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Measures and functions with prescribed homogeneous multifractal spectrum

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Abstract. In this paper we construct measures supported in $[0,1]$ with prescribed multifractal spectrum. Moreover, these measures are homogeneously multifractal (HM, for short), in the sense that their restriction on any subinterval of $[0,1]$ has the same multifractal spectrum as the whole measure. The spectra f that we are able to prescribe are suprema of a countable set of step functions supported by subintervals of $[0,1]$ and satisfy $f(h) \leq h$ for all $h \in [0,1]$. We also find a surprising constraint on the multifractal spectrum of a HM measure, that we call Darboux theorem for multifractal spectra of measures: the support of its spectrum within $[0,1]$ must be an interval. This result is optimal, because there exists a HM measure with spectrum supported by $[0,1] \cup \{2\}$. Using wavelet theory, we also build HM functions with prescribed multifractal spectrum.

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1. Introduction

The multifractal spectrum is now a widely spread issue in analysis. It allows one to describe the local behavior of a given Borel measure, or a function. Our goal is to investigate the possible forms that a multifractal spectrum can take. We obtain a new Darboux-like theorem for the spectrum of homogeneous multifractal measures, and we are able to construct measures with prescribed non-homogeneous and homogeneous multifractal spectrum obtained as suprema of countable sets of step functions, when the local dimensions of the measure are less than 1. Using wavelet methods, we extend our result to non-monotone functions.

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Before exposing our results, let us introduce how the local regularity is quantified. Recall that the support of a Borel positive measure, denoted by $\mathrm{Supp}(\mu)$, is the smallest closed set $E \subset \mathbb{R}^d$ such that $\mu(\mathbb{R}^d \setminus E) = 0$.

Definition 1.1. The local regularity of a positive Borel measure μ on $\mathbb{R}^d$ at a given $x_0 \in \mathrm{Supp}(\mu)$ is quantified by the (lower) local dimension $h_\mu(x_0)$ (also called local Hölder exponent), defined as

$$h_\mu(x_0) = \liminf_{r \to 0^+} \frac{\log \mu(B(x_0, r))}{\log r}, \tag{1}$$

where $B(x_0, r)$ denotes the open ball with center x_0 and radius r. When $x_0 \notin \mathrm{Supp}(\mu)$, by convention we set $h_\mu(x_0) = +\infty$.

Let $Z \in L^\infty_{loc}(\mathbb{R}^d)$, and $\alpha > 0$. The function Z belongs to the space $C^\alpha_{x_0}$ if there are a polynomial P of degree less than $[\alpha]$ and a constant $C > 0$ such that

$$\text{for every } x \text{ in a neighborhood of } x_0, \ \ |Z(x) - P(x - x_0)| \le C|x - x_0|^\alpha. \tag{2}$$

The pointwise Hölder exponent of Z at x_0 is $h_Z(x_0) = \sup\{\alpha \ge 0 : \ f \in C^\alpha_{x_0}\}$.

We mainly focus on real functions and measures, i.e. $d = 1$. Even more, we consider only measures and functions whose supports are included in $[0, 1]$: this has no influence on our dimension questions.

Observe that when $h_Z(x_0) \le 1$, the pointwise Hölder exponent of Z at x_0 is also given by the formula

$$h_Z(x_0) = \liminf_{x \to x_0} \frac{\log |Z(x) - Z(x_0)|}{\log |x - x_0|}. \tag{3}$$

Of course, there is a correspondence between the exponents and the spectrum of a measure μ and the same quantities for its integral $F_\mu(x) = \mu([0, x])$. Comparing formulae (1) and (3), we see that when $h_\mu(x_0) < 1$, $h_{F_\mu}(x_0) = h_\mu(x_0)$. Problems may occur when $h_\mu(x_0) \ge 1$ due to the presence of a polynomial in (2). Since we mainly consider exponents less than 1 for measures, the problem may rise only when $h_\mu(x) = 1$. In order to guarantee that $h_\mu(x) = 1$ is equivalent to $h_{F_\mu}(x) = 1$ in our results, we add to each measure another fixed measure $\widetilde{\mu}$ satisfying $h_{F_{\widetilde{\mu}}}(x) = 1$ for all x (see Proposition 1.16 for the existence of $\widetilde{\mu}$). Hence, in the sequel, **we work equivalently with continuous monotone functions on the interval** $[0, 1]$**, or with diffuse measures supported on** $[0, 1]$.

Definition 1.2. The multifractal spectrum of a measure μ (*resp.* a function Z) is the mapping d_μ (*resp.* d_Z) defined as

$$h \ge 0 \ \longmapsto \ d_\mu(h) := \dim E_\mu(h), \qquad (\textit{resp.} \ \ d_Z(h) := \dim E_Z(h))$$

where

$$E_\mu(h) := \{x : h_\mu(x) = h\}, \quad (\textit{resp.} \ \ E_Z(h) := \{x : h_Z(x) = h\}). \tag{4}$$

By convention, we set $\dim \emptyset = -\infty$. The support of the multifractal spectrum of μ (*resp.* of Z) is the set

$$\text{Support}\,(d_\mu) = \{h \geq 0 : d_\mu(h) \geq 0\} \quad (\,\textit{resp.}\ \text{Support}\,(d_Z) = \{h \geq 0 : d_Z(h) \geq 0\}). \tag{5}$$

Observe that $d_\mu(h) \geq 0$ as soon as there is a point in $E_\mu(h)$. The multifractal spectrum describes the geometrical distribution of the singularities of the measure or the function under consideration. It is natural to investigate the forms possibly taken by a multifractal spectrum, this is our goal in this paper.

A first question concerns the prescription of the local dimension of a measure or a function. The pointwise Hölder exponents of functions are well understood [12, 5, 18]: given a continuous function $f : [0,1] \to \mathbb{R}$, the mapping $x \mapsto h_f(x)$ is the liminf of a sequence of positive continuous functions, and, reciprocally, any liminf of a sequence of positive continuous functions is indeed the map $x \mapsto h_f(x)$ associated with some continuous function f.

It is much more delicate to deal with the local dimensions of measures, as stated by the following lemma.

Lemma 1.3. *Let μ be a probability measure supported on the closure of an open set $\Omega \subset \mathbb{R}^d$. If the mapping $x \mapsto h_\mu(x)$ is continuous, it is constant equal to d.*

Lemma 1.3 is quite easy (see Section 2), but it helps to understand that, from the local regularity standpoint, measures are less flexible than functions.

Definition 1.4. A measure μ supported in $[0,1]$ is homogeneously multifractal (for short, HM) when the restriction of μ to any non-degenerate subinterval of $[0,1]$ has the same multifractal spectrum, i.e. for any non-empty subinterval $U \subset [0,1]$,

$$\text{for any } h \geq 0, \quad \dim\{x \in U : h_\mu(x) = h\} = \dim\{x \in [0,1] : h_\mu(x) = h\} = d_\mu(h).$$

A similar definition applies to HM functions $Z : [0,1] \to \mathbb{R}$.

Some results related to the prescription of multifractal spectrum have already been obtained. The main result is due to Jaffard in [11], and concerns only general functions (non-monotone). Jaffard proved, by wavelet methods, that any supremum of a countable family of positive piecewise constant functions with support in $(0,+\infty)$ is the multifractal spectrum of a function $Z : [0,1] \to \mathbb{R}$, which is non-HM and non-monotone. This interesting result yields no information on the possible spectra of measures (which are integrals of positive non-decreasing functions), nor on HM measures or functions.

In this article, we obtain the first results for the prescription of multifractal spectrum of measures, both HM and non-HM, and also of HM functions. In addition, when the measures μ are homogeneously multifractal, we discover a surprising constraint on Support (d_μ).

Our first result is a Darboux-like theorem for multifractal spectra of HM measures (thus, it holds also for monotone functions) for exponents less than 1. Darboux's theorem for a real differentiable function $Z : \mathbb{R} \to \mathbb{R}$ asserts that the image

$Z'(I)$ of any interval $I \subset \mathbb{R}$ by the derivative of Z is an interval. We obtain a similar result for HM measures with exponents less than one: the support of the multifractal spectrum of μ, Support (d_μ) defined by (5), always contains an interval. In other words, there is **no spectrum gap for exponents less than 1 in the multifractal spectrum of HM measures**. The necessary connectedness of Support (d_μ) when μ is HM is a delicate issue. Establishing conditions under which Support (d_μ) computed using limit exponents (not liminf exponents, as we do here) is necessarily connected, would be very interesting and useful in some situations, for instance for self-similar measures with overlaps, self-affine measures, or Bernoulli convolutions, see [8] for instance and also [10, 9, 17] for the existence of an isolated exponent for the third convolution of the Cantor measure and generalizations. We prove the following.

Theorem 1.5. *For any non-atomic HM probability measure μ supported on $[0,1]$, Support (d_μ) $\cap$ $[0,1]$ is necessarily an interval of the form $[\alpha,1]$, where $0 \leq \alpha \leq 1$.*

Remark 1.6. Actually a more general theorem, Theorem 3.3, will be proved, which is much easier to apply than Theorem 1.5, because it requires only the density of $E_\mu(\alpha)$ on the support of the measure μ, where $\alpha = \inf(\text{Support}\,(d_\mu))$. In particular, Theorem 3.3 can certainly be applied to very large classes of self-similar or self-affine measure.

Theorem 1.5 is false for non-monotone functions: the non-differentiable Riemann function $\sum_{n\geq 1} \frac{\sin \pi n^2 x}{n^2}$ is HM, but its multifractal spectrum is supported by the set $[1/2, 3/4] \cup \{3/2\}$, which admits an isolated point [13]. This theorem is also false for exponents greater than one, as stated by the following complementary result proved in [3].

Proposition 1.7. *There is a HM measure μ such that Support $(d_\mu) = [0,1] \cup \{2\}$.*

Nevertheless, one can derive a theorem equivalent to Theorem 1.5 for the limsup exponents greater than 1 (i.e. when the liminf in (1) is replaced by a limsup) using an argument of inversion of measures, i.e. by considering the inverse measure μ^{-1} of μ defined as $\mu^{-1}([0, \mu([0,x])]) = x$. This inversion procedure transforms liminf exponents for μ into limsup exponents for μ^{-1}. The argument is due to Mandelbrot and Riedi in [16], and the details are left to the reader. The associated result using the upper multifractal spectrum defined as $\overline{d_\mu}(h) = \dim \left\{ x : \limsup_{r\to 0^+} \frac{\log \mu(B(x,r))}{\log r} = h \right\}$ is:

Theorem 1.8. *For any non-atomic HM measure supported on $[0,1]$, Support$(\overline{d_\mu}) \cap [1,+\infty)$ is necessarily an interval of the form $[1,\alpha]$ where $\alpha \in [1,+\infty]$.*

Our second main result deals with the prescription of multifractal spectrum of a non-HM measure. It is known (see Proposition 2.1) that the multifractal spectrum of a probability measure μ always satisfies $d_\mu(h) \leq \min(h,1)$, for every $h \geq 0$.

Let us introduce functions which are candidates to be a multifractal spectrum.

Definition 1.9. For every function $f : [0,1] \to [0,1] \cup \{-\infty\}$, we define Support$(f)$ $= \{x : f(x) > -\infty\}$ and Support$^*(f)$ as the smallest interval of the form $[h,1]$ containing Support(f).

This definition is analogous to the definition of the support of the multifractal spectrum of a measure or a function as defined in equation (5).

