quad (18)$$

Since the Haar measure on $C$ is a positive constant multiple of the Haar measure on $G$ restricted to $C$, on the compact connected Abelian group $C$ we would obtain a measurable function $f^{**} \notin L^1(C)$ such that for a set of positive measure of rotations (18) holds. This would contradict the first part of this proof concerning connected groups.

Theorem 1 says that if we do not have “too much torsion” in $\hat{G}$ then from $m(\Gamma_{f,b}) > 0$ it follows that $f \in L^1(G)$. In the next definition we define what we mean by “a lot of torsion” in a group.

**Definition 3.** We say that the group $G$ contains infinitely many multiple torsion if

1. either there is a prime number $p$ such that $G$ contains a subgroup algebraically isomorphic to the direct sum $(\mathbb{Z}/p) \oplus (\mathbb{Z}/p) \oplus \ldots$ (countably many copies of $\mathbb{Z}/p$),

2. or there are infinitely many different prime numbers $p_1, p_2, \ldots$ such that $G$ contains for any $j$ subgroups of the form $(\mathbb{Z}/p_j) \times (\mathbb{Z}/p_j)$.

**Theorem 4.** Suppose that $(n_k)$ is a strictly monotone increasing sequence of integers and $G$ is a compact Abelian group such that its dual group $\hat{G}$ contains infinitely many multiple torsion. Then there exists a measurable $f \notin L^1(G)$ such that

$$m(\Gamma_{f,0}) = m(\Gamma_{f,b}) = 1,$$

where $m$ is the Haar-measure on $G$. \quad (19)

**Proof.** First suppose that in Definition 3 property (i) holds for $\hat{G}$. Then for any $k$ we can select a subgroup $\hat{G}_k$ in $\hat{G}$ such that it is isomorphic to $(\mathbb{Z}/p)_k \times (\mathbb{Z}/p)_k \times \cdots \times (\mathbb{Z}/p)_k$. Suppose that the characters $\gamma_1, \ldots, \gamma_k$ are the generators of $\hat{G}_k$. 


Put $H_k = \bigcap_{j=1}^{k} \gamma_j^{-1}(1)$. Then $H_k$ is a closed subgroup of $G$. Since $y \in x + H_k$, that is $y - x \in H_k$ if and only if $\gamma_j(y) = \gamma_j(x)$ for $j = 1, \ldots, k$, which means that $\gamma_j(y - x) = \gamma_j(y)/\gamma_j(x) = 1$ for $j = 1, \ldots, k$ one can see that $G$ is tiled with $p^k$ many translated copies of $H_k$. The sets $x + H_k$ are all closed and therefore $H_k$ is a closed-open subgroup of $G$.

We also have
\[ m(H_k) = \frac{1}{p^k}. \tag{20} \]

Set $f_k(x) = p^k$ if $x \in H_k$ and $f_k(x) = 0$ otherwise.

Put $f = \sum_{k=1}^{\infty} f_k$. By the Borel-Cantelli lemma and (20) the function $f$ is $m$ a.e. finite. It is also clear that $f$ is measurable and $f \notin L^1(G)$.

Suppose $\alpha \in G$ is arbitrary. Set $X_k = \bigcup_{j=0}^{p^{-1}k} H_k - j\alpha$. Then $m(X_k) = p^{-k+1}$ and by the Borel-Cantelli lemma $m$ a.e. $x$ belongs to only finitely many $X_k$. If $x \notin X_k$ then $\forall j \in \mathbb{N}$, $x + j\alpha \notin H_k$ and hence
\[ f_k(x + j\alpha) = 0 \text{ for any } j \in \mathbb{N}. \tag{21} \]

Therefore, $\frac{f(x + n_k\alpha)}{k} \to 0$ for $m$ a.e. $x \in G$ and $\Gamma_{f,0} = G$.

