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IRREGULAR 1-SETS ON THE GRAPHS OF CONTINUOUS FUNCTIONS

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Abstract. One can define in a natural way irregular 1-sets on the graphs of several fractal functions, like Takagi's function, Weierstrass–Cellerier type functions and the typical continuous function. These irregular 1-sets can be useful during the investigation of level-sets and occupation measures of these functions. For example, we see that for Takagi's function and for certain Weierstrass–Cellerier functions the occupation measure is singular with respect to the Lebesgue measure and for almost every level the level set is finite.

1. Introduction

We denote by $\mathcal{H}^1$ the one-dimensional Hausdorff measure. An $\mathcal{H}^1$-measurable set $S \subset \mathbb{R}^2$ is called irregular (or purely unrectifiable) 1-set if $0 < \mathcal{H}^1(S) < \infty$ and S intersects every continuously differentiable curve in a set of $\mathcal{H}^1$-measure zero (here, equivalently, instead of continuously differentiable curves one could also use rectifiable curves as well). Besicovitch's theorem on the projection of irregular 1-sets states that the projection of an irregular 1-set

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S in almost all directions is of zero Lebesgue measure, see [1], or Theorem 6.13 in [10] (in this paper we will mainly use notation and terminology from K. Falconer's book [10]). A set is regular if it can be covered by countably many rectifiable curves (or equivalently, by countably many continuously differentiable curves) and an $\mathcal{H}^1$-null set. The projection theorem for regular 1-sets (see Theorem 6.10 in [10]) states that if S is a regular 1-set then the projection of S is of positive Lebesgue measure in all but at most one exceptional direction. This implies that if the projection of a 1-set is of zero Lebesgue measure in at least two directions then it is an irregular 1-set.

Clearly, the graph of a continuous function defined on a non-degenerate subinterval of $\mathbb{R}$ cannot be an irregular 1-set since there can be at most one direction in which its projection does not contain an interval. However, as we can see in this paper, quite often one can find on the graph of some continuous functions some irregular 1-subsets and these sets can provide useful information if we are interested in the level sets, or occupation measures of these functions.

The graphs of the fractal functions considered in this paper are not of finite $\mathcal{H}^1$-measure and hence they are not 1-sets. In the beautiful theory of 1-sets (see for example [10]) an arbitrary 1-set $S \subset \mathbb{R}^2$ can be decomposed into two (possibly empty) parts, S_{reg} and S_{irr} such that S_{irr} is irregular and S_{reg} is regular. In this paper we will see a naturally arising similar decomposition for Takagi's function and for some Weierstrass–Cellerier type functions.

The starting point of this paper was the following problem:

Suppose that $f(x) = \sum_{n=0}^{\infty} 2^{-n} \|2^n x\|$, where $\|x\|$ is the distance of x from the closest integer (that is, f is Takagi's function). What can we say for Lebesgue almost every $y \in f(\mathbb{R})$ about the cardinality of the level set

$$L_y = \{ x \in [0,1] : f(x) = y \} ?$$

The surprizing answer is that it is finite. This problem was proposed by the author for the Miklós Schweitzer Mathematical Competition of the János Bolyai Mathematical Society in 2006. During the competition three solutions arrived. Only one competitor – the winner of the competition – gave the correct answer, one student claimed that the level sets are almost everywhere countably infinite, another claimed that they are uncountably infinite.

In the sequel we will denote by $\mathcal{T}$ the restriction of Takagi's function onto the $[0,1]$ interval. In a rather natural way in Section 3 we will define an irregular 1-set S_{irr} on the graph of Takagi's function. Its complement with respect to the graph of the function is a set S_{reg} which can be covered by countably many graphs of monotone functions. In Theorem 9 we verify that the projection of S_{irr} onto the y-axis is of Lebesgue measure zero, while its projection onto the x-axis is of full measure. The set S_{reg} is "responsible" for the finite level sets. This result leads naturally to the study of the occupation measure

$$\mu(A) = \lambda\big\{ x \in [0,1] : \mathcal{T}(x) \in A \big\} = \lambda\big(\mathcal{T}^{-1}(A)\big) = \lambda\big(f^{-1}(A) \cap [0,1]\big),$$

where λ is the one-dimensional Lebesgue measure. From the projection properties of the irregular 1-set S_{irr} it follows that μ is singular with respect to the Lebesgue measure.

Occupation measures were studied extensively in probability theory, especially related to stochastic processes. With probability one for the Brownian motion W, the occupation measure $\mu(A) \stackrel{\mathrm{def}}{=} \lambda\big(W^{-1}(A)\big)$ is absolutely continuous with respect to the Lebesgue measure, it satisfies the local time (LT) condition.

D. Geman and J. Horowitz in their excellent survey paper [12] wrote the following: "... results can be successfully applied to random functions and fields, it is difficult to apply them to particular nonrandom functions. For example an interesting open problem is *to determine which functions representable as Fourier series* (for instance) *are (LT)*..."

Viewed from this angle it may be interesting to know that the occupation measures of Takagi's function and other continuous functions are singular.

J. Bertoin in [2] and [3] studied occupation measures and Hausdorff dimensions of level sets of some self affine functions. It turned out that, depending on certain parameter values, these functions are either (LT) or have singular occupation measure.

In [7] and [9] we studied micro tangent properties of continuous functions. Recall that Baire's category theorem holds in $C[0,1]$ with the supremum norm and a property is typical, or generic if the set of functions not having this property is of first category. Results of [7] imply that for the typical continuous function f the set of universal micro tangent points $\mathrm{UMT}(f)$ defines naturally an irregular 1-set on the graph of f. The set $\mathrm{UMT}(f)$ projects to a set of λ-measure zero in any direction different from the y-axis and the occupation measure of the typical continuous function is singular.

The level structure of the typical continuous function was characterized by Bruckner and Garg see Ch. 13 of [5], or [6]:

For the typical $f \in C[0,1]$

i) *there are two levels (corresponding to the maximum and minimum of the function) where L_y consists of one element,*

iia) *for all other y's in the range of f, L_y contains a non-empty nowhere dense perfect set and*

iib) *out of these y's there are countably many levels, corresponding to local extrema where the level set contains an additional isolated point.*

Hence for all but two y's in the range of the typical $f \in C[0,1]$, the level set L_y is uncountable.

The UMT (f) set of a continuous function is quite stable. In Theorem 10 we show that if $h \in C[0,1]$ is almost everywhere differentiable then for almost every $x \in [0,1]$ from $\big(x; f(x)\big) \in \mathrm{UMT}\,(f)$ it follows that $\big(x; f(x)+h(x)\big) \in \mathrm{UMT}\,(f+h)$. This implies that for the typical continuous function f the occupation measure of $f+h$ is also singular. In §12, pp. 18–19 of [12] it is proved that if f is (LT) and its occupation density is sufficiently nice then $f+h$ is also (LT) for any almost everywhere differentiable h, so absolute continuity can also be stable for occupation measures.

In Section 5 of this paper we consider a class of "perturbed" Weierstrass–Cellerier-type functions. Suppose that $\mathcal{F}_{\mathcal{W}}$ consists of those twice continuously differentiable functions f_{-1} on $[0,1]$ for which for any trigonometric polynomial P the function $f_{-1}+P$ is piecewise strictly monotone or constant. For example the set of functions which are analytic on an open set $G \supset [0,1]$ is a subset of $\mathcal{F}_{\mathcal{W}}$. Suppose that $f_{-1}(x)$ belongs to $\mathcal{F}_{\mathcal{W}}$. For $x \in [0,1]$ put

$$f(x) = f_{-1}(x) + \sum_{n=0}^{\infty} 2^{-n} \sin(2\pi 2^n x). \tag{1}$$

If f_{-1} equals identically zero then we obtain the Weierstrass type Cellerier nowhere differentiable function, defined by the Fourier series

$$\mathcal{W}(x) \overset{\text{def}}{=} f(x) = \sum_{n=0}^{\infty} 2^{-n} \sin(2\pi 2^n x). \tag{2}$$

For any f defined in (1) we can again define an irregular 1-set S_{irr} on the graph of f. The projection of this set onto the x-axis is again a set of full measure. By the Besicovitch projection theorem in almost every direction S_{irr} projects into a set of Lebesgue measure zero. This implies that $\mathcal{W}(x,c) \overset{\text{def}}{=} \mathcal{W}(x) + cx$ has purely singular occupation measure for almost every c. In Section 5.1 we will see that for these c the level sets L_y of $\mathcal{W}(x,c)$ are finite for almost every y.

Similarly to the Takagi function from Theorem 13, it follows that $S_{\mathrm{reg}} = \mathrm{graph}\,\big(\mathcal{W}(x,c)\big) \setminus S_{\mathrm{irr}}$ can again be covered by countably many strictly monotone functions and an $\mathcal{H}^1$-zero set.