Definition 1.10. The set $\mathcal{F}$ consists of functions $f : [0,1] \to [0,1] \cup \{-\infty\}$ satisfying the following: For each $f \in \mathcal{F}$, there exists a countable family of functions $(f_n)_{n\geq 1}$, $f_n : [0,+\infty] \to [0,1] \cup \{-\infty\}$ such that

- for every $n \geq 1$, Support(f_n) is a closed, possibly degenerate interval $I_n \subset [0,1]$,
- $\inf_{n\geq 1} \min(I_n) > 0$,
- f_n is constant over I_n, and for every $x \in I_n$, $f_n(x) \leq x$,
- for every $x \in \bigcup_{n\geq 1} I_n$, $f(x) = \sup_{n\geq 1} f_n(x)$.

The set $\mathcal{F}$ contains for instance the continuous functions and the lower semi-continuous functions (provided that they satisfy $f(x) \leq x$) supported by subintervals of $[0,1]$, and one can also allow functions f_n with degenerate, one-point supports. We prove the following:

Theorem 1.11. *For every $f \in \mathcal{F}$, there is a Borel probability measure μ, supported by $[0,1]$ such that the multifractal spectrum of μ satisfies:*

(i) *for all $h \in$ Support$(f) \setminus \{1\}$, $d_\mu(h) = f(h)$,*

(ii) *$d_\mu(h) = -\infty$ if $h \in \mathbb{R}^+ \setminus$Support$(f)$,*

(iii) *the set of points $\{x \in [0,1] : h_\mu(x) = 1\}$ has Lebesgue measure 1.*

In particular, Support (d_μ) =Support$(f) \cup \{1\}$.

The proof of Theorem 1.11 is somewhat classical: we concatenate measures μ_n whose spectra are (close to be) the functions f_n (used in Definition 1.10). When dealing with HM measures, the result is different:

Theorem 1.12. *For every $f \in \mathcal{F}$, there is a HM Borel probability measure μ, supported by $[0,1]$ such that the multifractal spectrum of μ satisfies:*

(i) *for all $h \in$ Support$^*(f) \setminus \{1\}$, $d_\mu(h) = \max(f(h), 0)$,*

(ii) *$d_\mu(h) = -\infty$ if $h \in \mathbb{R}^+ \setminus$Support$^*(f)$,*

(iii) *the set of points $\{x \in [0,1] : h_\mu(x) = 1\}$ has Lebesgue measure 1.*

In particular, Support (d_μ) =Support$^*(f)$.

The notation $d_\mu(h) = \max(f(h), 0)$ indicates that either $h \in$ *Support*(f), which implies $f(h) \geq 0$ and $d_\mu(h) = f(h)$, or $h \in$ Support$^*(f) \setminus$ Support(f), and in this second case $d_\mu(h) = 0$ (except for $h = 1$, for which $d_\mu(1) = 1$) (see Figure 1).

Observe that, by Theorem 1.5, in Theorem 1.12 the intersection of Support (d_μ) with $[0,1]$ must be an interval. This is why we introduced Support$^*(f)$, and why Theorems 1.11 and 1.12 differ. Theorem 1.12 is optimal for the class of functions $\mathcal{F}$ when Support (d_μ) is included in $[0,1]$.

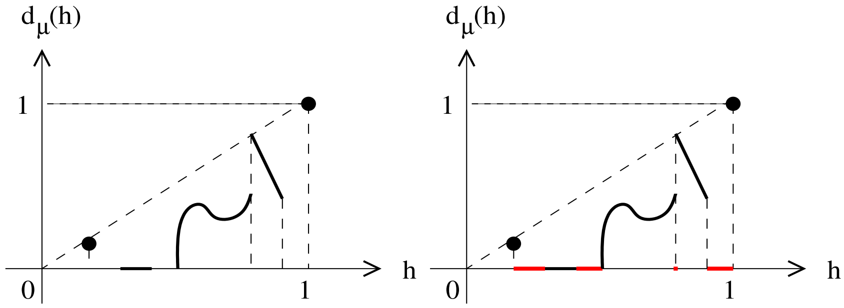

Figure 1. Spectrum of a non-HM measure (left) and a HM measure (right).

The proof of Theorem 1.12 is quite original. The method is the application of an iterative algorithm which allows us to superpose various affine spectra and to create a HM measure μ. Applying this algorithm iteratively, we loose control on the local dimensions of the limit measure μ, but only on a zero-dimensional set of exceptional points. It is really nice that Theorem 1.5 asserts that this is the normal situation (hence giving the optimality of Theorem 1.12), since an uncountable set of points with new exponents appears for HM measures. Theorems 1.5, 3.3 and Lemma 3.2 can also help us to verify that after this modification we still have a HM measure.

Remark 1.13. Although the measures built in Theorem 1.12 have the same multifractal spectrum on any interval, they do not have to satisfy in general any kind of multifractal formalism (their spectrum has no reason to be concave).

Remark 1.14. The key point to prove Theorems 1.11 and 1.12 is the explicit construction of a non-HM measure μ which has an affine spectrum, achieved in Theorem 4.1 in Section 4. **Theorem 4.1 can be admitted at first reading, the other proofs use only the existence of such measures.**

We have a similar result for the prescription of spectra of monotone functions.

Theorem 1.15. *For every $f \in \mathcal{F}$, there exists a HM strictly monotone increasing continuous function $Z : [0,1] \to \mathbb{R}$ satisfying:*

(i) *for all $h \in Support^*(f)$, $d_Z(h) = \max(f(h), 0)$,*
(ii) *$d_Z(h) = -\infty$ if $h \in \mathbb{R}^+ \backslash Support^*(f)$,*
(iii) *the set of points $\{x \in [0,1] : h_Z(x) = 1\}$ has Lebesgue measure 1.*

There is a subtle difference between Theorems 1.15 and 1.12 concerning the exponents larger than 1. This difference is due to the fact that (2) eliminates the polynomial trends, while (1) does not: hence if a monotone function Z is exactly linear with positive slope around a point x, one has $h_Z(x) = +\infty$, whilst for the corresponding measure μ (the derivative of F) one has $h_\mu(x) = 1$. Hence, to prove Theorem 1.15, it is enough to get rid of the points $\{x \in [0,1] : h_Z(x) > 1\}$ by adding the function constructed in the following proposition, proved in [3].

Proposition 1.16. *There exists* $Z : [0,1] \to [0,1]$ *a strictly monotone increasing HM function with* $h_Z(x) = 1$ *for all* $x \in [0,1]$.

It is also clear that Theorem 1.15 implies Theorem 1.12 if we consider the measure μ for which $\mu((a,b)) = (Z(b) - Z(a))/(Z(1) - Z(0))$, for all $0 \leq a < b \leq 1$.

Our final result concerns the prescription of the spectrum of HM non-monotone functions, which are more "flexible" than HM monotone functions. Combining Theorem 1.15 with wavelet methods, we obtain the following result.

Theorem 1.17. *Let* $0 < \alpha < \beta$ *be two real numbers, and consider* $f \in \mathcal{F}$. *There exists a continuous HM function* $Z : [0,1] \to [0,1]$ *satisfying:*

(i) *for all* $h \in (\alpha + (\beta - \alpha)\mathit{Support}^*(f)) \setminus \{\beta\}$, $d_Z(h) = \max\left(f\left(\frac{h-\alpha}{\beta-\alpha}\right), 0\right)$.

(ii) $d_Z(h) = -\infty$ *if* $h \notin \alpha + (\beta - \alpha)\mathit{Support}^*(f)$,

(iii) *the set of points* $\{x \in [0,1] : h_Z(x) = \beta\}$ *has Lebesgue measure 1.*

The fact that the functions Z constructed in Theorem 1.17 are non-monotone allows one to take any support for the multifractal spectrum of Z. Theorem 1.17 is certainly far from being optimal, and the most general form of the multifractal spectrum of a function is an open issue. This leads to some open questions:

Question 1: How to characterize and to prescribe local dimensions mapping $h \mapsto d_\mu(h)$ of probability measures μ? of HM probability measures μ?

Question 2: How to prescribe the multifractal spectrum with exponents larger than one for measures? Same questions for a HM spectrum?

Question 3: Characterize the most general form of the multifractal spectrum of a function, or of a measure? By Theorem 1.5 it is not the same for functions and measures. Can one state something more precise?

Question 4: Characterize the most general form of the multifractal spectrum of a function, or of a measure satisfying a multifractal formalism?

Question 5: What about higher dimensional results?

Let us finally mention that, since the writing of this paper, some of these questions have been addressed in [1].

The paper is organized as follows.

Section 2 contains preliminary results, and the proof of Lemma 1.3. In Section 3, we prove the Darboux Theorem 1.5 for HM measures, using a maximal inequality result (Proposition 3.1). Section 4 presents the construction of a non-HM monotone function with an affine increasing spectrum. As explained in Remark 1.14, the proof of Theorem 4.1 can be omitted at first reading. Using this construction, we prove Theorem 1.11 in Section 5. In Section 6, we develop an iterative algorithm which creates a sequence of monotone functions converging to a HM monotone function according to the constraints of Theorem 1.12. Finally, using wavelet methods, non-monotone HM functions with spectra which are dilated and translated versions of the function set $\mathcal{F}$ are built in Section 7.

2. Notation and preliminary results

The open ball centered at $x \in \mathbb{R}$ and of radius r is denoted by $B(x,r)$. By $\overline{B(x,r)}$ we denote the closed ball.

For a set $A \subset \mathbb{R}^d$ we denote its diameter by $|A|$ and by $\lambda(A)$ its d-dimensional Lebesgue measure. $\mathrm{int}(A)$ stands for the (open) interior of A.

The sum of two non-empty sets is $A+B = \{a+b : a \in A,\ b \in B\}$ for $A, B \subset \mathbb{R}^d$.

We refer to [6, 7, 14] for the standard definition of the Hausdorff measure $\mathcal{H}^s(E)$ and Hausdorff dimension $\dim(E)$ of a set E.

We will use the level sets $E_\mu(h)$ of the Hölder exponents (4), but also the following sets related to the Hölder exponents:

$$E_Z^{\leq}(h) = \{x \in [0,1] : h_Z(x) \leq h\} \quad \text{and} \quad E_\mu^{\leq}(h) = \{x \in [0,1] : h_\mu(x) \leq h\}. \tag{6}$$

Standard results on multifractal spectra of monotone functions and measures give upper bounds for their Hausdorff dimension.

Proposition 2.1. [4] *Let μ be a Borel probability measure on $[0,1]$.*
For every $h \in [0,1]$, $\dim E_\mu^{\leq}(h) \leq h$.
In particular, for every $h \in [0,1]$, $d_\mu(h) = \dim E_\mu(h) \leq h$.
The same holds for $E_Z^{\leq}(h)$ and $E_Z(h)$ for any monotone function $Z : [0,1] \to \mathbb{R}$.

From this proposition one deduces Lemma 1.3.

Proof of Lemma 1.3. Assume that the mapping $x \mapsto h_\mu(x)$ is continuous, and that $h_\mu(x_0) \neq d$ for an $x_0 \in \Omega$. By continuity, for some constants $0 < \varepsilon < M$, one has $\varepsilon \leq |h_\mu(x) - d| \leq M$ for all x in an open ball $B \subset \Omega$ around x_0 such that $\mu(B) > 0$.