If in Definition 3 property (ii) holds for $\widehat{G}$ then for any $k$ select $\widehat{G}_k$ in $\widehat{G}$ such that it is isomorphic to $(\mathbb{Z}/p_k) \times (\mathbb{Z}/p_k)$. We suppose that $\gamma_{1,k}$ and $\gamma_{2,k}$ are the generators of $\widehat{G}_k$. Put $H_k = \gamma_{1,k}^{-1}(1) \cap \gamma_{2,k}^{-1}(1)$. One can see, similar to the previous case, that $G$ is tiled by $p^k_2$ many translated copies of $H_k$.

Turning to a subsequence if necessary, we can suppose that
\[ \sum_{k=1}^{\infty} \frac{1}{p_k} < +\infty. \tag{22} \]

We also have
\[ m(H_k) = \frac{1}{p_k^2}. \tag{23} \]

Set $f_k(x) = p_k^2$ if $x \in H_k$ and $f_k(x) = 0$ otherwise.

Put $f = \sum_{k=1}^{\infty} f_k$. Again, it is clear that $f$ is $m$ a.e. finite, measurable and $f \notin L^1(G)$. For an arbitrary $\alpha \in G$ one can define $X_k = \bigcup_{j=0}^{p^{-1}k} H_k - j\alpha$. Then $m(X_k) = \frac{1}{p_k}$.

From (22) and from the Borel-Cantelli lemma it follows that $m$ a.e. $x$ belongs to only finitely many $X_k$. One can conclude the proof as we did it in the previous case. $\square$

It is natural to ask for a version of Theorem 4 for the non-conventional ergodic averages with $m(\Gamma_f) = 1$ in (19). For convergence of the non-conventional ergodic averages some arithmetic assumptions about $n_k$ are
needed.
We recall from [10] Definition 1.2 with some notational adjustment.

**Definition 5.** The sequence \((n_k)\) is ergodic mod \(q\) if for any \(h \in \mathbb{Z}\)

\[
\lim_{N \to \infty} \frac{\sum_{k=0}^{N} \chi_{h,q}(n_k)}{N + 1} = \frac{1}{q},
\]

(24)

Where \(\chi_{h,q}(x) = 1\) if \(x \equiv h \mod q\) and \(\chi_{h,q}(x) = 0\) otherwise.
A sequence \((n_k)\) is ergodic for periodic systems if it is ergodic mod \(q\) for every \(q \in \mathbb{N}\).

For ergodic sequences with essentially the same proof we can state the following version of Theorem 4:

**Theorem 6.** Suppose that \(n_k\) is a strictly monotone, ergodic sequence for periodic systems and \(G\) is a compact Abelian group such that its dual group \(\hat{G}\) contains infinitely many multiple torsion. Then there exist a measurable \(f \notin L^1(G)\) such that \(\Gamma_f = G\), and hence \(m(\Gamma_f) = 1\).

**Proof.** As we mentioned earlier the argument of the proof of Theorem 4 is applicable. One needs to add the observation that if \(x \in X_k\) then the ergodicity of \(n_k\) for periodic systems implies that \(M_{\alpha}^N f_k\) converges. If \(x \notin X_k\) then (21) can be used. Hence \(M_{\alpha}^N f\) converges for all \(\alpha \in G\) for a.e. \(x\). \(\square\)

In Theorem 4 we saw that if there is “lots of torsion” in \(\hat{G}\), that is, \(G\) is “highly disconnected” then there are measurable functions \(f\) not in \(L^1\) for which \(m(\Gamma_{f,0}) = 1\). Since the \(p\)-adic integers, \(\mathbb{Z}_p\) are the building blocks of 0-dimensional compact Abelian groups ([8, Theorem 25.22]) it is natural to consider them. If we take a countable product of \(\mathbb{Z}_p\) with \(p\) fixed then the dual group will be the direct sum of \(Z(p^n)\)’s and will contain a subgroup algebraically isomorphic to the direct sum \((Z/p) \oplus (Z/p) \oplus \ldots\).