There are many results where it can be proven that a certain property holds for almost every parameter value, but it is much more difficult to see that a certain parameter value satisfies this property. To see that $\mathcal{W}(x,0) = \mathcal{W}(x)$ has singular occupation measure, that is $c=0$ is not an exceptional value will be the subject of the forthcoming paper [8].

2. Notation and preliminary results

Points in $\mathbb{R}^2$ will be denoted by $(x;y)$ while the open interval with endpoints x and y will be denoted by (x,y). Given $A \subset \mathbb{R}^2$, by $|A|$, $\operatorname{int}(A)$, and $\operatorname{cl}(A)$ we mean its diameter, interior, and closure, respectively. The open ball of radius r centered at $(x;y)$ is denoted by $B\big((x;y),r\big)$, while the closed ball is denoted by $\overline{B}\big((x;y),r\big)$. The closed cube of side length $2\delta > 0$ centered at $(x;y)$ will be denoted by $Q\big((x;y),\delta\big)$, that is, $Q\big((x;y),\delta\big) = \big\{(x';y') : |x'-x| \leqq \delta \text{ and } |y'-y| \leqq \delta\big\}$. Let Q^2 be the closed cube of side length 2, centered at $(0;0)$, that is, $Q\big((0;0),1\big)$. The projections of the coordinate plane onto the x, or y axis are denoted by π_x, or π_y, respectively. The one-dimensional Hausdorff measure in $\mathbb{R}^2$ will be denoted by $\mathcal{H}^1$, the Lebesgue measure on $\mathbb{R}$ will be denoted by λ. The graph of a function $f : [a,b] \to \mathbb{R}$, that is the set $\Big\{\big(x;f(x)\big) : x \in [a,b]\Big\}$ is denoted by $\operatorname{graph}(f)$.

It is not difficult to see that Vitali's covering theorem is also valid for coverings by closed squares, that is, the following variant of Theorem 2.8 of [13] holds.

THEOREM 1. *Let μ be a Radon measure on $\mathbb{R}^2$, $A \subset \mathbb{R}^2$ and $\mathcal{Q}$ a family of closed squares such that each point of A is the centre of arbitrarily small squares of $\mathcal{Q}$, that is,*

$$\inf\Big\{r : Q\big((x;y),r\big) \in \mathcal{Q}\Big\} = 0 \quad \textit{for} \quad (x;y) \in A.$$

Then there are disjoint squares $Q_i \in \mathcal{Q}$ such that

$$\mu\Big(A \setminus \bigcup_i Q_i\Big) = 0.$$

The next definitions and theorems are from [7].

By $C[-1,1]_0$ we mean the set of those functions g in $C[-1,1]$ for which $g(0) = 0$.

DEFINITION 2. The micro tangent set system of $f \in C[0,1]$ at the point $x_0 \in (0,1)$ will be denoted by $f_{\mathrm{MT}}(x_0)$ and it is defined as follows. For $\delta_n > 0$ we put

$$(3) \quad F(f,x_0,\delta_n) = \frac{1}{\delta_n}\bigg(\Big(\operatorname{graph}(f) \cap Q\Big(\big(x_0;f(x_0)\big),\delta_n\Big)\Big) - \big(x_0;f(x_0)\big)\bigg),$$

that is, $F(f,x_0,\delta_n)$ is the $1/\delta_n$-times enlarged part of $\operatorname{graph}(f)$ belonging to $Q\Big(\big(x_0;f(x_0)\big),\delta_n\Big)$ translated into Q^2. The set F is a *micro tangent set* of f at x_0, that is, $F \in f_{\mathrm{MT}}(x_0)$ if there exists $\delta_n \searrow 0$ such that $F(f,x_0,\delta_n)$ converges to F in the Hausdorff metric.

Definition 3. We say that x_0 is a *universal* MT-*point* for f if graph $(g) \cap Q^2 \in f_{\mathrm{MT}}(x_0)$ for every $g \in C[-1,1]_0$. The collection of those points $\big(x_0; f(x_0)\big)$ for which x_0 is a universal MT-point of f will be denoted by UMT (f).

The next theorem is Theorem 5 from [7].

Theorem 4. *There is a dense G_δ set, $\mathcal{G}$ of $C[0,1]$ such that*

$$\lambda\Big(\pi_x\big(\mathrm{UMT}\,(f)\big)\Big) = 1$$

for all $f \in \mathcal{G}$. Furthermore, UMT (f) *is a dense G_δ subset in the relative topology of* graph (f). *Hence, for the typical continuous function in $C[0,1]$ almost every $x \in [0,1]$ is a universal* MT-*point and a typical point on the graph of f is in* UMT (f).

Definition 5. For a fixed $g \in C[-1,1]_0$ we denote by $\mathrm{GLMT}_g(f)$ the set of those $\big(x_0; f(x_0)\big)$ for which graph $(g) \cap Q^2$ belongs to $f_{\mathrm{MT}}(x_0)$.

Next we state Lemma 6 and Theorem 7 from [7].

Lemma 6. *Assume that g_0 denotes the identically zero function on $[-1,1]$. We have $\lambda\Big(\pi_y\big(\mathrm{GLMT}_{g_0}(f)\big)\Big) = 0$ for any $f \in C[0,1]$.*

Since UMT $(f) \subset \mathrm{GLMT}_{g_0}(f)$ we also have:

Theorem 7. *For any $f \in C[0,1]$, $\lambda\Big(\pi_y\big(\mathrm{UMT}\,(f)\big)\Big) = 0$. Hence any preimage of almost every y in the range of any (and especially the typical) continuous function is not a UMT-point.*

3. Takagi's function

For $x \in [0,1]$ we put

$$\mathcal{T}(x) \stackrel{\mathrm{def}}{=} \sum_{n=0}^{\infty} 2^{-n}\|2^n x\|, \quad \text{and} \quad \mathcal{T}_N(x) \stackrel{\mathrm{def}}{=} \sum_{n=0}^{N} 2^{-n}\|2^n x\|,$$

where $\|x\|$ is the distance of x from the closest integer.

Recall the following part of Theorem 10 of [7]:

Theorem 8. *Takagi's function, $\mathcal{T}(x)$ is "micro self-similar" in the sense that* graph $(\mathcal{T}) \in \mathcal{T}_{\mathrm{MT}}(x_0)$ *for almost every $x_0 \in [0,1]$.*

Our new result about Takagi's function is the following:

Theorem 9. *The occupation measure of Takagi's function is singular and for almost every y the level set L_y of the Takagi function is finite.*

The graph $\{(x;\mathcal{T}(x)) : x \in [0,1]\}$ *of Takagi's function can be decomposed into two sets* S_{irr} *and* S_{reg}. *The set* S_{irr} *is an irregular* 1*-set with* $\lambda(\pi_x(S_{\mathrm{irr}})) = 1$, $\lambda(\pi_y(S_{\mathrm{irr}})) = 0$ *and the set* S_{reg} *can be covered by the graphs of countably many monotone increasing functions.*

PROOF. Since $\mathcal{T}_N$ is piecewise linear its derivative exists at all but finitely many points. If $\mathcal{T}'_N(x)$ exists then it equals zero if and only if up to the $(N+1)$'st digit the number of 0's equals the number of 1's in the dyadic expansion of x.

It is well-known that for almost every $x \in [0,1]$ this happens for infinitely many N's. The $\mathcal{T}'_N$'s correspond to a symmetric random walk model, where an n'th digit 0 means a unit step in the negative and an n'th digit 1 means a unit step in the positive direction. This random walk is persistent by Pólya's theorem (p. 118 of [4]), that is, the particle doing the random walk returns with probability one infinitely often to the origin.

Denote by X_1 the set of those $x \in [0,1]$ for which $\mathcal{T}'_N(x)$ exists for all N and $\mathcal{T}'_N(x) = 0$ for infinitely many N. For $x \in X_1$ there exists an infinite sequence $N(x,1) < \dots < N(x,\kappa) < \dots$ such that $\mathcal{T}'_{N(x,\kappa)}(x) = 0$ for all $\kappa \in \mathbb{N}$, but if $N \neq N(x,\kappa)$ for all κ then $\mathcal{T}'_N(x) \neq 0$.