If $h_\mu(x) \leq d - \varepsilon$ when $x \in B$, then $\dim_H(\{x \in B : h_\mu(x) \leq d - \varepsilon\}) = d$, which is impossible by Propostion 2.1.

If $h_\mu(x) \geq d + \varepsilon$ when $x \in B$, then the argument is as follows. Fix $\eta > 0$. For every $x \in B$, there exists $0 < r_x < \eta$ such that $B(x, r_x) \subset B$ and $\mu(B(x, r_x)) \leq |B(x, r_x)|^{d+\varepsilon/2}$. Hence, $\{B(x, r_x)\}_{x \in B}$ forms a covering of B by balls centered at points of B. By Besicovitch's covering theorem, there is an integer $Q'(d)$ depending only on the dimension d such that one can find $Q'(d)$ families $\mathcal{F}_i$, $i = 1, ..., Q'(d)$ of disjoint balls amongst the balls $B(x, r_x)$ such that

$$B \subset \bigcup_{i=1,\dots,Q'(d)} \ \bigcup_{B' \in \mathcal{F}_i} B'.$$

All the balls within one $\mathcal{F}_i$ are disjoint, hence (using that $\lambda(B') = C_d 2^{-d} |B'|^d$ for any ball $B' \subset \mathbb{R}^d$, where C_d is the volume of the unit ball in $\mathbb{R}^d$)

$$\begin{aligned}
\mu(B) &\leq \sum_{i=1}^{Q'(d)} \mu\Big(\bigcup_{B' \in \mathcal{F}_i} B'\Big) = \sum_{i=1}^{Q'(d)} \sum_{B' \in \mathcal{F}_i} \mu(B') \leq \sum_{i=1}^{Q'(d)} \sum_{B' \in \mathcal{F}_i} |B'|^{d+\varepsilon/2} \\
&\leq \eta^{\varepsilon/2} \sum_{i=1}^{Q'(d)} \sum_{B' \in \mathcal{F}_i} |B'|^d \leq \frac{\eta^{\varepsilon/2} 2^d}{C_d} \sum_{i=1}^{Q'(d)} \sum_{B' \in \mathcal{F}_i} \lambda(B') \leq \frac{\eta^{\varepsilon/2} 2^d}{C_d} Q'(d) \lambda(B).
\end{aligned}$$

Letting η tend to zero we obtain $\mu(B)=0$, which is impossible. □

3. The Darboux theorem for HM measures

3.1. A maximal inequality. We need a variant of the maximal inequality used for the deduction of the Hardy–Littlewood maximal inequality. In the sequel, we always consider the case $d=1$, but the next proposition holds in any finite dimension, d.

Proposition 3.1. *Suppose that $I\subset[0,1]^d$ is an open ball, μ is a measure on $[0,1]^d$, $0<\beta\le 1$ and set*

$$M^*_{I,\beta}\,\mu(x):=\sup\Big\{\frac{\mu(B(x,r))}{(2r)^{d\beta}}:r>0,\ B(x,r)\subset I\Big\}.$$

Then, there exists a constant $Q(d)>0$ (depending on the dimension d only) such that for all $t>0$, the set $M^*_t=\{x\in[0,1]^d: M^*_{I,\beta}\,\mu(x)>t\}$ satisfies

$$\lambda(M^*_t)\le\frac{Q(d)\,\mu(I)\,|I|^{d(1-\beta)}}{t}.$$

Proof. For every $x\in M^*_t$, there is r_x such that $\mu(B(x,r_x))\ge t(2r_x)^{d\beta}$. Hence the family of balls $\{B(x,r_x)\}_{x\in M^*_t}$ forms a covering of M^*_t by balls centered at points of M^*_t. Recall that for a ball $B\subset\mathbb{R}^d$ of radius r, $\lambda(B)=C_dr^d=C_d2^{-d}|B|^d$. Using Besicovitch's covering theorem, there exists a constant $Q'(d)>0$ (depending only on the dimension) such that one can extract from this (possibly uncountable) family a countable system $B(x_i,r_i)\subset I$, $x_i\in M^*_t$ such that:

- the union of these balls covers M^*_t,
- no point $x\in\mathbb{R}^d$ is covered by more than $Q'(d)$ balls of the form $B(x_i,r_i)$,
- for every i, we have

$$\begin{aligned}\mu(B(x_i,r_i)) &> t(2r_i)^{d\beta}=t\cdot 2^{d\beta}\cdot(C_d)^{-\beta}\cdot\lambda(B(x_i,r_i))^{\beta}\\ &= t\cdot 2^{d\beta}\cdot(C_d)^{-\beta}\cdot\lambda(B(x_i,r_i))\cdot\lambda(B(x_i,r_i))^{\beta-1}\\ &\ge t\cdot 2^{d\beta}\cdot(C_d)^{-\beta}\cdot\lambda(B(x_i,r_i))\cdot\lambda(I)^{\beta-1}\\ &\ge t\cdot 2^{d}\cdot(C_d)^{-1}\cdot\lambda(B(x_i,r_i))\cdot|I|^{d(\beta-1)}.\end{aligned}\tag{7}$$

Since no point is covered by more than $Q'(d)$ balls $B(x_i,r_i)$, one can select an index set $\mathcal{I}$ such that the balls $B(x_i,r_i)$, $i\in\mathcal{I}$ are disjoint and the Lebesgue measure of $\cup_{i\in\mathcal{I}}B(x_i,r_i)$ is greater than $1/Q'(d)$ times that of M^*_t. Then summing (7) for $i\in\mathcal{I}$, one obtains

$$\begin{aligned}\frac{1}{Q'(d)}\lambda(M^*_t) &\le \sum_{i\in\mathcal{I}}\lambda(B(x_i,r_i))\le 2^{-d}C_d|I|^{d(1-\beta)}\frac{1}{t}\sum_{i\in\mathcal{I}}\mu(B(x_i,r_i))\\ &\le \frac{2^{-d}C_d\mu(I)|I|^{1-\beta}}{t}.\end{aligned}$$

Hence the result with $Q(d) = Q'(d)^{-1}2^{-d}C_d$. □

3.2. The minimum property of the spectrum for HM measures.

Lemma 3.2. *Suppose that μ is a non-atomic measure supported on $[0,1]$ and $0 \le \alpha < 1$. Assume that:*

- *for every $\varepsilon > 0$, $E_\mu^{\le}(\alpha+\varepsilon)$ is dense in $[0,1]$,*
- *$h_\mu(x) \ge \alpha$ for all $x \in [0,1]$.*

Then $E_\mu(\alpha)$ is dense in $[0,1]$.

Observe that Lemma 3.2 implies that for a non-atomic HM measure, if $\alpha_0 = \inf\{h_\mu(x) : x \in [0,1]\} < 1$, then necessarily $d_\mu(\alpha_0) \ge 0$.

Proof. Suppose $0 \le a < b \le 1$. Choose $x_1 \in (a,b)$ and $r_1 \in (0,1)$ such that $\mu(B(x_1,r_1)) > r_1^{\alpha+\frac{1}{2}}$. Since μ is non-atomic one can choose a small non-degenerate closed interval $I_1 \subset (a,b)$ such that $x_1 \in I_1$ and

$$\text{for any } x \in I_1 \text{ we have } \mu(B(x,r_1)) > r_1^{\alpha+\frac{1}{2}}.$$

Suppose that we have defined the non-degenerate nested intervals $I_1 \supset I_2 \supset ... \supset I_n$ and $r_1 > r_2 > ... > r_n > 0$ such that

$$\text{for any } x \in I_n \text{ we have } \mu(B(x,r_n)) > r_n^{\alpha+\frac{1}{n+1}} \text{ and } r_n < 1/n. \tag{8}$$

By our assumption we can choose $x_{n+1} \in \mathrm{int}(I_n)$ and $r_{n+1} \in (0, \frac{1}{n+1})$ such that

$$\mu(B(x_{n+1},r_{n+1})) > r_{n+1}^{\alpha+\frac{1}{n+2}}.$$

One can also choose a non-degenerate closed interval $I_{n+1} \subset I_n$ such that

$$\text{for any } x \in I_{n+1} \text{ we have } \mu(B(x,r_{n+1})) > r_{n+1}^{\alpha+\frac{1}{n+2}} \text{ and } r_{n+1} < 1/(n+1).$$

By induction we can define the infinite nested sequence of intervals I_n such that (8) holds for all n. Then any $x_\alpha \in \cap_{n=1}^\infty I_n \subset (a,b)$ satisfies $h_\mu(x_\alpha) = \alpha$. □

3.3. Continuity of the support of the multifractal spectrum, Support (d_μ).

Theorem 1.5 is a direct consequence of the next result.

Theorem 3.3. *Let μ be a non-atomic Borel probability measure supported in the interval $[0,1]$. Assume that there exists $0 \le \alpha < 1$ such that for every $\varepsilon > 0$, $\{x : h_\mu(x) \le \alpha+\varepsilon\}$ is dense in $[0,1]$ and $h_\mu(x) \ge \alpha$ for all $x \in [0,1]$. Then for every $\beta \in [\alpha,1]$, $d_\mu(\beta) \ge 0$ and $d_\mu(\beta) = -\infty$ for $\beta < \alpha$.*

Proof. The function $F_\mu(x) = \mu([0,x])$ is continuous, monotone increasing and hence λ-almost everywhere differentiable by Lebesgue's theorem. We denote by D_μ the set of those points $x \in (0,1)$ where (a finite) $F'_\mu(x)$ exists.

The case $\beta = \alpha$ is treated in Lemma 3.2. The case $\beta = 1$ is slightly different and will be discussed later.

Suppose $\alpha < \beta < 1$. We are going to build iteratively a nested sequence of intervals converging to one point x such that $h_\mu(x) = \beta$.

Put $\beta' = \frac{1+\beta}{2}$ (so that $\beta < \beta' < 1$). Clearly, for any $x \in D_\mu$ there exists $r_{\beta,x}$ such that $\mu(B(x,r)) \leq r^{\beta'}$ for all $0 < r < r_{\beta,x} < 1$.

Choose an $x_0 \in D_\mu$ and suppose that $r_0 \leq r_{\beta,x_0}$ satisfies

$$\mu(B(x_0,r_0)) \leq r_0^{\beta'} \quad \text{and} \quad r_0^{\beta'-\beta} < 1/10. \tag{9}$$

We start the induction.

Let $n \geq 1$ be fixed. Assume that one can construct two sequences of positive real numbers $\{(x_0,r_0),(x_1,r_1),...,(x_n,r_n)\}$ and $\{(\widetilde{x}_0,\widetilde{r}_0),(\widetilde{x}_1,\widetilde{r}_1),...,(\widetilde{x}_{n-1},\widetilde{r}_{n-1})\}$ satisfying the following properties:

(P0) the real numbers $x_0,x_1,...x_n,\widetilde{x}_0,\widetilde{x}_1,...,\widetilde{x}_{n-1}$ belong to the interval $[0,1]$,

(P1) the radii are decreasing with n, and they satisfy

$$r_0 > \widetilde{r}_0 > r_1 > \widetilde{r}_1 > ... > \widetilde{r}_{n-1} > r_n > 0,$$

(P2) for $i = 1,...,n$, one has

$$\overline{B(x_i,r_i)} \subset B(\widetilde{x}_{i-1},\widetilde{r}_{i-1}/3) \subset B(x_{i-1},r_{i-1}/3), \tag{10}$$

$$\mu(B(x_i,r_i)) \leq r_i^{\beta'}, \tag{11}$$

$$\mu(B(\widetilde{x}_{i-1},\widetilde{r}_{i-1})) = \widetilde{r}_{i-1}^{\beta}, \tag{12}$$

(P3) if $x \in \overline{B(x_i,r_i/3)}$ for some $i = 1,...,n$, and if $B(x,r) \not\subset B(x_i,r_i)$ but $B(x,r) \subset B(x_{i-1},r_{i-1})$, then

$$\mu(B(x,r)) < 300 \cdot r^\beta. \tag{13}$$

Observe that (P2) implies that for any $x \in \overline{B(x_i,r_i)}$ we have

$$\mu(B(x,2\widetilde{r}_{i-1})) \geq \mu(B(\widetilde{x}_{i-1},\widetilde{r}_{i-1})) = \widetilde{r}_{i-1}^{\beta}. \tag{14}$$

Let us explain why such a sequence is key to prove Theorem 3.3.

