A MAXIMAL INEQUALITY FOR THE TAIL OF THE BILINEAR HARDY-LITTLEWOOD FUNCTION

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Abstract. Let $(X, \mathcal{B}, \mu, T)$ be an ergodic dynamical system on a non-atomic finite measure space. Consider the maximal function $R^* : (f,g) \in L^p \times L^q \to R^*(f,g)(x) = \sup_n \dfrac{f(T^n x) g(T^{2n}x)}{n}$. We obtain the following maximal inequality. For each $1 < p \leq \infty$ there exists a finite constant C_p such that for each $\lambda > 0$, and nonnegative functions $f \in L^p$ and $g \in L^1$

$$\mu\{x : R^*(f,g)(x) > \lambda\} \leq C_p\left(\frac{\|f\|_p \|g\|_1}{\lambda} + \left(\frac{\|f\|_p \|g\|_1}{\lambda}\right)^{1/2}\right).$$

We also show that for each $\alpha > 2$ the maximal function $R^*(f,g)$ is a.e. finite for pairs of functions $(f,g) \in (L(\log L)^{2\alpha}, L^1)$.

1. Introduction

In [1] we proved the following maximal inequality about the maximal function $R^*(f,g)(x) = \sup_n \frac{f(T^n x) g(T^{2n}x)}{n}$. For each $1 < p \leq \infty$, there exists a finite constant C'_p such that for each $\lambda > 0$, for every $f \in L^p, f > 1$ and $g \in L^1, g > 1$

$$\mu\{x : R^*(f,g)(x) > \lambda\} \leq C'_p\Big(\frac{\|f\|_p^p \|g\|_1}{\lambda}\Big)^{1/2}. \tag{1}$$

Furthermore the constant C'_p behaves like $\frac{1}{p-1}$ when p tends to 1. To be more precise, we will use that there exists $\widetilde{C}'$ such that for any $1 < p < 2$ we have

$$C'_p \leq \frac{\widetilde{C}'}{p-1}. \tag{2}$$

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Inequality (1) was enough to prove the a.e. convergence to zero of the tail $\dfrac{f(T^n x)g(T^{2n}x)}{n}$ of the double recurrence averages $\dfrac{1}{n}\sum_{k=1}^{n} f(T^k x)g(T^{2k}x)$ for pairs of functions (f, g) in $L^p \times L^1$ (or $L^1 \times L^p$) as soon as $p > 1$. On the other hand, the tail was used to show that these averages do not converge a.e. for pairs of (L^1, L^1) functions.

During the 2007 Ergodic Theory workshop at UNC-Chapel Hill, J.P. Conze asked if this inequality could be made homogeneous with respect to f and g. In this paper first we derive from (1) the following homogeneous version.

Theorem 1. *For each $1 < p < \infty$ there exists a finite constant C_p such that for each $f, g \geq 0$ and for all $\lambda > 0$ we have*

$$\mu\{x : \sup_n \frac{f(T^n x)g(T^{2n}x)}{n} > \lambda\} \leq C_p\left[\left(\frac{\|f\|_p\|g\|_1}{\lambda}\right)^{1/2} + \frac{\|f\|_p\|g\|_1}{\lambda}\right], \tag{3}$$

and there exists $\widetilde{C}$ such that for any $1 < p < 2$ we have

$$C_p \leq \frac{\widetilde{C}}{p-1}. \tag{4}$$

At the same meeting a question was raised about the a.e. finiteness of $R^*(f, g)$ for pairs of functions in $(L \log L, L^1)$. Our second result is based on an adaptation of Zygmund's extrapolation method [3] (vol. II, ch. XII, pp. 119-120) to $R^*(f, g)$. With somewhat crude estimates we prove the following theorem.

Theorem 2. *If $\alpha > 2$ and the pair of nonnegative functions (f, g) belongs to $(L(\log L)^{2\alpha}, L^1)$ then $R^*(f, g) = \sup_n \frac{f(T^n x)g(T^{2n}x)}{n}$ is a.e. finite.*

2. Proofs

Proof of Theorem 1. First we can notice that the original inequality (1) is homogeneous with respect to the L^1 function g. Indeed, a simple change of variables shows that the case $g > t$ can easily be obtained from the case $g > 1$ with the same constant C'_p. So by approximating g with $g_n(x) = \max\{g(x), 1/n\}$ we can see that (1) holds if the assumption $g > 1$ is replaced by $g \geq 0$. Without loss of generality we can also suppose in the sequel that $\|g\|_1 = 1$.