If $\mathcal{T}'_{N_0}(x_0)$ does not exist then $x_0 = \frac{k_0}{2^{N_0+1}}$ with a suitable nonnegative integer k_0. For $x_0 = \frac{k_0}{2^{N_0+1}}$ we also have

$$\mathcal{T}_{N_0}(x_0) = \mathcal{T}_N(x_0) \quad \text{for} \quad N \geqq N_0 \quad \text{and hence} \quad \mathcal{T}_{N_0}(x_0) = \mathcal{T}(x_0). \tag{4}$$

Suppose $x_0 \in X_1$. For any $\kappa = 1, 2, \dots$ there exists $k_0(x_0,\kappa) \in \mathbb{Z}$ such that

$$x_0 \in I(x_0,\kappa) \overset{\mathrm{def}}{=} \left(\frac{k_0(x_0,\kappa)}{2^{N(x_0,\kappa)+1}}, \frac{k_0(x_0,\kappa)+1}{2^{N(x_0,\kappa)+1}} \right). \tag{5}$$

Since $\mathcal{T}'_{N(x_0,\kappa)}(x_0) = 0$ we have $\mathcal{T}'_{N(x_0,\kappa)}(x) = 0$ for all $x \in I(x_0,\kappa)$. We also have

$$\begin{cases} \mathcal{T}_{N'}\left(\dfrac{k_0(x_0,\kappa)}{2^{N(x_0,\kappa)+1}}\right) = \mathcal{T}\left(\dfrac{k_0(x_0,\kappa)}{2^{N(x_0,\kappa)+1}}\right), \\ \mathcal{T}_{N'}\left(\dfrac{k_0(x_0,\kappa)+1}{2^{N(x_0,\kappa)+1}}\right) = \mathcal{T}\left(\dfrac{k_0(x_0,\kappa)+1}{2^{N(x_0,\kappa)+1}}\right) \end{cases} \tag{6}$$

for all $N' \geqq N(x_0,\kappa)$. We have a rescaling property of $\mathcal{T}_N$, namely for $N > N(x_0,\kappa)$ and $x_1 \in I(x_0,\kappa)$ we have

$$\begin{aligned} \mathcal{T}_N(x_1) &= \mathcal{T}_{N(x_0,\kappa)}(x_0) \\ &+ 2^{-N(x_0,\kappa)-1}\mathcal{T}_{N-N(x_0,\kappa)-1}\left(2^{N(x_0,\kappa)+1}\left(x_1 - \frac{k(x_0,\kappa)}{2^{N(x_0,\kappa)+1}}\right)\right) \end{aligned} \tag{7}$$

and by letting $N \to \infty$, see Fig. 1,

$$(8)\quad \mathcal{T}(x_1) = \mathcal{T}_{N(x_0,\kappa)}(x_0) + 2^{-N(x_0,\kappa)-1}\mathcal{T}\left(2^{N(x_0,\kappa)+1}\left(x_1 - \frac{k(x_0,\kappa)}{2^{N(x_0,\kappa)+1}}\right)\right).$$

Given an interval $I = (a,b)$ set $\sigma_I(x) = \frac{x-a}{b-a}$, that is, σ_I maps (a,b) linearly onto $(0,1)$.

Observe that by (7) if $x_1 \in I(x_0,\kappa) \cap X_1$ then

$$I(x_1, \kappa+1) = \sigma^{-1}_{I(x_0,\kappa)}\big(I\big(\sigma_{I(x_0,\kappa)}(x_1), 1\big)\big).$$

Next we use that $\mathcal{T}$ and its partial sums are symmetric about the line $x = \frac{1}{2}$. If $x_1 \in I(x_0,1)$ then $\mathcal{T}_{N(x_0,1)}(x_1) = \mathcal{T}_{N(x_0,1)}(1-x_1)$, $N(1-x_0,1) = N(x_0,1)$, $I(1-x_0,1) = 1 - I(x_0,1)$.

We set $\mathcal{I}(x_0,1) = \big\{I(x_0,1), I(1-x_0,1)\big\}$ and call this the set of associated intervals at the first level.

Suppose that $\mathcal{I}(x_0,\kappa)$ consists of disjoint 2^κ many intervals, each of the same length and one of them is $I(x_0,\kappa)$. Furthermore,

$$(9)\qquad \begin{cases} \text{if } x_1 \in \cup\mathcal{I}(x_0,\kappa) \cap X_1 \text{ then } N(x_1,\kappa) = N(x_0,\kappa), \\ \mathcal{T}_{N(x_0,\kappa)}(x_1) = \mathcal{T}_{N(x_0,\kappa)}(x_0) \text{ and } \mathcal{T}'_{N(x_0,\kappa)}(x_1) = 0. \end{cases}$$

The system of intervals $\mathcal{I}(x_0,\kappa+1)$ will contain $I(x_0,\kappa+1)$ and for all $J \in \mathcal{I}(x_0,\kappa)$ we consider

$$\sigma_J^{-1}\big(I\big(\sigma_{I(x_0,\kappa)}(x_0), 1\big)\big) = \sigma_J^{-1}\big(\sigma_{I(x_0,\kappa)}\big(I(x_0,\kappa+1)\big)\big)$$

and

$$\sigma_J^{-1}\big(I\big(1 - \sigma_{I(x_0,\kappa)}(x_0), 1\big)\big) = \sigma_J^{-1}\big(1 - \sigma_{I(x_0,\kappa)}\big(I(x_0,\kappa+1)\big)\big).$$

This way we produce two disjoint subintervals of equal length in each $J \in \mathcal{I}(x_0,\kappa)$ and these intervals will be the two intervals of $\mathcal{I}(x_0,\kappa+1)$ in J.

$$(10)\quad \begin{cases} \text{If } x_1 \in \cup\mathcal{I}(x_0,\kappa+1) \cap X_1 \quad \text{then} \quad N(x_1,\kappa+1) = N(x_0,\kappa+1), \\ \mathcal{T}_{N(x_0,\kappa+1)}(x_1) = \mathcal{T}_{N(x_0,\kappa+1)}(x_0) \quad \text{and} \quad \mathcal{T}'_{N(x_0,\kappa+1)}(x_1) = 0. \end{cases}$$

We define

$$J(x_0,\kappa) = \big[\mathcal{T}_{N(x_0,\kappa)}(x_0), \mathcal{T}_{N(x_0,\kappa)}(x_0) + \lambda\big(I(x_0,\kappa)\big)\big].$$

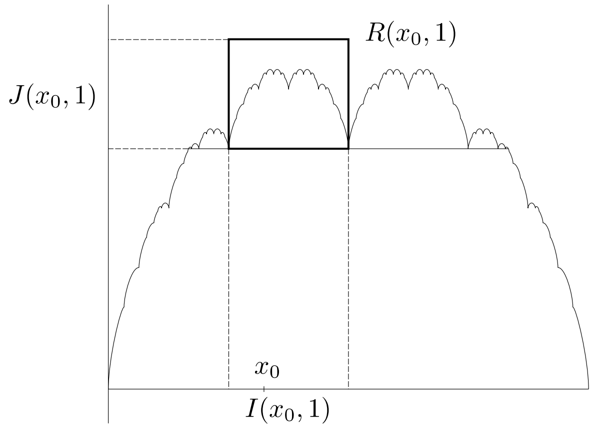

Fig. 1: $R(x_0,1)$ on the graph of $\mathcal{T}$

Set

$$R(x_0,\kappa) \stackrel{\text{def}}{=} \operatorname{cl}\big(I(x_0,\kappa)\big) \times J(x_0,\kappa)$$

(see Fig. 1). By elementary calculations for any $x_0 \in X_1$ we have

$$R(x_0,1) \subset [0,1] \times [0,1]. \tag{11}$$

Using mathematical induction and rescaling properties (7), (8) and property (11) we also have

$$R(x_0,1) \supset \ldots \supset R(x_0,\kappa) \supset R(x_0,\kappa+1) \supset \ldots . \tag{12}$$

Set

$$S_{\text{irr}} = \bigcap_{\kappa=1}^{\infty} \bigcup_{x_0 \in X_1} R(x_0,\kappa).$$

Since $\lambda\big(J(x_0,\kappa)\big) \to 0$ as $\kappa \to \infty$, S_{irr} is a subset of the graph of $\mathcal{T}$.

By the rescaling property (8) we also have

$$\Big\{\big(x;\mathcal{T}(x)\big) : x \in I(x_0,\kappa)\Big\} \subset R(x_0,\kappa).$$

Using (9) we can also see that for any $x_1 \in \cup\mathcal{I}(x_0,\kappa) \cap X_1$ we have $J(x_1,\kappa) = J(x_0,\kappa)$. Hence

$$\lambda\big(\mathcal{T}(X_1)\big) \leqq \lambda\big(\pi_y(S_{\text{irr}})\big) \tag{13}$$

$$\leqq \lambda\big(\cup_{x_1\in X_1} J(x_1,\kappa)\big) \leqq 2^{-\kappa}\lambda\big(\cup_{x_1\in X_1} I(x_1,\kappa)\big) \leqq 2^{-\kappa}.$$

Since this holds for all κ we have

$$\lambda\big(\mathcal{T}(X_1)\big) = \lambda\big(\pi_y(S_{\text{irr}})\big) = 0. \tag{14}$$

Since $\mu\big(\mathbb{R} \setminus \mathcal{T}(X_1)\big) = \lambda\big([0,1] \setminus X_1\big) = 0$ from (14) it follows that the occupation measure of Takagi's function is singular.