Then Theorem 4 is applicable.

If one considers an individual \(Z_p\) then its dual group is \(Z(p^\infty)\) with all elements of finite order, so still there seems to be “lots of torsion” in the dual group. It is also clear that arithmetic properties of \(n_k\) might matter if we consider \(Z_p\). For us it was quite surprising that if one considers ordinary ergodic averages, that is, \(n_k = k\) then \(Z_p\) behaves like a locally connected group and the following theorem is true.

**Theorem 7.** Suppose that \(n_k = k\), and \(p\) is a fixed prime number. We consider \(G = Z_p\), the group of \(p\)-adic integers. Then for any measurable function \(f : G \to \mathbb{R}\) from \(m(\Gamma_{f,k}) > 0\) it follows that \(f \in L^1(G)\).
Before turning to the proof of Theorem 7 we need some notation and a Claim simplifying the proof of Theorem 7. Denote by \( \Gamma_{f,b}^j \), \( j = -1, 0, 1, \ldots \) the set of those \( \alpha = (\alpha_0, \alpha_1, \ldots) \in \Gamma_{f,b} \) for which \( \alpha_{j+1} \neq 0 \) but \( \alpha_0 = \cdots = \alpha_j = 0 \). From \( m(\Gamma_{f,b}) > 0 \) it follows that there exists \( j_0 \) such that \( m(\hat{\Gamma}_{f,b}^{j_0}) > 0 \). Given a finite string \((x_0, \ldots, x_j)\) we denote by \([x_0, \ldots, x_j] \) the corresponding cylinder set in \( G \), that is,

\[
[x_0, \ldots, x_j] = \{(x'_0, x'_1, \ldots) \in G : (x'_0, \ldots, x'_j) = (x_0, \ldots, x_j)\}.
\]

**Claim 8.** If from \( m(\Gamma_{f,b}^{-1}) > 0 \) it follows that \( f \in L^1(G) \), then Theorem 7 is also true.

**Proof.** As mentioned above if \( m(\Gamma_{f,b}) > 0 \) then we can choose \( j_0 \) such that \( m(\hat{\Gamma}_{f,b}^{j_0}) > 0 \). Then for \( \alpha \in \Gamma_{f,b}^{j_0} \) for any cylinder \([x_0, \ldots, x_{j_0}]\) we have 

\[
[x_0, \ldots, x_{j_0}] + \alpha = [x_0, \ldots, x_{j_0}].
\]

If \( \sigma \) is the one-sided shift on \( Z_p \), that is, \( \sigma(x_0, x_1, \ldots) = (x_1, \ldots) \) then for \( \alpha \in \Gamma_{f,b}^{j_0} \) we have \( \sigma^{j_0+1}(x + \alpha) = \sigma^{j_0+1}x + \sigma^{j_0+1}\alpha \).

For an \( x' \in G \) we define the function \( f_{x_0,\ldots,x_{j_0}}(x') = f(x_0, \ldots, x_{j_0}, x') \), where \( (x_0, \ldots, x_{j_0}, x') \) is the concatenation of the finite string \((x_0, \ldots, x_{j_0})\) and \( x' \in G = Z_p \). Then \( \hat{\Gamma}_{f,x_0,\ldots,x_{j_0}} \supset \Gamma_{f,b}^{j_0} \) and we can apply the Claim for \( f_{x_0,\ldots,x_{j_0}} \) to obtain that \( f_{x_0,\ldots,x_{j_0}} \in L^1(G) \), that is, \( f \in L^1([x_0, \ldots, x_{j_0}]) \). Since this holds for any cylinder set \([x_0, \ldots, x_{j_0}]\) we obtain that \( f \in L^1(G) \). \( \square \)

**Proof of Theorem 7.** By Claim 8 we can assume that \( m(\Gamma_{f,b}^{-1}) > 0 \). We need to adjust the proof of Theorem 1 for the case of \( G = Z_p \). The key difficulty is the torsion in \( \hat{G} = Z(p^\infty) \) which makes it impossible to use a direct argument which lead to (16). Anyway, we start to argue as in the proof of Theorem 1, keeping in mind that now \( n_k = k \). We introduce the sets \( G_{\alpha,K}, B \subset \Gamma_{f,b}^{-1} \), \( L_k(f) \) as in (2), (3), and (4), respectively. We fix \( K \) and define the set \( H_\alpha \) and the auxiliary function \( h(x,\alpha) \) as in (5). We have (6) again.