If $\|f\|_p = 0$ we have nothing to prove. Otherwise, if we can show that (3) holds for $\widetilde{f} = f/\|f\|_p$ for all $\lambda > 0$, then this implies that it is true for f as well for all $\lambda > 0$. Thus, we just need to prove (3) for $f \in L^p$ with $\|f\|_p = 1$. We split the function f into the sum $f_1 + f_2$ where $f_1 = f$ if $f \leq 1$.

We have then

$$\mu\{x : \sup_n \frac{f(T^n x)g(T^{2n}x)}{n} > \lambda\} \leq$$

$$\mu\Big\{x : \sup_n \frac{f_1(T^n x)g(T^{2n}x)}{n} > \lambda/2\Big\} + \mu\Big\{x : \sup_n \frac{f_2(T^n x)g(T^{2n}x)}{n} > \lambda/2\Big\} = M_1 + M_2.$$

First we estimate M_1. Since

$$\sup_n \frac{f_1(T^n x)g(T^{2n}x)}{n} \le \sup_n \frac{g(T^{2n}x)}{n}$$

by the standard maximal inequality for the ergodic averages (see [2], for instance) we obtain

$$M_1 \le \frac{2\|g\|_1}{\lambda} = \frac{2}{\lambda} = \frac{2\|f\|_p\|g\|_1}{\lambda}.$$

To estimate M_2, we dominate f_2 with the function $h = \max\{f_2, 1\}$. By our remark about the assumption $g \ge 0$ the maximal inequality (1) is applicable and we obtain that $M_2 \le C'_p\left(\frac{\|h\|_p^p}{\lambda/2}\right)^{1/2}$, and (2) also holds for $1 < p < 2$. As $\|h\|_p \le \|\mathbf{1}\|_p + \|f_2\|_p \le \|\mathbf{1}\|_p + \|f\|_p = 2$ we have the estimate

$$M_2 \le 2^p\sqrt{2}C'_p\left(\frac{1}{\lambda}\right)^{1/2} = 2^p\sqrt{2}C'_p\left(\frac{\|f\|_p\|g\|_1}{\lambda}\right)^{1/2},$$

with C'_p satisfying (2) for $1 < p < 2$. By combining the upper bounds for M_1 and M_2 we obtain

$$\mu\Big\{x : \sup_n \frac{f(T^n x)g(T^{2n}x)}{n} > \lambda\Big\} \le \frac{2\|f\|_p\|g\|_1}{\lambda} + 2^p\sqrt{2}C'_p\left(\frac{\|f\|_p\|g\|_1}{\lambda}\right)^{1/2} \le$$

$$C_p\left(\left(\frac{\|f\|_p\|g\|_1}{\lambda}\right)^{1/2} + \frac{\|f\|_p\|g\|_1}{\lambda}\right)$$

with $C_p = \max\{2^p\sqrt{2}C'_p, 2\}$ and from (2) it follows that there exist $\widetilde{C}$ such that (4) holds for $1 < p < 2$. □

Proof of Theorem 2. The starting point is (3) and (4).

There exists a finite constant $\widetilde{C}$ such that for every $1 < p < 2$, for each $f, g \ge 0$ and for all $\lambda > 0$ we have

$$(5) \qquad \mu\Big\{x : \sup_n \frac{f(T^n x)g(T^{2n}x)}{n} > \lambda\Big\} \le \frac{\widetilde{C}}{p-1}\left[\left(\frac{\|f\|_p\|g\|_1}{\lambda}\right)^{1/2} + \frac{\|f\|_p\|g\|_1}{\lambda}\right].$$

We can again assume without loss of generality that $\|g\|_1 = 1$. We fix the function g and denote by $R^*(f)(x)$ the maximal function $\sup_n \frac{f(T^n x)g(T^{2n}x)}{n}$. Now we can rewrite (5) as

$$(6) \qquad \mu\Big\{x : R^*(f)(x) > \lambda\Big\} \le \frac{\widetilde{C}}{p-1}\left[\left(\frac{\|f\|_p}{\lambda}\right)^{1/2} + \frac{\|f\|_p}{\lambda}\right].$$

The important element for the extrapolation is the factor $\frac{1}{p-1}$ in the above inequality.