Lemma 3.4. *If there exist four infinite sequences* (x_n), (r_n), $(\widetilde{x}_n)$ *and* $(\widetilde{r}_n)$ *satisfying (P0-3) for all* $n \in \mathbb{N}$, *then Theorem* 3.3 *is proved.*

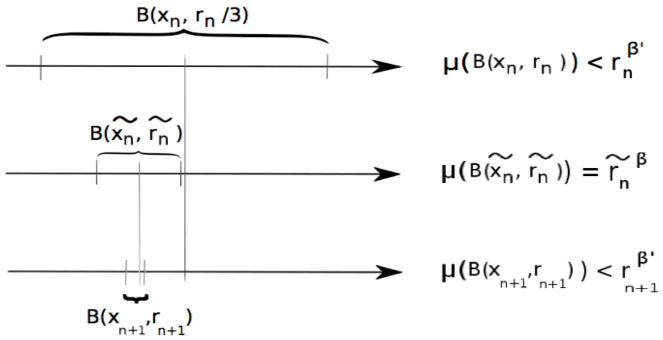

Figure 2. Nested sequence of balls in the construction

Proof. The sequences of radii obviously converge to zero by (P1) and (P2).

Let $\{x\} = \bigcap_{i=1}^{\infty} \overline{B(x_i, r_i)}$. The "nesting" relation (10) implies that x is always located in the "middle part" of the balls $B(x_i, r_i)$, more precisely in $B(x_i, r_i/3)$ and $B(\widetilde{x}_i, \widetilde{r}_i/3)$.

By definition of $h_\mu(x)$, and using (14), we obtain

$$h_\mu(x) \leq \liminf_{i \to +\infty} \frac{\log \mu(B(x, 2\widetilde{r}_i))}{\log |B(x, 2\widetilde{r}_i)|} \leq \beta. \tag{15}$$

Property (P3) allows us to control the behavior of $\mu(B(x,r))$, for every $r \in (0, r_1/3)$. Indeed, fix such a radius r, and choose $i \geq 1$ as the unique integer such that $r_i/3 < r \leq r_{i-1}/3$. By construction, one has $x \in B(x_i, r_i/3)$. Let R be the largest positive real number such that $B(x, R) \subset B(x_i, r_i)$. Obviously, one has $2r_i/3 \leq R \leq r_i \leq r_{i-1}/3$.

- if $r_i/3 < r \leq R$: using (11), one gets

$$\mu(B(x,r)) \leq \mu(B(x_i, r_i)) \leq (r_i)^{\beta'} \leq 3^{\beta'} r^{\beta'}.$$

- if $R \leq r \leq r_{i-1}/3$: then we are in the situation described by (P3), since $B(x,r) \not\subset B(x_i, r_i)$ but from $x \in B(x_{i-1}, r_{i-1}/3)$ it follows that $B(x,r) \subset B(x_{i-1}, r_{i-1})$. Hence (13) holds true.

In any case, one sees that $\mu(B(x,r)) \leq Cr^\beta$ for any $r \leq r_1$, for some constant C. This implies that $h_\mu(x) \geq \beta$. Combining this with (15), Lemma 3.4 is proved. □

It remains us to prove that as soon as we are given $(x_i,\ r_i)$ for $i = 0, ..., n$ and $(\widetilde{x}_i,\ \widetilde{r}_i)$ for $i = 1, ..., n-1$ fulfilling properties (P0-3), we can construct $(x_{n+1},\ r_{n+1})$, $(\widetilde{x}_n, \widetilde{r}_n)$ satisfying the same properties.

From $r_n \leq r_0$, (9), and (11), one deduces that

$$\mu(B(x_n, r_n)) \leq r_n^{\beta'} \quad \text{and} \quad r_n^{\beta' - \beta} < \frac{1}{10}. \tag{16}$$

By our assumption the set $\{x : h_\mu(x) < \beta\}$ is dense in $[0,1]$. Choose $\widetilde{x}_n \in B(x_n, r_n/6)$ such that $h_\mu(\widetilde{x}_n) < \beta$.

Lemma 3.5. *There exists a largest real number $\widetilde{r}_n$ such that $0 < \widetilde{r}_n < r_n$ and*

$$\mu(B(\widetilde{x}_n, \widetilde{r}_n)) = (\widetilde{r}_n)^\beta \tag{17}$$

$$\text{if } \widetilde{r}_n < r \text{ and } B(\widetilde{x}_n, r) \subset B(x_n, r_n), \text{ then } \mu(B(\widetilde{x}_n, r)) < r^\beta. \tag{18}$$

Moreover, one necessarily has $\widetilde{r}_n < r_n/10$.

Proof. If $r \geq r_n/10$ and $B(\widetilde{x}_n, r) \subset B(x_n, r_n)$ then from (16) we deduce that

$$\mu(B(\widetilde{x}_n, r)) \leq \mu(B(x_n, r_n)) \leq r_n^{\beta'} \leq \frac{1}{10} r_n^\beta \leq \frac{10^\beta}{10} r^\beta < r^\beta.$$

Hence, if there exists a suitable $\widetilde{r}_n$, then $\widetilde{r}_n < r_n/10$.

Since $h_\mu(\tilde{x}_n) < \beta$, there exists $0 < r < r_n/20$ such that $\mu(B(\widetilde{x}_n, r)) > r^\beta$. By continuity of the map $r \mapsto \mu(B(\widetilde{x}_n, r))$, we can choose $\widetilde{r}_n$ as the largest r such that (17) is satisfied. This choice of $\widetilde{r}_n$ implies relation (18). □

Observe that, since $\widetilde{x}_n \in B(x_n, r_n/6)$ and $\widetilde{r}_n < r_n/10$, one has

$$B(\widetilde{x}_n, \widetilde{r}_n) \subset B(x_n, r_n/3). \tag{19}$$

Lemma 3.6. *Let $x \in \overline{B(\widetilde{x}_n, \widetilde{r}_n/3)}$, $r \geq \widetilde{r}_n/3$ and assume that $B(x, r) \subset B(x_n, r_n)$. Then we have*

$$\mu(B(x, r)) < 5r^\beta. \tag{20}$$

Proof. By construction, $B(x, r) \subset B(\widetilde{x}_n, 4r)$.

• If $B(\widetilde{x}_n, 4r) \subset B(x_n, r_n)$: using (18), we obtain

$$\mu(B(x, r)) \leq \mu(B(\widetilde{x}_n, 4r)) \leq (4r)^\beta \leq 4r^\beta.$$

• If $B(\widetilde{x}_n, 4r) \not\subset B(x_n, r_n)$: the fact that $\widetilde{x}_n \in B(x_n, r_n/6)$ implies that $4r > 5r_n/6$, that is, $r > 5r_n/24$, and

$$\mu(B(x, r)) \leq \mu(B(x_n, r_n)) \leq r_n^{\beta'} \leq r_n^\beta < (24/5)^\beta r^\beta < 5r^\beta. \tag{21}$$

□

Set $I_{n+1} = B(\widetilde{x}_n, \widetilde{r}_n)$ and $\widehat{I}_{n+1} = B(\widetilde{x}_n, \widetilde{r}_n/3)$. Now, apply Proposition 3.1 with $I = I_{n+1}$ and $t = 50$. Since $\mu(I_{n+1}) = \mu(B(\widetilde{x}_n, \widetilde{r}_n)) = (\widetilde{r}_n)^\beta$, we have

$$\lambda\big(\big\{x : M^*_{I_{n+1},\beta}\, \mu(x) > 50\big\}\big) \leq \frac{Q(1) \cdot (\widetilde{r}_n)^\beta (2\widetilde{r}_n)^{1-\beta}}{50} < \lambda(\widehat{I}_{n+1}).$$

For the last inequality, we have used that the constant $Q(1)$ in Proposition 3.1 is less than 5, [14]. Hence, recalling that D_μ has full Lebesgue measure in $\widehat{I}_{n+1}$, we can pick $x_{n+1} \in \widehat{I}_{n+1} \cap D_\mu = B(\widetilde{x}_n, \widetilde{r}_n/3) \cap D_\mu$ such that for all $r > 0$ if $B(x_{n+1}, r) \subset B(\widetilde{x}_n, \widetilde{r}_n)$ then

$$\mu(B(x_{n+1}, r)) \leq 50(2r)^\beta \leq 100 r^\beta. \tag{22}$$

Using that $x_{n+1} \in D_\mu$, one can also choose $0 < r_{n+1} < \widetilde{r}_n/100$ such that

$$\overline{B(x_{n+1}, r_{n+1})} \subset B(\widetilde{x}_n, \widetilde{r}_n/3) \subset B(x_n, r_n/3) \text{ and } \mu(B(x_{n+1}, r_{n+1})) \leq r_{n+1}^{\beta'}. \tag{23}$$

Lemma 3.7. *For all $x \in \overline{B(x_{n+1}, r_{n+1}/3)}$, if $B(x,r) \not\subset B(x_{n+1}, r_{n+1})$ but $B(x,r) \subset B(x_n, r_n)$, then $\mu(B(x,r)) < 300 r^\beta$.*

Proof. The case $r \geq \widetilde{r}_n/3$ is a consequence of Lemma 3.6.

Let x be as in the statement, and assume that $r < \widetilde{r}_n/3$.

From $B(x,r) \not\subset B(x_{n+1}, r_{n+1})$, we deduce that $2r_{n+1}/3 < r$. Hence, $B(x,r) \subset B(x_{n+1}, 3r)$ and $B(x,r) \subset B(\widetilde{x}_n, \widetilde{r}_n)$.

- If $B(x_{n+1}, 3r) \subset B(\widetilde{x}_n, \widetilde{r}_n)$ then by (22), we have

$$\mu(B(x,r)) \leq \mu(B(x_{n+1}, 3r)) \leq 100 \cdot 3^\beta r^\beta < 300 r^\beta.$$

- If $B(x_{n+1}, 3r) \not\subset B(\widetilde{x}_n, \widetilde{r}_n)$ then one necessarily has $3r > 2\widetilde{r}_n/3$. Since $r < \widetilde{r}_n/3$ and $\widetilde{r}_n < r_n/10$, we have the inclusions $B(x,r) \subset B(x, \widetilde{r}_n/3) \subset B(x_n, r_n)$. Finally, by (20) used with $r = \widetilde{r}_n/3$, we infer

$$\mu(B(x,r)) \leq \mu(B(x, \widetilde{r}_n/3)) \leq 5(\widetilde{r}_n/3)^\beta < 5(9r/6)^\beta < 25 r^\beta.$$

□

Summarizing the above, (P0-3) hold when $i = n+1$. Iterating the procedure and applying the above technical lemmas, we complete our inductive construction.