Our aim is to establish that for a suitable \( \kappa_0 \)

\[
\sum_{\kappa \geq \kappa_0} p^\kappa m(L_{p^{\kappa+2},K}(f)) < \infty. \tag{25}
\]

Suppose that the function \( \varphi \) equals \( p^{\kappa+3}K \) on \( L_{p^{\kappa+2},K}(f) \setminus L_{p^{\kappa+3},K}(f), \kappa = \kappa_0, \kappa_0 + 1, \ldots \) and equals \( K \cdot p^{\kappa_0+2} \) on \( G \setminus L_{p^{\kappa_0+2},K}(f) \). Then \( \varphi \geq |f| \) and by (25)

\[
\int_G \varphi dm \leq K \cdot p^{\kappa_0+2}m(G) + \sum_{\kappa = \kappa_0}^{\infty} p^{\kappa+3} \cdot K m(L_{p^{\kappa+2},K}(f)) < + \infty. \tag{26}
\]

This implies that \( f \in L^1(G) \).
Hence we need to establish (25). Choose and fix \( \kappa_0 \in \mathbb{N} \) such that \( p^{\kappa_0} > K \) and suppose that \( \kappa \geq \kappa_0 \).

Then, keeping in mind that \( L_{k,K}(f) \supseteq L_{p^{\kappa+2},K}(f) \) for \( k \leq p^{\kappa+2} \) we have instead of (7)

\[
h(x - k\alpha, \alpha) = 1 \quad \text{for any } \alpha \in B, \, K < k < p^{\kappa+2} \text{ and } x \in L_{p^{\kappa+2},K}(f).
\] (27)

Let

\[
h_k(x, \alpha) = \frac{1}{p^\kappa} \sum_{k=p^\kappa}^{2p^\kappa-1} h(x - k\alpha, \alpha).
\] (28)

Then by (27)

\[
h_k(x - k\alpha, \alpha) = 1 \quad \text{for any } \alpha \in B, \, 0 \leq k < p^{\kappa+2} - 2p^\kappa \text{ and } x \in L_{p^{\kappa+2},K}(f)
\] (29)

Taking average on \( B \)

\[
\frac{1}{m(B)} \int_B h_k(x - k\alpha, \alpha) \, dm(\alpha) = 1
\] (30)

for \( \kappa \geq \kappa_0, \, 0 \leq k < p^{\kappa+2} - 2p^\kappa \) and \( x \in L_{p^{\kappa+2},K}(f) \).

Now we return to \( h(x, \alpha) \) and we define \( c_\gamma(\alpha) \) as in (9). Again, \( c_\gamma(\alpha) \) is a bounded, measurable function and (10) holds.

Denoting again by \( \gamma_0(x) \) the identically 1 character, the neutral element of \( \hat{G} \) we also have (11) satisfied. For \( h_\kappa(x, \alpha) \) we have

\[
h_\kappa(x, \alpha) \sim \sum_{\gamma \in \hat{G}} c_{\gamma,\kappa}(\alpha) \gamma(x) = \sum_{\gamma \in \hat{G}} c_{\gamma}(\alpha) \left( \frac{1}{p^\kappa} \sum_{k=p^\kappa}^{2p^\kappa-1} \gamma(-k\alpha) \right) \gamma(x).
\] (31)

Since \( \hat{G} = Z(p^\infty) \), the order of \( \gamma \) is a power of \( p \). We denote it by \( \text{ord}(\gamma) \).