Next we verify that S_{irr} is an irregular 1-set. Suppose that γ is the graph of a continuously differentiable curve. By (14) the projection of $\gamma \cap S_{\text{irr}}$ onto the y-axis is of zero Lebesgue measure. If we had $\lambda\big(\pi_x(\gamma \cap S_{\text{irr}})\big) > 0$ then one could find an x_1 which is a Lebesgue density point of $\pi_x(\gamma \cap S_{\text{irr}})$ and by Theorem 8, graph$\,(\mathcal{T}) \in \mathcal{T}_{\text{MT}}(x_1)$ as well. This would imply that one could find a small interval I such that $x_1 \in I$,

$$\frac{\lambda\big(\pi_x(\gamma \cap S_{\text{irr}}) \cap I\big)}{\lambda(I)} > 0.999$$

and on $\pi_x(\gamma \cap S_{\text{irr}}) \cap I$ the function $\mathcal{T}$ stays simultaneously close to a line segment and to a rescaled copy of its own graph. This is clearly impossible.

By (7), (11) and (12) we also have

$$\bigcup_{x_1 \in X_1} J(x_1, \kappa) \supset \bigcup_{x_1 \in X_1} J(x_1, \kappa') \quad \text{for all} \quad \kappa' > \kappa. \tag{15}$$

Suppose

$$y_0 \notin \bigcap_{\kappa=1}^{\infty} \bigcup_{x_1 \in X_1} J(x_1, \kappa) = \pi_y(S_{\text{irr}}). \tag{16}$$

By (15) there exists $\kappa(y_0)$ such that for all $\kappa \geqq \kappa(y_0)$ we have

$$y_0 \notin \bigcup_{x_1 \in X_1} J(x_1, \kappa). \tag{17}$$

Set

$$g_\kappa(x_0) \stackrel{\text{def}}{=} \mathcal{T}_{N(x_0,\kappa)}(x_0) \quad \text{for} \quad x_0 \in X_1.$$

The functions g_1 and g_2 are pictured on Fig. 2. For $x \in [0,1] \setminus \cup_{x_0} I(x_0, \kappa)$ we set $g_\kappa(x) = \mathcal{T}(x)$.

By (6) for $x_1 \in I(x_0, \kappa) \cap X_1$ we have

$$g_\kappa(x_0) = g_\kappa(x_1) = \mathcal{T}_{N(x_0,\kappa)}\left(\frac{k_0(x_0,\kappa)}{2^{N(x_0,\kappa)+1}}\right) = \mathcal{T}\left(\frac{k_0(x_0,\kappa)}{2^{N(x_0,\kappa)+1}}\right). \tag{18}$$

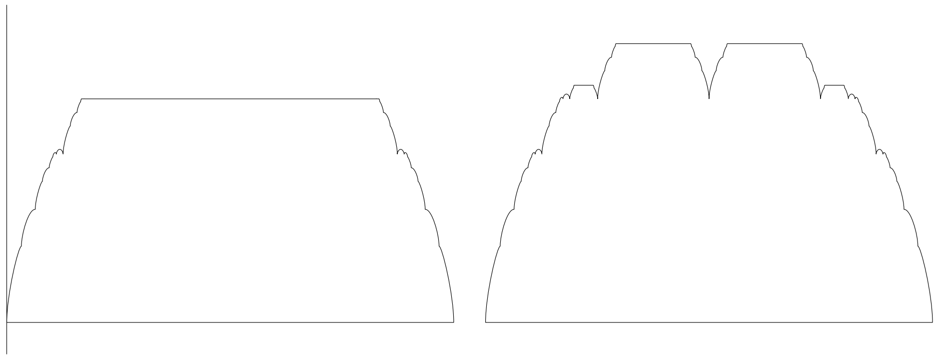

Fig. 2: g_1 *and* g_2

For any $x \in I(x_0, \kappa)$ set $g_\kappa(x) = g_\kappa(x_0)$. By (18) the function g_κ is constant on $I(x_0, \kappa)$. If $x \in I(x_0, \kappa)$ and $N(x_0, \kappa) \leqq N$ set $g_{\kappa,N}(x) = g_\kappa(x)$. If there is no x_0 with $N(x_0, \kappa) \leqq N$ such that $x \in I(x_0, \kappa)$ then set $g_{\kappa,N}(x) = \mathcal{T}_N(x)$. By using (4) observe that $g_{\kappa,N}(x)$ is continuous. It is also clear that

$$\left| g_{\kappa,N}(x) - g_\kappa(x) \right| \leqq \sum_{n=N+1}^{\infty} 2^{-n}.$$

Therefore $g_{\kappa,N}$ converges monotone increasingly and uniformly to g_κ. Moreover, $g_{1,0}(x) = \mathcal{T}_0(x)$ is monotone increasing on $[0, 1/2]$ and decreasing on $[1/2, 1]$. This implies that if $x_0 \in [0, 1/2] \cap X_1$ and $N < N(x_0, 1)$ then $\mathcal{T}_N'(x) > 0$ while if $x_0 \in [1/2, 1] \cap X_1$ and $N < N(x_0, 1)$ then $\mathcal{T}_N'(x) < 0$. Since $\mathcal{T}_N(x)$ is piecewise monotone and piecewise differentiable we obtain that $g_{1,N}$ is monotone increasing on $[0, 1/2]$ and monotone decreasing on $[1/2, 1]$. Since $g_{1,N} \to g_1$ this limit function is also monotone increasing on $[0, 1/2]$ and decreasing on $[1/2, 1]$, see Fig. 2. The set $g_1(X_1)$ is countable and if $y_0 \notin g_1(X_1)$ then $g_1^{-1}\big(\{y_0\}\big)$ has at most two elements, one in $[0, 1/2]$ and another in $[1/2, 1]$. For $x \in [0, 1] \setminus \cup_{x_0} I(x_0, 1)$ we have $g_1(x) = \mathcal{T}(x)$.

Suppose $\kappa \geqq 2$. By (13) we have $\lambda\big(\cup_{x_0} J(x_0, \kappa - 1)\big) \leqq 2^{-\kappa+1}$. Using the Borel–Cantelli lemma we obtain that almost every y belongs to only finitely many of the different intervals $J(x_0, \kappa - 1)$ (if $J(x_0, \kappa - 1) = J(x_0', \kappa - 1)$ then these intervals count as one).

By using the similarity property (7) one can see by induction that in general $g_\kappa(X_1)$ is countable and if y belongs to only finitely many different $J(x_0, \kappa - 1)$ then $g_\kappa^{-1}\big(\{y\}\big)$ has finitely many elements.

By (16) and (17) if $\kappa \geqq \kappa(y_0)$ then $g_\kappa^{-1}\big(\{y_0\}\big) = g_{\kappa(y_0)}^{-1}\big(\{y_0\}\big)$. Hence, $\mathcal{T}^{-1}\big(\{y_0\}\big) = g_{\kappa(y_0)}^{-1}\big(\{y_0\}\big)$ is finite. This implies the statement about the level sets.

The set

$$S_{\rm reg} \stackrel{\rm def}{=} \Big\{\big(x;\mathcal{T}(x)\big) : x \in [0,1] \setminus \pi_x(S_{\rm irr})\Big\}$$

is covered by the graphs of the functions g_κ. Therefore $S_{\rm reg}$ can be covered by the graphs of countably many monotone functions. $\square$

4. Typical continuous functions

THEOREM 10. *Suppose* $f \in C[0,1]$ *and* $h \in C[0,1]$ *is differentiable at* $x_0 \in (0,1)$. *If* $\big(x_0; f(x_0)\big) \in \mathrm{UMT}\,(f)$ *then*

$$\big(x_0; f(x_0) + h(x_0)\big) \in \mathrm{UMT}\,(f+h).$$

PROOF. Denote by $C[-1,1]_{0,0}$ the set of those $g \in C[-1,1]_0$ which have no local extrema or interval of constancy on the boundary of Q^2 and do not go through any vertex of Q^2. Clearly, we can approximate functions in $C[-1,1]_0$ by functions in $C[-1,1]_{0,0}$. Hence, it is sufficient to show that for any $g \in C[-1,1]_{0,0}$ the set $\mathrm{graph}\,(g) \cap Q^2$ is a micro tangent set of $f+h$ at x_0.

For $1 \geqq \delta_0 > 0$ set $g_{\delta_0}(x) = \delta_0 g\big(\frac{x}{\delta_0}\big) - h'(x_0)x$ when $x \in [-\delta_0, \delta_0]$. For $x \in [-1, -\delta_0)$ set $g_{\delta_0}(x) = \delta_0 g(-1) - h'(x_0)(-\delta_0)$ and for $x \in (\delta_0, 1]$ set $g_{\delta_0}(x) = \delta_0 g(1) - h'(x_0)\delta_0$. Then $g_{\delta_0} \in C[-1,1]_0$. Choose $1 \geqq \delta_0 > 0$ such that

$$(19) \qquad \big|g_{\delta_0}(x)\big| < 1 \quad \text{for} \quad x \in [-1,1].$$

This implies that $\mathrm{graph}\,\big(g_{\delta_0}\big) \subset Q^2$.