Our goal is to prove that for $\alpha > 2$ there is C_α such that for any $f \in L(\log L)^{2\alpha}$ we have for each $\lambda > 0$

(7)
$$\mu\big\{x : R^*(f)(x) > \lambda\big\} \leq C_\alpha \left(\frac{\big[1 + \big(\int |f|(\log^+ |f|)^{2\alpha}\big)^{1/2}\big]}{\lambda^{1/2}} + \frac{\big[1 + \int |f|(\log^+ |f|)^{\alpha}\big]}{\lambda} \right).$$

Let γ_j be a positive sequence of numbers such that $\sum_{j=0}^{\infty} \gamma_j = 1$.

The function f being in $L(\log L)^{2\alpha}$ we have $\sum_{j=1}^{\infty} j^{2\alpha} 2^j \mu\big\{x : 2^j \leq f < 2^{j+1}\big\} < \infty$. We denote by t_j the quantity $\mu\big\{2^j \leq f < 2^{j+1}\big\}$, by f_j the function $2^j \mathbf{1}_{\{x: 2^j \leq f < 2^{j+1}\}}$ and by p_j the number $1 + \frac{1}{j}$. We set $f_0(x) = f(x)$ if $0 \leq f(x) < 2$, otherwise we put $f_0(x) = 0$. Then

$$f \leq 2 \sum_{j=0}^{\infty} f_j. \tag{8}$$

We also have

$$\mu\big\{x : R^*(f_0)(x) > \frac{\lambda\gamma_0}{2}\big\} \leq \tag{9}$$
$$\mu\big\{x : R^*(2 \cdot \mathbf{1}_X)(x) > \frac{\lambda\gamma_0}{2}\big\} \leq \frac{4\|g\|_1}{\lambda\gamma_0} = \frac{4}{\lambda\gamma_0}$$

by the standard maximal inequality for the ergodic averages.

For $j \geq 1$ by (6) used with $p_j = 1 + \frac{1}{j}$ we obtain

$$\mu\big\{x : R^*(f_j)(x) > \frac{\lambda\gamma_j}{2}\big\} \leq \tag{10}$$
$$\widetilde{C} \frac{1}{(1 + (1/j)) - 1} \left(\frac{2^{j/2}[t_j]^{1/2p_j}}{(\lambda\gamma_j/2)^{1/2}} + \frac{2^j [t_j]^{1/p_j}}{\lambda\gamma_j/2} \right) \leq$$
$$2\widetilde{C} \left(\frac{j 2^{j/2}[t_j]^{1/2p_j}}{(\lambda\gamma_j)^{1/2}} + \frac{j 2^j [t_j]^{1/p_j}}{\lambda\gamma_j} \right).$$

We choose $\gamma_0 = 1/2$ and $\gamma_j = \frac{C_\gamma}{j(\log(j+1))^\beta}$ with $\beta > 1$ and C_γ such that $\sum_{j=0}^{\infty} \gamma_j = 1$.

Set $\widehat{C} = 2\widetilde{C} \max\{\frac{1}{C_\gamma^{1/2}}, \frac{1}{C_\gamma}\}$.

Using (8) and adding (9) and (10) for all j we obtain

$$\mu\{x : R^*(f)(x) > \lambda\} \le \sum_{j=0}^{\infty} \mu\{R^*(f_j) > \frac{\lambda\gamma_j}{2}\} \le \tag{11}$$

$$\frac{8}{\lambda} + 2\widetilde{C}\Bigg(\sum_{j=1}^{\infty} \frac{j2^{j/2}\big[t_j\big]^{1/2p_j}}{(\lambda\gamma_j)^{1/2}} + \sum_{j=1}^{\infty} \frac{j2^j\big[t_j\big]^{1/p_j}}{(\lambda\gamma_j)}\Bigg) \le$$

$$\frac{8}{\lambda} + \widehat{C}\Bigg(\frac{\sum_{j=1}^{\infty} j^{3/2}[\log(j+1)]^{\beta/2}2^{j/2}\big[t_j\big]^{1/2p_j}}{\lambda^{1/2}} + \frac{\sum_{j=1}^{\infty} j^2[\log(j+1)]^{\beta}2^j\big[t_j\big]^{1/p_j}}{\lambda}\Bigg) =$$

$$\frac{8}{\lambda} + \widehat{C}(\frac{A_1}{\lambda^{1/2}} + \frac{A_2}{\lambda}).$$