A \( \gamma \in \hat{G} \) of order \( p^r, \, r > 0 \) is of the form

\[
\gamma(x) = \exp \left( \frac{2\pi il}{p^r} (x_0 + px_1 + \cdots + p^{r-1}x_{r-1}) \right)
\] (32)

for \( x = (x_0, x_1, \ldots) \in G = \mathbb{Z}_p \) with \( l \) not divisible by \( p \).

Since \( B \subset \Gamma_{f,b}^{-1} \), for \( \alpha \in B \) we have \( \alpha_0 \neq 0 \) which implies \( \gamma(-\alpha) \neq 1 \) and if \( \gamma \) is of order \( p^r, \, r > 0 \) then \( \gamma(-\alpha) \in \mathbb{C} \) is also of order \( p^r, \, r > 0 \). Hence for \( \text{ord}(\gamma) = p^r \leq p^\kappa \) and \( \alpha \in B \) we have

\[
\sum_{k=p^\kappa}^{2p^\kappa-1} \gamma(-k\alpha) = \sum_{k=p^\kappa}^{2p^\kappa-1} \gamma^k(-\alpha) = \gamma(-p^\kappa \alpha) \frac{1 - \gamma(p^\kappa \alpha)}{1 - \gamma(-\alpha)} = 0.
\] (33)
This way we can get rid of some characters with small torsion in the Fourier-series of \( h_\kappa(x, \alpha) \).

Recalling that \( c_{\gamma_0}(\alpha) = \int_G h(x, \alpha) \cdot 1 \, dm(\alpha) = 0 \) by (10) we have in (31)

\[
c_{\gamma_0,\kappa}(\alpha) = 0 \quad \text{if} \quad \alpha \in B.
\]

(34)

Using (31) again we have

\[
h_\kappa(x - k\alpha, \alpha) \sim \sum_{\gamma \in \hat{G}} c_{\gamma,\kappa}(\alpha) \gamma(-k\alpha) \gamma(x)
\]

(35)

and by (30) for any \( 0 \leq k < p^{\kappa+2} - 2p^\kappa \)

\[
m(L_{p^{\kappa+2},K}(f)) \leq \int_G \left| \frac{1}{m(B)} \int_B h_\kappa(x - k\alpha, \alpha) dm(\alpha) \right|^2 \, dm(x)
\]

(36)

\[
= \int_G |\varphi_{\kappa,k}(x)|^2 \, dm(x),
\]

where \( \varphi_{\kappa,k}(x) = \frac{1}{m(B)} \int_B h_\kappa(x - k\alpha, \alpha) dm(\alpha) \) is a bounded measurable function.

Recall that by (31) we can express the Fourier-coefficients of \( h_\kappa \) by those of \( h \), that is

\[
c_{\gamma,\kappa}(\alpha) = \int_G h_\kappa(x, \alpha) \gamma(-x) \, dm(x) = c_\gamma(\alpha) \frac{1}{p^\kappa} \sum_{k=p^\kappa}^{2p^\kappa-1} \gamma(-k\alpha).
\]

(37)

Therefore,

\[
\hat{\varphi}_{\kappa,k}(\gamma) = \int_G \frac{1}{m(B)} \int_B h_\kappa(x - k\alpha, \alpha) dm(\alpha) \gamma(-x) \, dm(x)
\]

\[
= \frac{1}{m(B)} \int_B \int_G h_\kappa(x - k\alpha, \alpha) \gamma(-x) dm(x) dm(\alpha)
\]

(38)

\[
= \frac{1}{m(B)} \int_G \chi_B(\alpha) \cdot \int_G h_\kappa(u, \alpha) \gamma(-u - k\alpha) dm(u) dm(\alpha)
\]

\[
= \frac{1}{m(B)} \int_G \chi_B(\alpha) \gamma(-k\alpha) c_{\gamma,\kappa}(\alpha) dm(\alpha).
\]

If \( \gamma \neq \gamma_0 \) and \( \text{ord}(\gamma) \leq p^\kappa \) then by (33) and (37) we have \( c_{\gamma,\kappa}(\alpha) = 0 \) for any \( \alpha \in B \), and hence \( \hat{\varphi}_{\kappa,k}(\gamma) = 0 \).