Finally, we discuss the case $\beta = 1$, indicating the minor adjustments in the proof. If there exists $x \in D_\mu$ such that $F'_\mu(x) > 0$, then

$$h_\mu(x) = \liminf_{r \to 0} \frac{\log \mu(B(x,r))}{\log 2r} = \liminf_{r \to 0} \frac{\log |F_\mu(x+r) - F_\mu(x-r)|}{\log 2r} = 1,$$

and we are done. Hence, we suppose that $F_\mu(x)' = 0$ for all $x \in D_\mu$.

Choose an $x_0 \in D_\mu$ and r_0 such that instead of (9) we have

$$\mu(B(x_0, r_0)) \leq r_0/20.$$

Assume that $n \geq 0$ and (x_0, r_0), ..., (x_n, r_n), $(\widetilde{x}_1, \widetilde{r}_1), ..., (\widetilde{x}_n, \widetilde{r}_n)$ are given as before and that (10) holds. Instead of (11) and (12) we have

$$\mu(B(x_i, r_i)) \leq r_i/20 \text{ and } \mu(B(\widetilde{x}_{i-1}, \widetilde{r}_{i-1})) = \widetilde{r}_{i-1} \text{ for } i = 1, ..., n.$$

This implies that (14) and (13) hold true with $\beta = 1$. Further, instead of (16) we use

$$\mu(B(x_n, r_n)) \leq r_n/20. \tag{24}$$

We select $\widetilde{x}_n$ and $\widetilde{r}_n$ such that (17) holds with $\beta = 1$.

By (24), from $B(\widetilde{x}_n, r) \subset B(x_n, r_n)$ one deduces $\mu(B(\widetilde{x}_n, r)) \leq \mu(B(x_n, r_n)) \leq r_n/20$. Hence, for $r \geq r_n/10$ we have $\mu(B(\widetilde{x}_n, r)) < r$. Therefore, (19) holds in this case as well.

Finally, Lemma 3.1 with $t = 50$ allows us to select some real number x_{n+1}, and we keep on arguing as before. One only needs to remove the inequality containing $r_n^{\beta'}$ from (21) and in (23) instead of $r_{n+1}^{\beta'}$ we have to use $r_{n+1}/20$. We also have to keep in mind that $x_{n+1} \in D_\mu$ and our assumption implies $F'_\mu(x_{n+1}) = 0$. □

Remark 3.8. It is not difficult to modify the above proof so that at each step, in $B(x_n, r_n)$, two balls $B(x'_{n+1}, r'_{n+1})$ and $B(x''_{n+1}, r''_{n+1})$ are found with the required properties. Iterating this remark, one concludes that $E_\mu(\beta)$ is uncountable.

4. A non-HM monotone function with an affine spectrum

In this section, we work with monotone functions rather than measures: although the result is the same at the end, here functions are more convenient to deal with.

We construct a monotone function whose spectrum is affine on an interval strictly included in $[0, 1]$ and compatible with the conditions of a spectrum (Proposition 2.1). In the next sections, we explain how the superposition of functions built in Theorem 4.1 yield HM and non-HM measures with prescribed spectrum. The function $\mathbf{1}^*_{[\alpha_0,\beta_0]}(h)$ equals 1 if $h \in [\alpha_0, \beta_0]$ and equals $-\infty$ otherwise.

Theorem 4.1. *Let $0 < \alpha_0 \leq \beta_0 < 1$. Let $0 < d < \alpha_0$ and $\eta > 0$ satisfy*

$$d(1 + \eta\beta_0) \leq \beta_0 \text{ and } d(1 + \eta\alpha_0) \leq \alpha_0. \tag{25}$$

Then there exists a monotone continuous function Z with the following properties: $Z(x) = 0$ when $x \leq 0$, $Z(x) = 1$ when $x \geq 1$, $d_Z(+\infty) = 1$ and

$$d_Z(h) = d(1 + \eta h)\mathbf{1}^*_{[\alpha_0,\beta_0]}(h) \text{ for } h \in [0, \infty). \tag{26}$$

Moreover, Z can be constructed with the additional properties:

(i) *$\{x : h_Z(x) < +\infty\} = \{x : h_Z(x) < 1\} = \{x : h_Z(x) \leq \beta_0\}$ is located on a Cantor set $\mathcal{C}$, strictly included in $[0, 1]$,*

(ii) *$[0, 1] \setminus \mathcal{C}$ consists of a countable number of open intervals whose maximal length is less than 1/10,*

(iii) *there exists $0 < r_0 < 1$ such that for every $x \in [0, 1]$ and $0 < r < r_0$,*

$$\omega_{B(x,r)}(Z) = |Z(x + r) - Z(x - r)| \leq (2r)^{\alpha_0}. \tag{27}$$

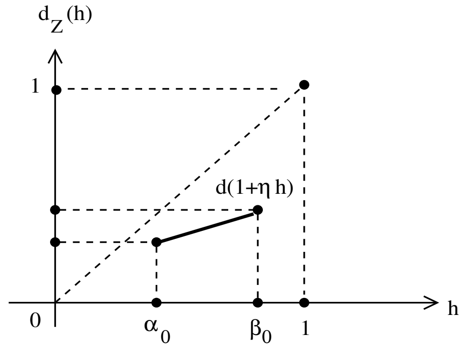

Figure 3. Function in the space $\mathcal{F}^*$

Definition 4.2. We denote by $\mathcal{F}^*$ the class of functions d_Z appearing in (26) with all possible choices of parameters $(\alpha_0, \beta_0, d, \eta)$ satisfying the assumptions of Theorem 4.1 (see Figure 3).

It will be useful to keep in mind that after division by α_0 the second inequality in (25) is of the form: $d\Big(\frac{1}{\alpha_0} + \eta\Big) \leq 1$.

The rest of this section is devoted to the proof of Theorem 4.1.

We assume $\alpha_0 < \beta_0$ and indicate (between parenthesis) during the proof the places where the case $\alpha_0 = \beta_0$ requires a slightly different argument.

For every integer $n \geq 1$, let us denote by

$$\alpha_{n,0} := \alpha_0 < \alpha_{n,1} < \alpha_{n,2} < \cdots < \alpha_{n,n} < \alpha_{n,n+1} := \beta_0$$

the unique set of $n+2$ real numbers equally spaced in the interval $[\alpha_0, \beta_0]$.

(For the $\alpha_0 = \beta_0$ case, choose $\alpha_0' \in (\alpha_0, 1)$ such that

$$\frac{(1-\alpha)\alpha'}{(1-\alpha_0)\alpha_0} > 1 - \alpha_0 \text{ for } \alpha, \alpha' \in [\alpha_0, \alpha_0']. \tag{28}$$

For this case we set $\alpha_{n,0} = \alpha_0 = \beta_0$, $\alpha_{n,1} = \alpha_0 + \dfrac{\alpha_0' - \alpha_0}{n+1}$, $\alpha_{n,2} = \alpha_0 + \dfrac{2(\alpha_0' - \alpha_0)}{n+1}$ and we do not define $\alpha_{n,i}$, for $i \geq 3$.)

For every integer $n \geq 1$, we set

$$\gamma_{n,i} = d(1 + \eta \alpha_{n,i})(1 - 10^{-n}).$$

By (25) for any $\alpha \in [\alpha_0, \beta_0]$, $\dfrac{d(1+\eta\alpha)}{\alpha} = d\Big(\dfrac{1}{\alpha} + \eta\Big) \leq d\Big(\dfrac{1}{\alpha_0} + \eta\Big) \leq 1$, hence

$$\gamma_{n,i} < \alpha_{n,i}. \tag{29}$$

The function Z will be obtained as the sum of an infinite number of functions Z_n, $n \geq 1$, whose increments of order $\alpha_{n,i}$, $i \in \{1, 2, \cdots, n\}$, have their cardinality controlled. Some notation for dyadic intervals are needed.

Definition 4.3. Let (k, j) be two positive integers, and $0 < \alpha \leq 1$ be a real number. We set $I_{j,k} = [k2^{-j}, (k+1)2^{-j})$ and

$$a_{j,k}(\alpha) = (k+1)2^{-j} - 2^{-j/\alpha} \quad \text{and} \quad I_{j,k}(\alpha) = [a_{j,k}(\alpha), (k+1)2^{-j}].$$

Then the length of the interval $I_{j,k}(\alpha)$ is $2^{-j/\alpha}$.

4.1. First step. Let us begin with the function Z_1. Let

$$\varepsilon_0 = \min\{\alpha_0, 1 - \beta_0\}/2 > 0. \tag{30}$$

(When $\alpha_0 = \beta_0$, we also suppose that

$$1 - \varepsilon_0 > \alpha_{n,2}.) \tag{31}$$

Consider $\alpha_{1,1}$ (which belongs to (α_0, β_0)), and choose an integer J_1 so large that

$$2^{100} < 2^{\left[J_1 \frac{\gamma_{1,1}}{\alpha_{1,1}}\right]+1} < 2^{J_1}/10,\ J_1 \leq 2^{\varepsilon_0 J_1},\ 2^{-J_1/\beta_0} < \frac{2^{-J_1}}{100},\ \text{and } 2^{-1} \cdot 2^{J_1 \frac{\beta_0 - \alpha_0}{2}} > 1. \tag{32}$$

The first inequality can be satisfied since $\gamma_{1,1} < \alpha_{1,1}$ by (29).

(When $\alpha_0 = \beta_0$, we need to argue differently, since in this case $\alpha_{1,1} = \alpha_0 + \dfrac{\alpha_0' - \alpha_0}{2}$ does not belong to (α_0, β_0). In (32) the last inequality should be replaced by $2^{-1}2^{J_1 \frac{\alpha_0' - \alpha_0}{2}} > 1$.)

We denote by $\mathcal{T}_1$ the set of integers

$$\mathcal{T}_1 := \left\{ k \in \{1, \cdots, 2^{J_1} - 1\} : \ k \text{ is a multiple of } 2^{\left[J_1(1 - \frac{\gamma_{1,1}}{\alpha_{1,1}})\right]} \right\}.$$

We also put $\mathcal{T}_{1,1} = \mathcal{T}_1$. Then

$$\frac{1}{2} \cdot 2^{J_1 - \left[J_1(1 - \frac{\gamma_{1,1}}{\alpha_{1,1}})\right]} < \#\mathcal{T}_{1,1} < 2 \cdot 2^{J_1 - \left[J_1(1 - \frac{\gamma_{1,1}}{\alpha_{1,1}})\right]}.$$

By (32) we also have $\#\mathcal{T}_{1,1} = 2^{J_1 \frac{\gamma_{1,1}}{\alpha_{1,1}}(1 - \varepsilon_{1,1})}$ with $\varepsilon_{1,1} \leq \frac{1}{10}$.

The function Z_1 is obtained as follows.

- for every $k \in \{0, 1, \cdots, 2^{J_1} - 1\}$ such that $k \notin \mathcal{T}_1$, for every $x \in I_{J_1,k}$, we set

$$Z_1(x) = x/2.$$

  Hence, Z_1 is just affine on $I_{J_1,k}$, with slope $1/2$.