First we estimate A_1. Denote by J_1 the set of those j for which $t_j^{1/2p_j} \le 3^{-j}$. Then

$$\sum_{j\in J_1} j^{3/2}[\log(j+1)]^{\beta/2}2^{j/2}\big[t_j\big]^{1/2p_j} \le \sum_{j=1}^{\infty} j^{3/2}[\log(j+1)]^{\beta/2}2^{j/2}3^{-j} \stackrel{\text{def}}{=} C_s. \tag{12}$$

If $j \notin J_1$ then $t_j^{1/2p_j} > 3^{-j}$, that is,

$$3 > t_j^{\frac{-\frac{1}{j}}{2p_j}} = t_j^{\frac{1-(1+\frac{1}{j})}{2p_j}} = t_j^{\frac{1}{2p_j}-\frac{1}{2}},$$

which implies $t_j^{1/2p_j} < 3t_j^{1/2}$. Hence

$$\sum_{j\notin J_1} j^{3/2}[\log(j+1)]^{\beta/2}2^{j/2}\big[t_j\big]^{1/2p_j} \le 3\sum_{j=1}^{\infty} j^{3/2}[\log(j+1)]^{\beta/2}2^{j/2}\big[t_j\big]^{1/2} \stackrel{\text{def}}{=} B_1. \tag{13}$$

Suppose that $\alpha > \delta > 2$. By rewriting and applying the Cauchy–Schwartz inequality we obtain with a suitable constant C_δ that

$$B_1 = 3\sum_{j=1}^{\infty}\big[j^{3/2}j^{-\delta}\big]j^{\delta}\big[\log(j+1)]^{\beta/2}2^{j/2}\big[t_j\big]^{1/2} \le$$

$$3\Big[\sum_{j=1}^{\infty} j^{3-2\delta}\Big]^{1/2}\Big[\sum_{j=1}^{\infty} j^{2\delta}\big[\log(j+1)\big]^{\beta}2^j t_j\Big]^{1/2} =$$

$$C_\delta\Big[\sum_{j=1}^{\infty} j^{2\delta}\big[\log(j+1)\big]^{\beta}2^j t_j\Big]^{1/2} \stackrel{\text{def}}{=} B_2.$$

There exists $C_{\delta,\alpha,\beta}$ such that for all $j = 1, 2, ...$

$$\big[\log(j+1)\big]^{\beta} \le C_{\delta,\alpha,\beta}\, j^{2(\alpha-\delta)}.$$

Hence,

$$B_1 \leq B_2 \leq C_\delta C_{\delta,\alpha,\beta}\left(\int |f|(\log^+|f|)^{2\alpha}d\mu\right)^{1/2}. \tag{14}$$

To estimate A_2 a similar argument can be used. We denote by J_1' the set of those j for which $t_j^{1/p_j} \leq 3^{-j}$. Then

$$\sum_{j\in J_1'} j^2[\log(j+1)]^\beta 2^j\left[t_j\right]^{1/p_j} \leq \sum_{j=1}^{\infty} j^2[\log(j+1)]^\beta 2^j 3^{-j} \stackrel{\text{def}}{=} C_s'. \tag{15}$$

If $j \notin J_1'$ then $t_j^{1/p_j} > 3^{-j}$, that is,

$$3 > t_j^{\frac{-\frac{1}{j}}{p_j}} = t_j^{\frac{1-(1+\frac{1}{j})}{p_j}} = t_j^{\frac{1}{p_j}-1},$$

which implies $t_j^{1/p_j} < 3t_j$. Hence,

$$\sum_{j\notin J_1'} j^2[\log(j+1)]^\beta 2^j\left[t_j\right]^{1/p_j} \leq 3\sum_{j=1}^{\infty} j^2[\log(j+1)]^\beta 2^j t_j \stackrel{\text{def}}{=} B_1'. \tag{16}$$