Recall from (34) that if \( \alpha \in B \) then \( c_{\gamma_0,\kappa}(\alpha) = 0 \). Hence \( \hat{\varphi}_{\kappa,k}(\gamma_0) = 0 \) holds in this case as well.
Now suppose that \( \gamma^{p} \neq \gamma_{0} \). Then \( \text{ord}(\gamma) \geq p^{k+1} \) and for \( k = 0, \ldots, p^{k+1} - 1 \) the characters \( \gamma^{k} \) are different.

By using the Parseval-formula we can continue (36) to obtain for any \( 0 \leq k < p^{k+2} - 2p^{k} \) that

\[
m(\mathcal{L}_{p^{k+2},K}(f)) \leq \sum_{\gamma \in \mathcal{G}} |\hat{\varphi}_{\kappa,k}(\gamma)|^{2}
\]

\[
= \sum_{\gamma \in \mathcal{G}, \gamma^{p^{k}} \neq \gamma_{0}} \frac{1}{(m(B))^{2}} \cdot \left| \int_{\mathcal{G}} \chi_{B}(\alpha) \gamma(-k\alpha) c_{\gamma,\kappa}(\alpha) dm(\alpha) \right|^{2}.
\]

Since \( p \geq 2 \) implies \( p^{k+2} \geq 3p^{k} \) we can use (29) and (39) for \( k = 0, \ldots, p^{k} - 1 \). Adding equation (39) for all \( \kappa \geq \kappa_{0} \) and \( k = 0, \ldots, p^{k} - 1 \) we need to estimate

\[
\sum_{\kappa \geq \kappa_{0}}\sum_{k=0}^{p^{k}-1} p^{k} m(\mathcal{L}_{p^{k+2},K}(f))
\]

\[
\leq \sum_{\kappa \geq \kappa_{0}} \sum_{\gamma \in \mathcal{G}, \gamma^{p^{k}} \neq \gamma_{0}} \frac{1}{(m(B))^{2}} \cdot \left| \int_{\mathcal{G}} \chi_{B}(\alpha) c_{\gamma,\kappa}(\alpha) \gamma(-k\alpha) dm(\alpha) \right|^{2}.
\]

Using (31) and (37) first we estimate for \( \kappa \geq \kappa_{0} \)

\[
\sum_{k=0}^{p^{k}-1} \left| \int_{\mathcal{G}} \chi_{B}(\alpha) c_{\gamma,\kappa}(\alpha) \gamma(-k\alpha) dm(\alpha) \right|^{2}
\]

\[
= \sum_{k=0}^{p^{k}-1} \left| \int_{\mathcal{G}} \chi_{B}(\alpha) c_{\gamma}(\alpha) \frac{1}{p^{k}} \sum_{k'=p^{k}}^{2p^{k}-1} \gamma(-(k'+k)\alpha) dm(\alpha) \right|^{2} = **
\]

in the last expression \( k' + k \) can take values between \( p^{k} \) and \( 3p^{k} - 2 \). If \( p \geq 3 \) then \( 3p^{k} - 2 \leq p^{k+1} - 1 \) so for the moment we suppose that \( p \geq 3 \). In the end of this proof we will point out the little adjustments which we need for the case \( p = 2 \).