Since $\big(x_0; f(x_0)\big) \in \mathrm{UMT}\,(f)$ there exists $\delta_n \searrow 0$ such that $F(f, x_0, \delta_n)$ converges to the graph of g_{δ_0} in the Hausdorff metric, which by (19) also means uniform convergence for the corresponding functions.

It is also clear that both $F(h, x_0, \delta_n)$ and $F(h, x_0, \delta_0\delta_n)$ converge to the part of the line $y = h'(x_0)x$ which is in Q^2. Then it is not difficult to see (using $g \in C[-1,1]_{0,0}$) that $F(f+h, x_0, \delta_0\delta_n)$ converges in the Hausdorff metric to $\mathrm{graph}\,(g) \cap Q^2$. $\square$

THEOREM 11. *If* $f \in C[0,1]$ *then*

$$(20) \qquad \mathcal{H}^1\big(\mathrm{UMT}\,(f)\big) \leqq 2.$$

PROOF. Suppose that $g_0(x)=0$ for all $x\in[-1,1]$. As in the proof of Theorem 7 based on Lemma 6 (see [7]) we use again the fact that $\mathrm{UMT}(f)\subset \mathrm{GLMT}_{g_0}(f)$. Therefore, $\mathrm{graph}(g_0)=\mathrm{graph}(g_0)\cap Q^2\subset f_{\mathrm{MT}}(x_1)$ for any $\big(x_1;f(x_1)\big)\in \mathrm{UMT}(f)$. By the definition of $f_{\mathrm{MT}}(x_1)$ there exists a sequence $\delta_{n,x_1}\searrow 0$ such that $F(f,x_1,\delta_{n,x_1})$ converges to $\mathrm{graph}(g_0)$ in the Hausdorff metric. This means that we can choose cubes Q_{n,x_1} of side length $2\delta_{n,x_1}$ such that the graph of f over $\big(x_1-\delta_{n,x_1},x_1+\delta_{n,x_1}\big)$, that is the set $\big\{\big(x;f(x)\big):x\in\big(x_1-\delta_{n,x_1},x_1+\delta_{n,x_1}\big)\big\}$ is covered by the interior of Q_{n,x_1}. Therefore, given $\rho>0$ we can find a Vitali cover of $\mathrm{UMT}(f)$ with cubes of the form Q_{n,x_1} satisfying

$$\big|Q_{n,x_1}\big|=2\sqrt{2}\delta_{n,x_1}<4\delta_{n,x_1}\quad\text{and}\quad\big|Q_{n,x_1}\big|<\rho. \tag{21}$$

By Theorem 1 we can select disjoint cubes $Q(k)\overset{\mathrm{def}}{=}Q_{n_k,x_{1,k}}$ such that

$$\mathcal{H}^1\big(\mathrm{UMT}(f)\setminus\cup_k Q(k)\big)=0.$$

Since the cubes $Q(k)$ are disjoint and contain $\big(x;f(x)\big)$ when $x\in\pi_x\big(Q(k)\big)$ the intervals $\pi_x\big(Q(k)\big)=\big[x_{1,k}-\delta_{n_k,x_{1,k}},x_{1,k}+\delta_{n_k,x_{1,k}}\big]$ are non-overlapping and we have $\sum_k 2\delta_{n_k,x_{1,k}}\leqq 1$. This implies by (21)

$$\sum_k\big|Q(k)\big|<\sum_k 4\delta_{n_k,x_{1,k}}\leqq 2.$$

Since $\rho>0$ is arbitrary we obtain (20). □

REMARK 12. Taking a typical continuous function $f(x)$ one can apply Theorem 10 with $h_c(x)=cx$ where $c\in\mathbb{R}$ is an arbitrary constant. Since $\mathrm{UMT}(f+h_c)\subset\mathrm{GLMT}_{g_0}(f+h_c)$, by Lemma 6, $\lambda\Big(\pi_y\big(\mathrm{UMT}(f+h_c)\big)\Big)=0$ holds for any $c\in\mathbb{R}$. By Theorem 10 this implies that the projection of $\mathrm{UMT}(f)$ onto any line non-parallel with the x-axis is of λ measure zero. On the other hand, by Theorem 4 its projection onto the x-axis is of λ-measure one. Theorem 11 implies that $\mathcal{H}^1\big(\mathrm{UMT}(f)\big)\leqq 2<\infty$ (in fact, one can verify that $\mathcal{H}^1\big(\mathrm{UMT}(f)\big)=1$). Hence, $\mathrm{UMT}(f)$ for the typical continuous function is an irregular 1-set. By a result of R. D. Mauldin and S. C. Williams [14] the graph of the typical continuous function is of Hausdorff dimension one, but is not of σ-finite-$\mathcal{H}^1$-measure. For the typical continuous function f the set $\mathrm{graph}(f)\setminus\mathrm{UMT}(f)$ is "large", unlike S_{reg} for the Takagi, or for the Weierstrass function. Indeed, by (20), $\mathcal{H}^1\big(\mathrm{UMT}(f)\big)\leqq 2<\infty$ and the union of countably many rectifiable curves is of σ-finite $\mathcal{H}^1$-measure hence $\mathrm{graph}(f)\setminus\mathrm{UMT}(f)$ cannot be covered by countably many rectifiable curves and by an $\mathcal{H}^1$-zero set for the typical continuous function f.

5. Weierstrass–Cellerier type functions

In this section we work with functions f defined in (1). We also put

$$f_N(x) = f_{-1}(x) + \sum_{n=0}^{N} 2^{-n} \sin(2\pi 2^n x).$$

If $f_{-1}(x)$ equals identically zero then, following the notation of (2), we denote f_N by $\mathcal{W}_N$. We will need the estimate

$$(22) \qquad \left| \mathcal{W}_N''(x) \right| = \left| \sum_{n=0}^{N} -4\pi^2 2^n \sin(2\pi 2^n x) \right| < 8\pi^2 2^N.$$

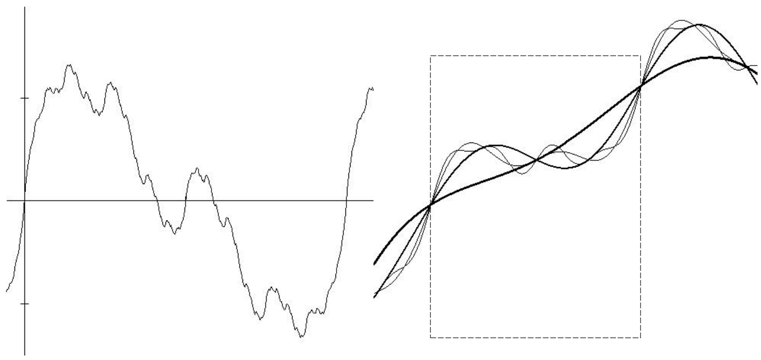

Fig. 3: $\mathcal{W}(x)$ and a small part of $\mathcal{W}_N$ for $N = 3, 4, 5, 6$

Set

$$X_1 = \Big\{ x \in [0,1] : \liminf_{N\to\infty} f_N'(x) = -\infty,\ \limsup_{N\to\infty} f_N'(x) = +\infty \Big\}$$

$$= \Big\{ x \in [0,1] : \liminf_{N\to\infty} \mathcal{W}_N'(x) = -\infty,\ \limsup_{N\to\infty} \mathcal{W}_N'(x) = +\infty \Big\}.$$

By results of [7] concerning the function $\mathcal{W}$ defined in (2),

$$\liminf_{N\to\infty} f_N'(x) = \liminf_{N\to\infty} (f_N - f_{-1})'(x) = -\infty$$

and

$$\limsup_{N\to\infty} f_N'(x) = \limsup_{N\to\infty} (f_N - f_{-1})'(x) = +\infty$$

for almost every x. For these x's infinitely often $f'_N(x)$ and $f'_{N+1}(x)$ are not of the same sign. Since

$$\left| f'_{N+1}(x) - f'_N(x) \right| = \left| \left(2^{-N} \sin(2\pi 2^N x) \right)' \right| = \left| 2\pi \cos\left(2\pi 2^N x\right) \right| \leqq 2\pi$$

for every $x \in X_1$ there exist infinitely many N's such that $\left| f'_N(x) \right| \leqq 2\pi$.

THEOREM 13. *The set* $S_{\mathrm{irr}} = \left\{ \left(x; f(x) \right) : x \in X_1 \right\}$ *is an irregular* 1*-set, the set* $S_{\mathrm{reg}} = \left\{ \left(x; f(x) \right) : x \in [0,1] \setminus X_1 \right\}$ *can be covered by the union of the graphs of countably many strictly monotone functions and by a set of zero* $\mathcal{H}^1$*-measure.*

REMARK 14. If one wishes, the above set of zero measure can be added to S_{irr} and this way we can obtain a modified S^*_{irr} and S^*_{reg}. Then the set S^*_{reg} can be covered by the union of the graphs of countably many strictly monotone functions.