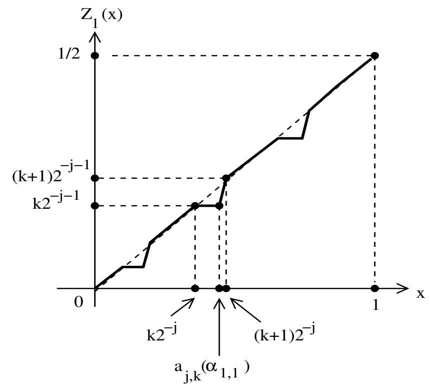

Figure 4. Sketch of the graph of Z_1

- for every integer $k \in \mathcal{T}_1$, we set for every $x \in I_{J_1,k}$,

$$Z_1(x) = \begin{cases} 2^{-1}k2^{-J_1} & \text{if } x \in [k2^{-J_1}, a_{J_1,k}(\alpha_{1,1})) \\ 2^{-1}\left((k+1)2^{-J_1} + 2^{J_1\frac{1-\alpha_{1,1}}{\alpha_{1,1}}}(x-(k+1)2^{-J_1})\right) & \text{if } x \in I_{J_1,k}(\alpha_{1,1}). \end{cases}$$

Hence Z_1 is first constant on $[k2^{-J_1}, a_{J_1,k}(\alpha_{1,1}))$, and then affine with a large slope $2^{J_1\frac{1-\alpha_{1,1}}{\alpha_{1,1}}}$ on the interval $I_{J_1,k}(\alpha_{1,1})$.

A quick analysis shows that the function Z_1 is continuous, piecewise affine, with $Z_1(0) = 0$ and $Z_1(1) = 1/2$, and that $Z_1(x) \leq x/2$. Observe that the oscillations of Z_1 on the intervals $I_{J_1,k}(\alpha_{1,1})$, $k \in \mathcal{T}_1$, satisfy

$$\omega_{I_{J_1,k}(\alpha_{1,1})}(Z_1) = 2^{-1}2^{-J_1} = 2^{-1}|I_{J_1,k}(\alpha_{1,1})|^{\alpha_{1,1}}.$$

We remark that the cardinality of $\mathcal{T}_1$ is $2^{J_1-\left[J_1(1-\frac{\gamma_{1,1}}{\alpha_{1,1}})\right]} \sim 2^{J_1\frac{\gamma_{1,1}}{\alpha_{1,1}}}$.

4.2. Construction of Z_n. Let $n \geq 2$, and assume that $Z_1, Z_2, \cdots, Z_{n-1}$ are constructed. We also suppose that the sets of integers $\mathcal{T}_{n-1,i}$ satisfy

$$\#\mathcal{T}_{n-1,i} = 2^{J_{n-1}\frac{\gamma_{n-1,i}}{\alpha_{n-1,i}}(1-\varepsilon_{n-1,i})}, \text{ where } \varepsilon_{n-1,i} \leq 10^{-(n-1)}. \tag{33}$$

(When $\alpha_0 = \beta_0$, we have $\mathcal{T}_{n-1,i}$ only for $i = 1$. In this case in the sequel we consider only the index $i = 1$, instead of $i = 1, ..., n$.)

Choose an integer J_n satisfying the following conditions:

$$4n10^n \cdot 2^{J_{n-1}10^{2n}/\alpha_0} \leq J_n \cdot \gamma_{n,0} \leq J_n \text{ and } 2n2^{J_n(1-10^{-n})} \leq 2^{J_n - J_{n-1}/\alpha_0}, \tag{34}$$

moreover

$$4nJ_n \leq 2^{\varepsilon_0 J_n}, \ \ 2^{J_n(\frac{1}{\beta_0}-1)} > 1 \quad \text{and} \quad 2^{-n}2^{J_n\frac{\beta_0-\alpha_0}{n+1}} > 1. \tag{35}$$

(When $\alpha_0 = \beta_0$, we need to replace the last inequality by $2^{-n}2^{J_n\frac{\alpha_0'-\alpha_0}{n+1}} > 1$.)
One has

$$[J_n \cdot \gamma_{n,0}] \leq [J_n \cdot \gamma_{n,i}] \leq \left[J_n \frac{\gamma_{n,i}}{\alpha_{n,i}}\right],$$

and by (25)

$$\frac{\gamma_{n,i}}{\alpha_{n,i}} = d(\frac{1}{\alpha_{n,i}} + \eta)(1-10^{-n}) \leq d(\frac{1}{\alpha_0}+\eta)(1-10^{-n}) \leq (1-10^{-n}).$$

Hence by (34) for every $i \in \{1, \cdots, n\}$, one obtains

$$\left[J_n \frac{\gamma_{n,i}}{\alpha_{n,i}}\right] \geq [J_n\gamma_{n,0}] \geq 2^{J_{n-1}/\alpha_0} \ \text{ and } \ \sum_{i=1}^{n} 2^{\left[J_n\frac{\gamma_{n,i}}{\alpha_{n,i}}\right]} \leq 2^{J_n - J_{n-1}/\alpha_0}.$$

Simultaneously for all exponents $\alpha_{n,i}$, $i \in \{1,2,...,n\}$, our aim is now to spread as uniformly as possible the intervals I on which Z_n has oscillations of order $|I|^{\alpha_{n,i}}$ (as we performed during the construction of Z_1). This is achieved as follows.

Let $p_1 = 3$, $p_2 = 5$, ..., p_n be the first n odd prime numbers. For every $i \in \{1,2,...,n\}$, we denote by $\mathcal{T}_{n,i}$ the set of integers

$$\mathcal{T}_{n,i} := \left\{ k \in \{1,\cdots,2^{J_n}\} : \left\{ \begin{array}{l} k-p_i \text{ is a multiple of } 2^{\left[J_n(1-\frac{\gamma_{n,i}}{\alpha_{n,i}})\right]}, \text{ and} \\ \text{there exists an integer } 0 < i' \leq n-1 \text{ such that} \\ k2^{-J_n} \text{ belongs to } I_{J_{n-1},K}(\alpha_{n,i'}) \text{ for some } K \in \mathcal{T}_{n-1,i'} \end{array} \right. \right\}. \tag{36}$$

(When $\alpha_0 = \beta_0$, we use only $p_1 = 3$, consider only $\mathcal{T}_{n,1}$ and use only $i' = 1$ in the definition of $\mathcal{T}_{n,i} = \mathcal{T}_{n,1}$.)

To estimate the cardinality of $\mathcal{T}_{n,i}$ we have

$$\frac{1}{2}\sum_{i'=1}^{n-1} (\#\mathcal{T}_{n-1,i'})\, 2^{-J_{n-1}/\alpha_{n,i'}} \cdot 2^{J_n - \left[J_n(1-\frac{\gamma_{n,i}}{\alpha_{n,i}})\right]}$$
$$< \ \#\mathcal{T}_{n,i} \ < \ 2\cdot \sum_{i'=1}^{n-1} (\#\mathcal{T}_{n-1,i'})\, 2^{-J_{n-1}/\alpha_{n,i'}} \cdot 2^{J_n - \left[J_n(1-\frac{\gamma_{n,i}}{\alpha_{n,i}})\right]} < 4n2^{J_n\frac{\gamma_{n,i}}{\alpha_{n,i}}},$$

that is, $\#\mathcal{T}_{n,i} \sim \ 2\cdot \sum_{i'=1}^{n-1} (\#\mathcal{T}_{n-1,i'})\, 2^{-J_{n-1}/\alpha_{n,i'}} \cdot 2^{J_n\frac{\gamma_{n,i}}{\alpha_{n,i}}}$. Hence, using (33) and (34) we have

$$\#\mathcal{T}_{n,i} = 2^{J_n\frac{\gamma_{n,i}}{\alpha_{n,i}}(1-\varepsilon_{n,i})} \quad \text{where} \quad \varepsilon_{n,i} \leq 10^{-n}. \tag{37}$$

Finally, we set (when $\alpha_0 = \beta_0$, we have $\mathcal{T}_n = \mathcal{T}_{n,1}$)

$$\mathcal{T}_n = \bigcup_{i=1}^{n} \mathcal{T}_{n,i}. \tag{38}$$

Lemma 4.4. *If $1 \le i < j \le n$, then $\mathcal{T}_{n,i} \cap \mathcal{T}_{n,j} = \emptyset$. Moreover, if $k \in \mathcal{T}_{n,i}$ and $k' \in \mathcal{T}_{n,j}$ such that $k \neq k'$ (one may have $i = j$), then $I_{J_n,k}(\alpha_{n,i}) \bigcap I_{J_n,k'}(\alpha_{n,j}) = \emptyset$.*

Proof. Suppose that an integer q belongs to $\mathcal{T}_{n,i} \cap \mathcal{T}_{n,j}$. Hence q can be written

$$q = p_i + m_i 2^{\left[J_n(1-\frac{\gamma_{n,i}}{\alpha_{n,i}})\right]} = p_j + m_j 2^{\left[J_n(1-\frac{\gamma_{n,j}}{\alpha_{n,j}})\right]}.$$

Assume that $2^{\left[J_n(1-\frac{\gamma_{n,i}}{\alpha_{n,i}})\right]} \ge 2^{\left[J_n(1-\frac{\gamma_{n,j}}{\alpha_{n,j}})\right]}$, the other case is similar. Then

$$0 \neq |p_j - p_i| = 2^{\left[J_n(1-\frac{\gamma_{n,j}}{\alpha_{n,j}})\right]} \left| m_j - m_i 2^{\left[J_n(1-\frac{\gamma_{n,i}}{\alpha_{n,i}})\right] - \left[J_n(1-\frac{\gamma_{n,j}}{\alpha_{n,j}})\right]} \right| > 2^{\left[J_n(1-\frac{\gamma_{n,j}}{\alpha_{n,j}})\right]}. \tag{39}$$

Obviously, by Bertrand's postulate (proved by Chebyshev) about prime numbers we have $0 < |p_j - p_i| \le p_n \le 2^{n+1}$, while $2^{\left[J_n(1-\frac{\gamma_{n,j}}{\alpha_{n,j}})\right]} \gg 2^{n+1}$. Hence, it is impossible to realize (39).

Finally, if $k \neq k'$, since $\alpha_{n,i}$ and $\alpha_{n,j}$ are smaller than one, it is clear from the construction that $I_{J_n,k}(\alpha_{n,i}) \cap I_{J_n,k'}(\alpha_{n,j}) = \emptyset$. □

The non-decreasing mapping Z_n is obtained as follows.

- for $k \in \{0, 1, \cdots, 2^{J_n} - 1\}$ such that $k \notin \mathcal{T}_n$, for every $x \in I_{J_n,k}$, we set:

$$Z_n(x) = 2^{-n} x.$$

- When the integer k belongs to some $\mathcal{T}_{n,i}$, we set: for every $x \in I_{J_n,k}$,

$$Z_n(x) = \begin{cases} 2^{-n} k 2^{-J_n} & \text{if } x \in [k2^{-J_n}, a_{J_n,k}(\alpha_{n,i})), \\ 2^{-n}\Big((k+1)2^{-J_n} + 2^{J_n \frac{1-\alpha_{n,i}}{\alpha_{n,i}}}(x - (k+1)2^{-J_n})\Big) & \text{if } x \in I_{J_n,k}(\alpha_{n,i}). \end{cases}$$

As above, the function Z_n is continuous, piecewise affine, and it obviously satisfies $Z_n(0) = 0$, $Z_n(1) = 2^{-n}$ and $Z_n(x) \le 2^{-n}$. Moreover, the oscillations of Z_n on the intervals $I_{J_n,k}(\alpha_{n,i})$, $k \in \mathcal{T}_{n,i}$, satisfy

$$\omega_{I_{J_n,k}(\alpha_{n,i})}(Z_n) = 2^{-n} 2^{-J_n} = 2^{-n} |I_{J_n,k}(\alpha_{n,i})|^{\alpha_{n,i}}.$$

In addition, a key remark is that Z_n is not a linear function with slope 2^{-n} only on some intervals $I_{J_n,k}$ that are included in intervals $I_{J_{n-1},k'}$ on which Z_{n-1} has large oscillation. Hence, these nested intervals are the intermediary steps of the construction of a Cantor set.