For \( p^{k} \leq j \leq 3p^{k} - 2 \leq p^{k+1} - 1 \) we denote by \( w_{j}' \) the number of those couples \( (k, k') \) for which \( 0 \leq k \leq p^{k} - 1, p^{k} \leq k' \leq 2p^{k} - 1 \) and \( k + k' = j \). Obviously, \( w_{j}' \leq p^{k} \). Set \( w_{j} = w_{j}' / p^{k} \leq 1 \). We select these \( w_{j} \) for all \( \kappa_{0} \leq \kappa \leq \text{ord}(\gamma) \). For those values of \( j \) for which we have not defined \( w_{j} \) yet we set \( w_{j} = 0 \).

By using this notation we can continue ** from (41)

\[
** \leq \sum_{j=p^{k}}^{p^{k+1}-1} w_{j} \left| \int_{\mathcal{G}} \chi_{B}(\alpha) c_{\gamma}(\alpha) \cdot \gamma(-j\alpha) dm(\alpha) \right|^{2}
\]

(42)
Using (41) and (42) while continuing the estimation of (40) we obtain
\[
\sum_{\kappa \geq \kappa_0} p^\kappa \mu(L_{p^\kappa+2,K}(f)) \leq
\]
\[
\sum_{\kappa \geq \kappa_0} \sum_{\gamma \in G} \frac{1}{(m(B))^2} p^{\kappa+1-1} \int_G |\chi_B(\alpha) c_\gamma(\alpha) \gamma(-j\alpha) dm(\alpha)|^2.
\]
Since for a fixed $\gamma$ the characters $\gamma^{-j}$ are different, for different values $0 \leq j < \text{ord}(\gamma)$ by Parseval's Theorem we infer
\[
\sum_{j=1}^{\text{ord}(\gamma)-1} |\int_G \chi_B(\alpha) c_\gamma(\alpha) \gamma(-j\alpha) dm(\alpha)|^2 \leq \int_G |\chi_B(\alpha) c_\gamma(\alpha)|^2 dm(\alpha).
\]
Using this in (43) we obtain
\[
\sum_{\kappa \geq \kappa_0} p^\kappa \mu(L_{p^\kappa+2,K}(f)) \leq \frac{1}{(m(B))^2} \sum_{\gamma \in G} \int_G |\chi_B(\alpha) c_\gamma(\alpha)|^2 dm(\alpha)
\]
\[
= \frac{1}{(m(B))^2} \int_G \chi_B(\alpha) \sum_{\gamma \in G} |c_\gamma(\alpha)|^2 dm(\alpha)
\]
\[
= \frac{1}{(m(B))^2} \int_G \chi_B(\alpha) \int_G |h(x,\alpha)|^2 dm(x) dm(\alpha) < +\infty.
\]
This completes the proof if $p \geq 3$.

In case of $p = 2$ the intervals $p^\kappa \leq j \leq 3p^\kappa - 2$ are not disjoint, but $3p^\kappa - 2 \leq p^{\kappa+2} - 1$. Instead of (43) we could obtain
\[
\sum_{\kappa \geq \kappa_0} p^{\kappa+1} \mu(L_{p^{\kappa+1},K}(f)) \leq 2 \sum_{\gamma \in G} \sum_{j=1}^{2\text{ord}(\gamma)-1} \frac{1}{(m(B))^2} \int_G |\chi_B(\alpha) c_\gamma(\alpha) \gamma(-j\alpha) dm(\alpha)|^2.
\]
For a fixed $\gamma$ the characters $\gamma^{-j}(\alpha)$, $j \leq 2\text{ord}(\gamma) - 1$ are not different but for each $j \leq 2\text{ord}(\gamma) - 1$ there is at most one other $j' \leq 2\text{ord}(\gamma) - 1$ such that $\gamma^{-j} = \gamma^{-j'}$, hence
\[
\sum_{j=1}^{2\text{ord}(\gamma)-1} |\int_G \chi_B(\alpha) c_\gamma(\alpha) \gamma(-j\alpha) dm(\alpha)|^2 \leq 2 \int_G |\chi_B(\alpha) c_\gamma(\alpha)|^2 dm(\alpha).
\]
The conclusion of the proof is similar to the $p \geq 3$ case.

References