PROOF OF THEOREM 13. The projection of S_{irr} onto the x-axis is of full measure. Thus $\mathcal{H}^1(S_{\mathrm{irr}}) > 0$. Since $\sin(k\pi) = 0$ for $k \in \mathbb{Z}$ we have

$$f_N \left(\frac{k}{2^{N+1}} \right) = f \left(\frac{k}{2^{N+1}} \right) \tag{23}$$

for any $N \geqq 0$ and $k \in \{0, \dots, 2^{N+1}\}$.

Denote by $M_{-1,0}$, $M_{-1,1}$ and $M_{-1,2}$ the bounds of $|f_{-1}|$, $\left| f'_{-1} \right|$ and $\left| f''_{-1} \right|$ on $[0,1]$, respectively. Suppose $\left[\frac{k_0}{2^{N_0}}, \frac{k_0+1}{2^{N_0}} \right) \subset [0,1)$. By Lagrange's Mean Value Theorem

$$\left| f_{-1} \left(\frac{k_0+1}{2^{N_0}} \right) - f_{-1} \left(\frac{k_0}{2^{N_0}} \right) \right| \leqq M_{-1,1} \cdot \frac{1}{2^{N_0}}. \tag{24}$$

We also have

$$\left| \sum_{n=N_0+1}^{\infty} 2^{-n} \sin(2\pi 2^n x) \right| \leqq \frac{2}{2^{N_0+1}} = \frac{1}{2^{N_0}}. \tag{25}$$

By (22)

$$\left| f''_N(x) \right| \leqq \left| f''_{-1}(x) \right| + \left| \mathcal{W}''_N(x) \right| \leqq M_{-1,2} + 8\pi^2 2^N. \tag{26}$$

Suppose $x_0 \in X_1$ and $\left| f'_{N_0}(x_0) \right| \leqq 2\pi$. If $x, x_0 \in \left[\frac{k_0}{2^{N_0}}, \frac{k_0+1}{2^{N_0}} \right) \subset [0,1)$ then by (26) and the Mean Value Theorem

$$\left| f'_{N_0}(x) \right| \leqq \left| f'_{N_0}(x) - f'_{N_0}(x_0) \right| + \left| f'_{N_0}(x_0) \right| \tag{27}$$

$$\leqq \left(M_{-1,2} + 8\pi^2 2^{N_0}\right) 2^{-N_0} + 2\pi < M_{-1,2} + 100.$$

By (27) and again by the Mean Value Theorem

$$\left| f_{N_0}(x) - f_{N_0}\left(\frac{k_0}{2^{N_0}}\right) \right| < (M_{-1,2} + 100) 2^{-N_0} \tag{28}$$

for $x \in \left[\frac{k_0}{2^{N_0}}, \frac{k_0+1}{2^{N_0}}\right) \subset [0,1)$.

Using (23), (25) and (28) we obtain

$$\left| f(x) - f\left(\frac{k_0}{2^{N_0}}\right) \right| \leqq (M_{-1,2} + 101) 2^{-N_0} \tag{29}$$

for $x \in \left[\frac{k_0}{2^{N_0}}, \frac{k_0+1}{2^{N_0}}\right) \subset [0,1)$.

Suppose N_0' is given. For each $x_1 \in X_1$ we can choose the smallest $N_0(x_1, N_0') \geqq N_0'$ such that there exists $x_0(x_1, N_0')$ for which both x_1 and $x_0(x_1, N_0')$ belong to

$$I(x_1, N_0') \stackrel{\text{def}}{=} \left(\frac{k_0(x_1, N_0')}{2^{N_0(x_1, N_0')}}, \frac{k_0(x_1, N_0') + 1}{2^{N_0(x_1, N_0')}} \right) \subset [0,1)$$

and $\left| f'_{N_0(x_1,N_0')}\left(x_0(x_1, N_0')\right) \right| \leqq 2\pi$. Since $f'_N(k_0/2^{N_0}) \to \infty$ as $N \to \infty$ points of the form $k_0/2^{N_0}$ do not belong to X_1.

By the definition of X_1 for $x_1 = x_0$ there are infinitely many $N_0 \geqq N_1'$ for which $\left| f'_{N_0}(x_0) \right| \leqq 2\pi$ but it may happen that for other x_0's there are smaller N_0's, this motivates our definition of $N_0(x_1, N_0')$.

For fixed N_0' and arbitrary $x_1, x_2 \in X_1$ the dyadic grid intervals $I(x_1, N_0')$ and $I(x_2, N_0')$ either coincide, or are disjoint by the minimality of $N_0(x_1, N_0')$ and $N_0(x_2, N_0')$. The countably many disjoint different intervals of the form $I(x_1, N_0')$ cover X_1, denote this covering by $\mathcal{I}(N_0')$. Set

$$J(x_1, N_0') = \left[f_{N_0(x_1,N_0')}\left(\frac{k(x_1, N_0')}{2^{N_0(x_1,N_0')}} \right) - (M_{-1,2} + 101) 2^{-N_0}, \right. \tag{30}$$

$$\left. f_{N_0(x_1,N_0')}\left(\frac{k(x_1, N_0')}{2^{N_0(x_1,N_0')}} \right) + (M_{-1,2} + 101) 2^{-N_0} \right].$$

Then by (29)

$$\left\{ f(x) : x \in I(x_1, N_0') \right\} \subset J(x_1, N_0') \tag{31}$$

and rectangles of the form $R(x_1, N_0') = I(x_1, N_0') \times J(x_1, N_0')$ cover the set

$$S_{\text{irr}} \stackrel{\text{def}}{=} \big\{ \big(x; f(x)\big) \,:\, x \in X_1 \big\}.$$

The system of different rectangles of the form $R(x_1, N_0')$ will be denoted by $\mathcal{R}(N_0')$. By (29) the diameter of $R(x_1, N_0')$ is less than $\lambda\big(I(x_1, N_0')\big)\, 2(M_{-1,2} + 102)$ and $\sum_{I \in \mathcal{I}(N_0')} \lambda(I) = 1$ implies that the sum of the diameters of $R \in \mathcal{R}(N_0')$ is less than $2(M_{-1,2} + 102)$. The diameter of any R in $\mathcal{R}(N_0')$ is less than $2^{-N_0'} 2(M_{-1,2} + 102)$. Hence using the coverings of S_{irr} by the rectangles in $\mathcal{R}(N_0')$ we see that $\mathcal{H}^1(S_{\text{irr}}) < +\infty$. Since the projection of S_{irr} onto the x-axis differs from $[0,1]$ by a set of Lebesgue measure zero we verified that S_{irr} is a 1-set.

Set

$$X_1^* = \bigcup_{N_0=1}^{\infty} \bigcap_{N_0' \geqq N_0} \bigcup_{I \in \mathcal{I}(N_0')} I.$$

Then $X_1 \subset X_1^*$.

Next we interrupt the proof of Theorem 13 by two claims.

Claim 15. *If $H \subset X_1^*$ is Lebesgue measurable then*

$$\mathcal{H}^1\Big(\big\{ \big(x; f(x)\big) \,:\, x \in H \big\}\Big) \leqq \lambda(H) 2(M_{-1,2} + 102). \tag{32}$$

Especially,

$$\text{if} \quad \lambda(H) = 0 \quad \text{then} \quad \mathcal{H}^1\Big(\big\{ \big(x; f(x)\big) \,:\, x \in H \big\}\Big) = 0. \tag{33}$$

Proof of the claim. Suppose that the open set $G \subset \mathbb{R}$ contains H and $\lambda(G) \leqq \lambda(H) + \varepsilon$. Choose N_0 such that for $N_0' \geqq N_0$ any $I \subset \mathcal{I}(N_0')$ is of diameter less than ε, that is, $2^{-N_0'} \leqq 2^{-N_0} < \varepsilon$. Cover H by intervals $I \in \cup_{N_0' \geqq N_0} \mathcal{I}(N_0')$ such that $I \subset G$. Since the system of intervals $\cup_{N_0' \geqq N_0} \mathcal{I}(N_0')$ consists of dyadic grid intervals we can select a countable disjoint subsystem $I_1, I_2, \ldots$ which covers H and $I_j \subset G$ for all j. For each I_j the set $\big\{ \big(x; f(x)\big) \,:\, x \in I_j \big\}$ is a subset of a rectangle of diameter less than or equal to $\lambda(I_j) 2(M_{-1,2} + 102)$. This implies (32). $\square$

Claim 16. *Suppose $x_0 \in \big[\frac{k_0}{2^{N_0}}, \frac{k_0+1}{2^{N_0}}\big]$, $\Delta_0 = f_{N_0}'(x_0)$. Then*

i) *for any $x_1 \in \big[\frac{k_0}{2^{N_0}}, \frac{k_0}{2^{N_0}} + \frac{1}{2 \cdot 2^{N_0}}\big]$ and $x_2 \in \big[\frac{k_0+1}{2^{N_0}} - \frac{1}{4 \cdot 2^{N_0}}, \frac{k_0+1}{2^{N_0}}\big]$ we have*

$$\left| \frac{f(x_2) - f(x_1)}{x_2 - x_1} \right| \geqq \big(|\Delta_0| - (M_{-1,2} + 100)\big) - 8; \tag{34}$$

ii) *for any $x_1 \in \big[\frac{k_0+1}{2^{N_0}} - \frac{1}{2 \cdot 2^{N_0}}, \frac{k_0+1}{2^{N_0}}\big]$ and $x_2 \in \big[\frac{k_0}{2^{N_0}}, \frac{k_0}{2^{N_0}} + \frac{1}{4 \cdot 2^{N_0}}\big)$ we also have* (34).