4.3. Construction of Z.

Definition 4.5. We define the mapping $Z : [0,1] \to [0,1]$ by the formula

$$\text{for every } x \in [0,1], \quad Z(x) = \sum_{n=1}^{+\infty} Z_n(x).$$

Immediate properties of Z are gathered in the next Proposition.

Proposition 4.6. *The mapping Z is continuous (as uniform limit of continuous functions), strictly increasing, and satisfies $Z(0) = 0$ and $Z(1) = 1$.*

This obviously follows from the construction, and from the fact that

$$\text{for every } N \geq 1, \quad \|Z - Z_N\|_\infty = \Big\| \sum_{n \geq N+1} Z_n \Big\|_\infty \leq 2^{-N+2}.$$

Moreover, let us call $\mathcal{C}$ the (closed) Cantor set

$$\mathcal{C} = \bigcap_{N \geq 1} \bigcup_{n \geq N} \bigcup_{i=1}^{n} \bigcup_{k \in \bigstar_n} I_{J_n,k}(\alpha_{n,i}), \tag{40}$$

$\bigstar_n$ meaning that the intervals $I_{J_n,k}(\alpha_{n,i})$ considered are only for those k which appear in the construction of Z_n.

Lemma 4.7. *If $x \notin \mathcal{C}$, then $h_Z(x) = +\infty$.*

Proof. If $x \notin \mathcal{C}$, then there exists $r > 0$ such that the distance between x and $\mathcal{C}$ is larger than r. In particular, x does not belong to any interval of the form $I_{J_n,k}(\alpha_{n,i})$ appearing in the definition of $\mathcal{C}$, for n larger than some integer $N > 0$. In other words, Z is affine in a neighborhood of x, hence the conclusion. □

Lemma 4.7 and our construction yield items (i) and (ii) of Theorem 4.1 concerning the size of the complement of the Cantor set.

4.4. Local regularity properties of Z. To find the multifractal spectrum of Z, we start by studying its local oscillations.

For every N, k, i and every $r > 0$, let us define the intervals $I_{J_N,k}(i,r)$ by:

$$I_{J_N,k}(i,r) = I_{J_N,k}(\alpha_{N,i}) + B(0,r).$$

Definition 4.8. For $i_0 = 1, ..., N+1$ let us introduce the sets

$$\begin{aligned}
\mathcal{E}_{N,i_0,r} &= \{x \in [0,1] : \omega_{B(x,r)}(Z) \geq (2r)^{\alpha_{N,i_0}}\}, \\
\mathcal{E}'_{N,i_0} &= \bigcup_{i : i < i_0} \bigcup_{k \in \mathcal{T}_{N,i}} I_{J_N,k}(\alpha_{N,i}), \\
\mathcal{E}''_{N,i_0,r} &= \bigcup_{i : i < i_0} \bigcup_{k \in \mathcal{T}_{N,i}} I_{J_N,k}(i,r).
\end{aligned}$$

(When $\alpha_0 = \beta_0$, we consider these sets only for $i_0 = 1, 2$, this restriction applies in Lemma 4.9 as well.)

Heuristically, $\mathcal{E}'_{N,i_0}$ contains all the intervals on which Z_N has an exponent less than α_{N,i_0}, and $\mathcal{E}''_{N,i_0,r}$ is the r-neighborhood of these points. Also, $\mathcal{E}''_{N,1,r} = \emptyset$.

We also remark that $\mathcal{C} = \bigcap_{N=1}^{\infty} \mathcal{E}'_{N,N+1}$. (When $\alpha_0 = \beta_0$ then $\mathcal{C} = \bigcap_{N=1}^{\infty} \mathcal{E}'_{N,2}$.)

The next Lemma, very technical, allows us to "locate" the elements around which Z has an oscillation of a given size.

Lemma 4.9. *Let $r \in (0, 2^{-J_5-1})$, and let $N \geq 5$ be the unique integer such that*

$$2^{-J_{N+1}} \leq 2r < 2^{-J_N}. \tag{41}$$

Let $i_0 \in \{1, ..., N+1\}$. One has:

(i) *If $2^{-\frac{J_N}{\alpha_0(1-\beta_0)}} < 2r < 2^{-\frac{J_N}{\alpha_{N,i_0}}}$, then $\mathcal{E}_{N,i_0,r} \subset \mathcal{E}''_{N,i_0,r}$.*

(ii) *If $2r > 2^{-\frac{J_N}{\alpha_{N,i_0}}}$, or if $2r < 2^{-\frac{J_N}{\alpha_0(1-\beta_0)}}$, then $\mathcal{E}_{N,i_0,r} = \emptyset$.*

(iii) *Moreover, if $x \in \mathcal{E}'_{N,i_0}$, then there is $0 < r \leq 2^{-J_N/\alpha_{N,i_0}}$ such that $x \in \mathcal{E}_{N,i_0,r}$.*

Proof. We are going to investigate the possible values of $\omega_{B(x,r)}(Z)$, for all possible values of $x \in [0,1]$ and $r > 0$. We also emphasize that $0 < \varepsilon_0$ by (30) is so small that $0 < \alpha_0 - \varepsilon_0 < \beta_0 + \varepsilon_0 < 1$. Recall also that we supposed that N is the only integer satisfying (41).

We fix $i_0 \in \{1, ..., N+1\}$, and we look for the locations of the elements of $\mathcal{E}_{N,i_0,r}$. Obviously, $\omega_{B(x,r)}(Z) = \sum_{n\geq 1} \omega_{B(x,r)}(Z_n)$. Let us compare the terms in this sum according to the value of n.

(i) $\mathbf{n \geq N+1}$: by construction, $B(x,r)$ is covered by at most $(2r/2^{-J_n}) + 2 \leq 4r2^{J_n}$ dyadic intervals of generation J_n, on which the oscillation of Z_n is exactly $2^{-J_n}2^{-n}$. Hence $\omega_{B(x,r)}(Z_n) \leq 4r2^{J_n}2^{-J_n}2^{-n} \leq 4 \cdot 2^{-n}r$. Since $N > 4$ summing over $n \geq N+1$ yields

$$\sum_{n\geq N+1} \omega_{B(x,r)}(Z_n) \leq 8 \cdot 2^{-N} r \leq (2r)/4 \leq (2r)^{\alpha_{N,i_0}}/4. \tag{42}$$

(ii) $\mathbf{n \leq N-1}$: by construction, Z_n has a maximal slope of $2^{-n}2^{J_n \frac{1-\alpha_{n,j}}{\alpha_{n,j}}}$, for some integer $j \in \{1, \cdots, n\}$, which by $2^{-n} < 1$, $1 - \alpha_{n,j} < 1$, $\alpha_{n,0} < \alpha_{n,j}$ verifies

$$2^{-n}2^{J_n \frac{1-\alpha_{n,j}}{\alpha_{n,j}}} < 2^{J_n \frac{1}{\alpha_{n,0}}} = 2^{J_n/\alpha_0}. \tag{43}$$

By (34) and (35), for every $n < N$ one has

$$J_N \geq 2^{J_n/\alpha_0} \quad \text{and} \quad 4NJ_N \leq 2^{\varepsilon_0 J_N}. \tag{44}$$

By (43) we have $\omega_{B(x,r)}(Z_n) \leq 2^{J_n/\alpha_0} 2r$. By (41) and (44) we obtain

$$\sum_{n=1}^{N-1} \omega_{B(x,r)}(Z_n) \leq \sum_{n=1}^{N-1} 2^{J_n/\alpha_0}(2r) \leq 4NJ_N(2r)/4 \leq 2^{\varepsilon_0 J_N} 2r/4 \leq (2r)^{1-\varepsilon_0}/4.$$

Since $1-\varepsilon_0 > \beta_0 = \alpha_{N,N+1} \geq \alpha_{N,i_0}$, we deduce that

$$\sum_{n=1}^{N-1} \omega_{B(x,r)}(Z_n) \leq (2r)^{\alpha_{N,i_0}}/4. \tag{45}$$

(When $\alpha_0 = \beta_0$, we also use (31).)

(iii) $\mathbf{n = N}$: It remains to study the oscillation of Z_N on $B(x,r)$. By (42) and (45), if $x \in \mathcal{E}_{N,i_0,r}$, then one necessarily has

$$\omega_{B(x,r)}(Z_N) \geq (2r)^{\alpha_{N,i_0}}/2. \tag{46}$$

Our goal is now to identify the elements satisfying (46).

By (41), $B(x,r)$ contains at most one dyadic number $k2^{-J_N}$. Recall that $\mathcal{T}_N$ defined by (38) contains the integers k such that Z_N possesses large oscillations on intervals of the form $I_{J_N,k}(\alpha_{N,i})$. If $B(x,r)$ does not meet any interval of the form $I_{J_N,k}(\alpha_{N,i})$, then the oscillation of Z_N on $B(x,r)$ is less than $2^{-N}2r$ (since 2^{-N} is the value of the slope of Z_N on such intervals), and (46) cannot be realized.

We thus assume that $B(x,r)$ intersects an interval of the form $I_{J_N,k}(\alpha_{N,i})$, where $i \in \{0,1,...,N\}$ and $k \in \mathcal{T}_{N,i}$. Observe that $B(x,r)$ can intersect at most two such intervals, and when it does, $2r \sim 2^{-J_N}$ and $\omega_{B(x,r)}(Z_N) \leq 2 \cdot 2^{-N} \cdot 2^{-J_N}$. Thus, in this case, $x \notin \mathcal{E}_{N,i_0,r}$.

We thus assume that $B(x,r)$ intersects exactly one interval of the form $I_{J_N,k}(\alpha_{N,i})$, where $i \in \{0,1,...,N\}$ and $k \in \mathcal{T}_{N,i}$.

1. If $\mathbf{2r > 2^{-J_N/\alpha_{N,i_0}}}$: then by definition of Z_N the oscillation on the intersection $I_{J_N,k}(\alpha_{N,i}) \cap B(x,r)$ is less than the oscillation of Z_N on one interval of size 2^{-J_N}, i.e. less than $2^{-N}2^{-J_N} \leq 2^{-N}(2r)^{\alpha_{N,i_0}} \leq (2r)^{\alpha_{N,i_0}}/4$. Hence, x cannot belong to $\mathcal{E}_{N,i_0,r}$.