Since $\alpha > 2$ there exists $C_{\alpha,\beta}'$ such that for all $j = 1, 2, ...$

$$\left[\log(j+1)\right]^\beta \leq C_{\alpha,\beta}' j^{(\alpha-2)}.$$

Hence,

$$B_1' \leq 3\cdot C_{\alpha,\beta}'\sum_{j=1}^{\infty} j^\alpha 2^j t_j \leq 3\cdot C_{\alpha,\beta}'\int |f|(\log^+|f|)^\alpha d\mu. \tag{17}$$

By (11-17) we have

$$\mu\{x : R^*(f)(x) > \lambda\} \leq$$

$$\widehat{C}\left(\frac{C_s + C_\delta C_{\delta,\alpha,\beta}(\int |f|(\log^+|f|)^{2\alpha}d\mu)^{1/2}}{\lambda^{1/2}} + \frac{(8/\widehat{C}) + C_s' + 3C_{\alpha,\beta}'\int |f|(\log^+|f|)^\alpha d\mu}{\lambda}\right)$$

this implies (7) with a suitable C_α.

□

Remark 1. Using the fact that we are dealing with finite measure spaces one can assume that $\mu(X) = 1$. With this assumption one can obtain a more compact form of (3):

$$\mu\big\{x : \sup_n \frac{f(T^n x)g(T^{2n}x)}{n} > \lambda\big\} \leq 2C_p\left[\left(\frac{\|f\|_p\|g\|_1}{\lambda}\right)^{1/2}\right]. \tag{18}$$

Indeed, we can distinguish two cases. If $\frac{\|f\|_p\|g\|_1}{\lambda} \geq 1$ then (18) is always true. If $\frac{\|f\|_p\|g\|_1}{\lambda} < 1$, then $\left(\frac{\|f\|_p\|g\|_1}{\lambda}\right)^{1/2} > \left(\frac{\|f\|_p\|g\|_1}{\lambda}\right)$ and (18) is also true.

Remark 2. Equation (7) implies also that for the pair of nonnegative functions (f, g) in $(L(\log L)^{2\alpha}, L^1)$ we have

$$\lim_n \frac{f(T^n x)g(T^{2n}x)}{n} = 0. \tag{19}$$

Indeed, consider a sequence of bounded functions $0 \leq f_M \leq f$ converging monotone increasingly to $f \in L(\log L)^{2\alpha}$. Then we have

$$\lim_n \frac{f_M(T^n x)g(T^{2n}x)}{n} = 0. \tag{20}$$

Given $\varepsilon \in (0, 1)$ choose M so large that

$$I(M, \varepsilon, 1/2) \stackrel{\text{def}}{=} \left(\int \frac{2}{\varepsilon^2}|f - f_M|(\log^+ \frac{2}{\varepsilon^2}|f - f_M|)^{2\alpha} d\mu\right)^{1/2} < 1, \text{ and} \tag{21}$$

$$I(M, \varepsilon, 1) \stackrel{\text{def}}{=} \int \frac{2}{\varepsilon^2}|f - f_M|(\log^+ \frac{2}{\varepsilon^2}|f - f_M|)^{\alpha} d\mu < 1.$$

Then

$$\mu\{x : \limsup_{n\to\infty} \frac{f(T^n x)g(T^{2n}x)}{n} > \varepsilon\} \leq$$

$$\mu\{x : \limsup_{n\to\infty} \frac{(f - f_M)(T^n x)g(T^{2n}x)}{n} > \frac{\varepsilon}{2}\} + \mu\{x : \limsup_{n\to\infty} \frac{f_M(T^n x)g(T^{2n}x)}{n} > \frac{\varepsilon}{2}\} \leq$$

(by using (20))

$$\mu\{x : R^*((f - f_M), g)(x) > \frac{\varepsilon}{2}\} = \mu\{x : R^*(\frac{2}{\varepsilon^2}(f - f_M), g)(x) > \frac{1}{\varepsilon}\} \leq$$

(by using (7) and (21))

$$C_\alpha\sqrt{\varepsilon}(2 + I(M, \varepsilon, 1/2) + I(M, \varepsilon, 1)) \leq 4C_\alpha\sqrt{\varepsilon}.$$

Since this holds for any $\varepsilon \in (0, 1)$ we obtained (19).

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