PROOF OF THE CLAIM. The two cases are similar. We prove only case i). By (26) and the Mean Value Theorem applied to f'_{N_0} we have

$$\left| f'_{N_0}(x) - f'_{N_0}(x_0) \right| = \left| f'_{N_0}(x) - \Delta_0 \right|$$
$$\leqq (M_{-1,2} + 8\pi^2 2^{N_0}) 2^{-N_0} \leqq (M_{-1,2} + 100)$$

for any $x \in \left[\frac{k_0}{2^{N_0}}, \frac{k_0+1}{2^{N_0}}\right]$. Therefore,

$$\left| f'_{N_0}(x) \right| > |\Delta_0| - (M_{-1,2} + 100) \tag{35}$$

holds for any $x \in \left[\frac{k_0}{2^{N_0}}, \frac{k_0+1}{2^{N_0}}\right]$. This implies that if assumption i) holds then by the Mean Value Theorem

$$\begin{cases} \left| \dfrac{f_{N_0}(x_2) - f_{N_0}(x_1)}{x_2 - x_1} \right| \geqq |\Delta_0| - (M_{-1,2} + 100) \\ \text{and} \quad \dfrac{1}{2^{N_0}} \geqq |x_2 - x_1| \geqq \dfrac{1}{4 \cdot 2^{N_0}}. \end{cases} \tag{36}$$

By (25) we have $\left| f(x_i) - f_{N_0}(x_i) \right| \leqq \frac{1}{2^{N_0}}$ for $i = 1, 2$. Therefore,

$$\left| \frac{f_{N_0}(x_2) - f_{N_0}(x_1)}{x_2 - x_1} - \frac{f(x_2) - f(x_1)}{x_2 - x_1} \right| \leqq \frac{2 \cdot 2^{-N_0}}{(1/4) 2^{-N_0}} = 8.$$

This and (36) imply (34). $\square$

Now we continue the proof of Theorem 13. Suppose that γ is the graph of a continuously differentiable curve.

Let X_γ denote the projection of $\gamma \cap S_{\text{irr}}$ onto the x-axis. Then $X_\gamma \subset X_1$. Since $\left\{ \left(x; f(x) \right) : x \in X_\gamma \right\} = \gamma \cap S_{\text{irr}}$ by Claim 15 from $\lambda(X_\gamma) = 0$ it follows that $\mathcal{H}^1(\gamma \cap S_{\text{irr}}) = 0$. So we need to show that $\lambda(X_\gamma) = 0$. Proceeding towards a contradiction suppose that $\lambda(X_\gamma) > 0$. Since γ is continuously differentiable the (possibly infinite) derivative of $f|_{X_\gamma}$ would exist at all accumulation points of X_γ.

By the Denjoy–Young–Saks theorem (Ch. IX., §3 of [15]) if $X_{\gamma,\infty}$ denotes the set of those x's where $f|_{X_\gamma}$ has infinite derivative then $\lambda(X_{\gamma,\infty}) = 0$ and hence $\lambda(X_\gamma \setminus X_{\gamma,\infty}) > 0$. Suppose $x_0 \in X_\gamma \setminus X_{\gamma,\infty}$ is a Lebesgue density point of $X_\gamma \setminus X_{\gamma,\infty}$. Since γ is differentiable at $\left(x_0; f(x_0) \right)$ and $x_0 \notin X_{\gamma,\infty}$ we have

$$\lim_{x \to x_0,\ x \in X_\gamma} \frac{f(x) - f(x_0)}{x - x_0} = D_\gamma(x_0) \in \mathbb{R}. \tag{37}$$

Using that x_0 is a density point of $X_\gamma \setminus X_{\gamma,\infty}$, $x_0 \in X_1$ and (37) we can choose N_0 such that

$$\lambda\left(\left[\frac{k_0}{2^{N_0}}, \frac{k_0+1}{2^{N_0}}\right) \cap (X_\gamma \setminus X_{\gamma,\infty})\right) > \frac{3}{4} \cdot \frac{1}{2^{N_0}}, \tag{38}$$

$$\left| f'_{N_0}(x_0)\right| = |\Delta_0| > \left| D_\gamma(x_0)\right| + 9 + M_{-1,2} + 100 \tag{39}$$

and

$$\left|\frac{f(x)-f(x_0)}{x-x_0}\right| < \left|D_\gamma(x_0)\right| + 1 \quad \text{for} \quad x \in \left[\frac{k_0}{2^{N_0}}, \frac{k_0+1}{2^{N_0}}\right] \cap (X_\gamma \setminus X_{\gamma,\infty}). \tag{40}$$

Without limiting generality we can suppose that $x_0 \in \left[\frac{k_0}{2^{N_0}}, \frac{k_0}{2^{N_0}} + \frac{1}{2\cdot 2^{N_0}}\right]$ (otherwise at the next step one needs to use ii) of Claim 16). By i) of Claim 16 for any $x_2 \in \left[\frac{k_0+1}{2^{N_0}} - \frac{1}{4\cdot 2^{N_0}}, \frac{k_0+1}{2^{N_0}}\right]$ we have

$$\left|\frac{f(x_2)-f(x_0)}{x_2-x_0}\right| \geqq |\Delta_0| - M_{-1,2} - 100 - 8 > \left|D_\gamma(x_0)\right| + 1. \tag{41}$$

By (38) there exists $x_2 \in \left[\frac{k_0+1}{2^{N_0}} - \frac{1}{4\cdot 2^{N_0}}, \frac{k_0+1}{2^{N_0}}\right) \cap (X_\gamma \setminus X_{\gamma,\infty})$ and hence (40) and (41) contradict. This implies that S_{irr} is an irregular 1-set.

Next we show that $S^*_{\mathrm{reg}} = \left\{\left(x; f(x)\right) : x \in [0,1] \setminus X^*_1\right\}$ can be covered by countably many graphs of strictly monotone functions. Set

$$X_{3,M} \overset{\mathrm{def}}{=} [0,1] \setminus \bigcup_{I(x_1,M)\in\mathcal{I}(M)} I(x_1,M) = [0,1] \setminus \bigcup_{I\in\mathcal{I}(M)} I,$$

that is, the set of those points which are not covered by an interval of the form $I(x_1,M)$.

Claim 17. *Either f_M is constant on $[0,1]$ or the set $\left\{\left(x; f(x)\right) : x \in X_{3,M}\right\}$ can be covered by the graphs of finitely many strictly monotone increasing functions.*

Proof of the claim. We can suppose that f_M is not constant on $[0,1]$. By the choice of $\mathcal{F}_{\mathcal{W}}$ the function $f_M(x)$ is piecewise strictly monotone, that is, $[0,1]$ can be decomposed into finitely many closed intervals on which f_M is strictly monotone. Suppose that I_1 is such an interval. Without limiting generality we suppose that f_M is strictly monotone increasing on I_1. Each $x_1 \in I_1 \cap X_1$ is covered by an interval

$$I(x_1,M) = \left(\frac{k_0(x_1,M)}{2^{N_0(x_1,M)}}, \frac{k_0(x_1,M)+1}{2^{N_0(x_1,M)}}\right).$$

If $I(x_1,M)\subset I_1$ and $N_0(x_1,M)=M$ then by the strict monotonicity of f_M we have

$$0<f_M\left(\frac{k_0(x_1,M)+1}{2^{N_0(x_1,M)}}\right)-f_M\left(\frac{k_0(x_1,M)}{2^{N_0(x_1,M)}}\right) \tag{42}$$
$$=f\left(\frac{k_0(x_1,M)+1}{2^{N_0(x_1,M)}}\right)-f\left(\frac{k_0(x_1,M)}{2^{N_0(x_1,M)}}\right).$$

Suppose $I(x_1,M)\subset I_1$ and $N_0(x_1,M)>M$. Then for any $x_2\in I(x_1,M)$ and $M\leqq N<N_0(x_1,M)$ we have $\left|f_N'(x_2)\right|>2\pi$ and $f_M'(x_2)>2\pi$. Since $\left|f_{N+1}'(x_2)-f_N'(x_2)\right|=\left|2\pi\cos\left(2\pi 2^{N+1}x\right)\right|\leqq 2\pi$ this also implies that $f_N'(x_2)>2\pi$ holds for any $x_2\in I(x_1,M)$ and $M\leqq N<N_0(x_1,M)$ and we also have $f_{N_0(x_1,M)}'(x_2)>0$ as well.

Thus

$$0<f_{N_0(x_1,M)}\left(\frac{k_0(x_1,M)+1}{2^{N_0(x_1,M)}}\right)-f_{N_0(x_1,M)}\left(\frac{k_0(x_1,M)}{2^{N_0(x_1,M)}}\right) \tag{43}$$
$$=f\left(\frac{k_0(x_1,M)+1}{2^{N_0(x_1,M)}}\right)-f\left(\frac{k_0(x_1,M)}{2^{N_0(x_1,M)}}\right).$$

Denote by $\widehat{f}_N$ the function which equals $f_N(x)$ for

$$x\notin E_N\overset{\text{def}}{=}\cup_{x_1\in X_1,\ N_0(x_1,M)\leqq N}\,I(x_1,M),$$

and on the intervals $I(x_1,M)$ with $N_0(x_1,M)\leqq N$ the function $\widehat{f}_N$ is linear and equals f_N at the endpoints of $I(x_1,M)$. Then $\widehat{f}_N$ is strictly monotone increasing on $I_1\cap X_{3,M}$ for any $N\geqq M$, in fact $\widehat{f}_N$ is strictly monotone on the interval $\left[\inf\{x\in I_1\cap X_{3,M}\},\sup\{x\in I_1\cap X_{3,M}\}\right]\subset I_1$, since it is continuous and has positive derivative at all but finitely many points in this interval. Since $\widehat{f}_N(x)=f_N(x)\to f(x)$ as $N\to\infty$ for any $x\in[0,1]\setminus\cup_{N=1}^{\infty}E_N=X_{3,M}$ we obtain that f is monotone increasing on $I_1\cap X_{3,M}$.

Since $I_1\setminus X_1$ is of zero Lebesgue measure by (42) and (43) one can see that f is strictly monotone increasing on $I_1\cap X_{3,M}$. Hence $S_{\text{reg},M}^*=\big\{\big(x;f(x)\big):x\in X_{3,M}\big\}$ can be covered by the graphs of finitely many strictly monotone functions. This proves Claim 17. $\square$

Observe that for any N_0

$$[0,1]\setminus X_1^*\subset\bigcup_{M\geqq N_0}X_{3,M}$$

and there can be at most one M such that f_M is constant on $[0,1]$. Hence, choosing a sufficiently large N_0 for any $M \geqq N_0$ we can apply Claim 17 and $S^*_{\text{reg}} \subset \cup_{M \geqq N_0} S^*_{\text{reg},M}$ can be covered by the graphs of countably many strictly monotone increasing functions. Since $\lambda(X_1^* \setminus X_1) = 0$ by Claim 15, $\mathcal{H}^1\big(\big\{(x; f(x)) : x \in X_1^* \setminus X_1\big\}\big) = 0$ and hence $S_{\text{reg}} = \big\{(x; f(x)) : x \in [0,1] \setminus X_1\big\}$ can be covered by the graphs of countably many strictly monotone functions and by an $\mathcal{H}^1$-null set. □

THEOREM 18. *If $f_{-1} \in \mathcal{F}_{\mathcal{W}}$ then the occupation measure of any f defined by* (1) *is nonatomic.*

PROOF. If we had y_0 such that $\mu\big(\{y_0\}\big) = \lambda\big(\big\{x : f(x) = y_0\big\}\big) > 0$ then S_{irr} would intersect the line $y = y_0$ in a set of positive $\mathcal{H}^1$-measure, contradicting the irregularity of S_{irr}. □

THEOREM 19. *If $f_{-1} \in \mathcal{F}_{\mathcal{W}}$ and the occupation measure of f defined in* (1) *is singular then almost every level set of $f(x)$ is finite.*

PROOF. The occupation measure of f will be denoted by μ. Since μ is singular there exists $Y_\mu \subset \mathbb{R}$ such that $\lambda(Y_\mu) = 0$ and $\mu(Y_\mu) = 1$, that is, there exists $X_\mu \subset [0,1]$ with $\lambda(X_\mu) = 1$ such that $X_\mu = f^{-1}(Y_\mu)$. Then, by using notation from the proof of Theorem 13, $f(X_\mu \cap X_1^*) \subset Y_\mu$. By $\lambda(X_1^* \setminus X_\mu) = 0$ and by Claim 15 we obtain $\mathcal{H}^1\big(\big\{(x; f(x)) : x \in X_1^* \setminus X_\mu\big\}\big) = 0$ and hence $\lambda\big(f(X_1^* \setminus X_\mu)\big) = 0$ as well. Therefore, $\lambda\big(f(X_1^*)\big) = 0$. Suppose $\varepsilon > 0$ and

$$\text{(44)} \qquad \text{choose an open} \quad G \supset f(X_1^*) \quad \text{such that} \quad \lambda(G) < \varepsilon.$$

For each $x_1 \in X_1 \subset X_1^*$ there exists $N_0''(x_1)$ such that for $N_0' \geqq N_0''(x_1)$ by (30) the interval $J(x_1, N_0') \subset G$ and (31) holds as well.

Choose M such that

$$\text{(45)} \qquad \lambda\big(\big\{x_1 \in X_1 : N_0''(x_1) \leqq M\big\}\big) > 1 - \frac{\varepsilon}{2(M_{-1,2} + 102)}.$$

There can be at most one M for which f_M is constant on $[0,1]$. We can suppose that M is chosen so that f_M is not constant on $[0,1]$. Then consider the covering $\mathcal{I}(M)$ of $X_1 \subset X_1^*$. This covering can be split into two subcoverings. We denote by $\mathcal{I}_1(M)$ the set of those $I \in \mathcal{I}(M)$ for which there exists $x_1 \in X_1$ such that $I = I(x_1, M)$ and $N_0''(x_1) \leqq M$. Those intervals in $\mathcal{I}(M)$ which do not belong to $\mathcal{I}_1(M)$ will form the system $\mathcal{I}_2(M)$. By (45)

$$\lambda(\cup_{I \in \mathcal{I}_1(M)} I) > 1 - \frac{\varepsilon}{2(M_{-1,2} + 102)}$$

and hence by the disjointness of the intervals in $\mathcal{I}(M)$ we have

$$\lambda\big(\cup_{I\in\mathcal{I}_2(M)} I\big) < \frac{\varepsilon}{2(M_{-1,2}+102)}. \tag{46}$$

By (30) for any $I\in\mathcal{I}_2(M)\subset\mathcal{I}(M)$ we have

$$\lambda\big(f(I)\big) \leqq 2(M_{-1,2}+101)\lambda(I). \tag{47}$$

Hence by (46) and (47)

$$\lambda\big(f\big(\cup_{I\in\mathcal{I}_2(M)} I\big)\big) < \varepsilon. \tag{48}$$

By the choice of M

$$\text{for } I\in\mathcal{I}_1(M) \quad \text{we have} \quad f(I)\subset G. \tag{49}$$

Set $Y(M)=f(\cup_{I\in\mathcal{I}(M)}I)$. Using (44), (48) and (49) we infer

$$\lambda\big(Y(M)\big) < 2\varepsilon. \tag{50}$$

Suppose $y\notin Y(M)$. Then

$$f^{-1}\big(\{y\}\big) \subset [0,1]\setminus\cup_{I\in\mathcal{I}(M)}I = X_{3,M}.$$

By Claim 17 the graph of f restricted to $X_{3,M}$ can be covered by the graphs of finitely many strictly monotone increasing functions. This implies that $f^{-1}\big(\{y\}\big)$ is finite. Since $\varepsilon>0$ can be arbitrary by (50) we obtain that for almost every y the level set $\big\{x: f(x)=y\big\}$ is finite. $\square$

5.1. Consequences for the Weierstrass–Cellerier function. Taking $f_0(x)\equiv 0$ we have $f(x)=\mathcal{W}(x)$. By the projection theorems for irregular 1-sets (Theorem 6.13 of [10]) for almost every direction the projection of S_{irr} is of zero Lebesgue measure. Denote by $X_{1,\mathcal{W}}$ the set X_1 corresponding to $\mathcal{W}$. By the above projection property for almost every $c\in\mathbb{R}$ the projection of the set $X_{\text{irr},c}=\big\{\big(x;\mathcal{W}(x)+cx\big): x\in X_{1,\mathcal{W}}\big\}$ onto the y-axis is of zero Lebesgue measure. This implies that the occupation measure of $\mathcal{W}(x)+cx$ is singular for almost every $c\in\mathbb{R}$. Furthermore, by Theorem 19 for these c almost every level set of $\mathcal{W}(x)+cx$ is finite. In the forthcoming paper [8] I will show that here there are no exceptional sets and especially for $\mathcal{W}(x)$ almost every level set is finite.

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