2. Next we assume:

$$\mathbf{2r \leq 2^{-J_N/\alpha_{N,i_0}}}. \tag{47}$$

2A. If $\mathbf{2r < 2^{\frac{-J_N}{(1-\beta_0)\alpha_0}}}$: Keeping in mind that $1 > 1-\alpha_{N,i}$, $1-\beta_0 \leq 1-\alpha_{N,i_0}$ and $\alpha_0 \leq \alpha_{N,i}$ this implies that

$$2r < 2^{\frac{-J_N}{(1-\beta_0)\alpha_0}} < 2^{\frac{-J_N(1-\alpha_{N,i})}{(1-\alpha_{N,i_0})\alpha_{N,i}}}. \tag{48}$$

(At this point, when $\alpha_0 = \beta_0$, extra care is needed. In this case $i_0 = 1$, or 2. To obtain (48) one can use (28) with $\alpha = \alpha_{N,i_0}$ and $\alpha' = \alpha_{N,i}$ to obtain

$$\frac{(1-\alpha_{N,i_0})\alpha_{N,i}}{(1-\alpha_0)\alpha_0} > 1-\alpha_0 > 1-\alpha_{N,i}. \qquad)$$

By (48), one gets

$$\omega_{B(x,r)}(Z_N) < 2^{-N}2^{J_N \frac{1-\alpha_{N,i}}{\alpha_{N,i}}} 2r < 2^{-N}(2r)^{-1+\alpha_{N,i_0}}(2r) \leq (2r)^{\alpha_{N,i_0}}/4,$$

and $x \notin \mathcal{E}_{N,i_0,r}$. Since this argument applies for any $x \in [0,1]$ for $2r < 2^{\frac{-J_N}{(1-\beta_0)\alpha_0}}$ we have $\mathcal{E}_{N,i_0,r} = \emptyset$.

2B. If $\mathbf{2^{\frac{-J_N}{(1-\beta_0)\alpha_0}} < 2r < 2^{-J_N/\alpha_{N,i_0}}}$ and $\alpha_{\mathbf{N,i}} \geq \alpha_{\mathbf{N,i_0}}$: the slope of the function Z_N on the interval $I_{J_N,k}(\alpha_{N,i})$ is exactly $2^{-N}2^{J_N \frac{1-\alpha_{N,i}}{\alpha_{N,i}}}$. The ball $B(x,r)$ can also intersect $I_{J_N,k-1}$, or $I_{J_N,k+1}$ but by (35) on these intervals the slope of Z_N equals $2^{-N} < 2^{-N}2^{J_N \frac{1-\alpha_{N,i}}{\alpha_{N,i}}}$. Consequently, the oscillation of Z_N on $B(x,r)$ is less than $2^{-N}2^{J_N \frac{1-\alpha_{N,i}}{\alpha_{N,i}}} 2r$, which by (47) and $\alpha_{N,i_0}/\alpha_{N,i} \leq 1$ is no more than

$$2^{-N}(2r)^{-\alpha_{N,i_0}\frac{1-\alpha_{N,i}}{\alpha_{N,i}}}(2r) \leq (2r)^{(\alpha_{N,i}-1)\frac{\alpha_{N,i_0}}{\alpha_{N,i}}+1}/4 \leq (2r)^{\alpha_{N,i}}/4 \leq (2r)^{\alpha_{N,i_0}}/4.$$

Once again, $x \notin \mathcal{E}_{N,i_0,r}$.

2C. If $\mathbf{2^{\frac{-J_N}{(1-\beta_0)\alpha_0}} < 2r < 2^{-J_N/\alpha_{N,i_0}}}$ and $\alpha_{\mathbf{N,i}} < \alpha_{\mathbf{N,i_0}}$: observe that if $x \notin \mathcal{E}''_{N,i_0,r}$, then $B(x,r)$ does not intersect any interval of the form $I_{J_N,k}(\alpha_{N,i})$ with $\alpha_{N,i} \leq \alpha_{N,i_0}$. Hence, $x \notin \mathcal{E}_{N,i_0,r}$. We deduce that necessarily $\mathcal{E}_{N,i_0,r} \subset \mathcal{E}''_{N,i_0,r}$.

This concludes the proof of the parts (i) and (ii) of Lemma 4.9.

To obtain part (iii) of the lemma, assume that $x \in \mathcal{E}'_{N,i_0}$, for some $i_0 \in \{1,...,N+1\}$.

(When $\alpha_0 = \beta_0$, this means $i_0 = 2$, since $\mathcal{E}'_{N,1} = \emptyset$.)

Then there exists $i < i_0$ and $k \in \mathcal{T}_{N,i}$ such that $x \in I_{J_N,k}(\alpha_{N,i})$. Choose $r = 2^{-J_N/\alpha_{N,i}} < 2^{-J_N/\alpha_{N,i_0}}$. Then $B(x,r) \supset I_{J_N,k}(\alpha_{N,i})$ and

$$\omega_{B(x,r)}(Z_N) \geq 2^{-N} \cdot 2^{-J_N} = 2^{-N}(2^{-J_N/\alpha_{N,i}})^{\alpha_{N,i_0}}(2^{-J_N/\alpha_{N,i}})^{\alpha_{N,i}-\alpha_{N,i_0}}.$$

Using (35), one concludes that

$$\omega_{B(x,r)}(Z_N) \geq 2^{-N}2^{J_N \frac{\alpha_{N,i_0}-\alpha_{N,i}}{\alpha_{N,i}}} \cdot r^{\alpha_{N,i_0}} \geq 2^{-N}2^{J_N \frac{\beta_0-\alpha_0}{N+1}} r^{\alpha_{N,i_0}} > r^{\alpha_{N,i_0}}.$$

Consequently, $x \in \mathcal{E}_{N,i_0,r}$. □

From these considerations, one deduces easily item (iii) of Theorem 4.1: for every $x \in [0,1]$, for every $r > 0$ small enough, $Z(x+r) - Z(x-r) \leq 2r^{\alpha_0}$.

Indeed, consider $0 < r < 2^{-J_5-1}$ and the associated integer $N > 4$ such that (41) holds. By Lemma 4.9 we have $\mathcal{E}_{N,1,r} \subset \mathcal{E}''_{N,1,r} = \emptyset$. Hence, for any $x \in [0,1]$,

$$\omega_{B(x,r)}(Z) = Z(x+r) - Z(x-r) < (2r)^{\alpha_{N,1}} < (2r)^{\alpha_{N,0}} = (2r)^{\alpha_0}. \qquad (49)$$

4.5. Upper bound for the spectrum of Z. We need to consider only points within the Cantor set $\mathcal{C}$.

Let $\alpha_0 \leq \alpha \leq \beta_0$, and assume that a real number x satisfies $h_Z(x) \leq \alpha$. (When $\alpha_0 = \beta_0$, we need to consider only the case $\alpha = \alpha_0$.) By definition, for every $\varepsilon > 0$ such that $0 < \alpha_0 - \varepsilon < \beta_0 + \varepsilon < 1$, one can find a strictly increasing sequence of integers p such that x belongs to an $\mathcal{E}_{N_p, i_{N_p}, 2^{-p}}(Z)$, where i_n is the largest integer satisfying $\alpha_{n,i_n} \leq \alpha + \varepsilon$, ($i_n$ depends on α and ε, but we omit the subscripts for clarity). Otherwise we would have for every small $r > 0$

$$\omega_{B(x,r)} < (2r)^{\alpha + \frac{\varepsilon}{2}},$$

which contradicts the fact that $h_Z(x) \leq \alpha$.

Hence, we have the inclusion

$$E_Z^{\leq}(\alpha) = \{x : h_Z(x) \leq \alpha\} \subset \limsup_{p \to +\infty} \mathcal{E}_{N_p, i_{N_p}, 2^{-p}} = \bigcap_{P \geq 1} \bigcup_{p \geq P} \mathcal{E}_{N_p, i_{N_p}, 2^{-p}}. \qquad (50)$$

Using this, we prove that the Hausdorff dimension of $E_Z^{\leq}(\alpha)$ is less than $d(1+\eta\alpha)$. Let $s > d(1+\eta\alpha)$.

Obviously, by (50), $E_Z^{\leq}(\alpha)$ is covered for every $P \geq 1$ by the union $\bigcup_{p \geq P} \mathcal{E}_{N_p, i_{N_p}, 2^{-p}}$. Let us count the number of intervals in $\mathcal{E}_{N_p, i_{N_p}, 2^{-p}}$. Recalling (37) we deduce using the decomposition above that $\mathcal{E}_{N_p, i_{N_p}, 2^{-p}}$ is covered by the union over each i such that $i < i_{N_p}$ of

$$2 \cdot 2^{J_{N_p} \frac{\gamma_{N_p,i}}{\alpha_{N_p,i}}(1-\varepsilon_{N_p,i})}$$

intervals of the form $I_{J_{N_p},k}(i, 2^{-p})$ (here the value of i depends on the value of 2^{-p}). But as noticed above, in order to have $h_Z(x) \leq \alpha$, one necessarily has $x \in I_{J_{N_p},k}(\alpha_{N_p,i})$ (not only $x \in I_{J_{N_p},k}(i, 2^{-p})$).

Hence, the s-Hausdorff pre-measure $\mathcal{H}^s_\delta$ (which is obtained by using covering of size less than $\delta \geq 2^{-p}$) of $\mathcal{E}_{N_p, i_{N_p}, 2^{-p}}$ is bounded from above by

$$\begin{aligned} \mathcal{H}^s_\delta(\mathcal{E}_{N_p, i_{N_p}, 2^{-p}}) &\leq \sum_{i=1}^{i_{N_p}-1} |I_{J_{N_p},k}(\alpha_{N_p,i})|^s \cdot 2 \cdot 2^{J_{N_p} \frac{\gamma_{N_p,i}}{\alpha_{N_p,i}}(1-\varepsilon_{N_p,i})} \\ &\leq C \sum_{i=1}^{i_{N_p}-1} 2^{J_{N_p}\left(\frac{\gamma_{N_p,i}}{\alpha_{N_p,i}}(1-\varepsilon_{N_p,i}) - \frac{s}{\alpha_{N_p,i}}\right)}. \end{aligned}$$

Then,

$$\begin{aligned} \mathcal{H}^s_\delta(E_Z^{\leq}(\alpha)) &\leq \sum_{p \geq P} \mathcal{H}^s_\delta(\mathcal{E}_{N_p, i_{N_p}, 2^{-p}}) \\ &\leq C \sum_{p \geq P} \sum_{i=1}^{i_{N_p}-1} 2^{J_{N_p}\left(\frac{\gamma_{N_p,i}}{\alpha_{N_p,i}}(1-\varepsilon_{N_p,i}) - \frac{s}{\alpha_{N_p,i}}\right)}. \end{aligned}$$

This series converges since $s > d(1+\eta\alpha) > \gamma_{N_p,i_{N_p}} > \gamma_{N_p,i}$ and $\varepsilon_{N_p,i} \leq 10^{-N_p}$ (for every indices p and i). We conclude that $\mathcal{H}^s_\delta(E^{\leq}_Z(\alpha)) = 0$, hence $\dim_{\mathcal{H}} E^{\leq}_Z(\alpha) \leq s$. Since this holds for every $s > d(1+\eta\alpha)$ we obtained our upper bound for the spectrum of Z when $\alpha \in [\alpha_0, \beta_0]$.

By (27), $h_Z(x) \geq \alpha_0$ for any $x \in [0,1]$ and hence $\{x : h_Z(x) < \alpha_0\} = \emptyset$.

On the other hand, in order to have $h_Z(x) < +\infty$ one needs $x \in \cap_{N=1}^{\infty} \mathcal{E}'_{N,N+1}$. But then by Lemma 4.9 there exists $r_N \to 0$ such that $x \in \mathcal{E}_{N,N+1,r_N}$. Since $\alpha_{N,N+1} = \beta_0$ we obtain that in this case $h_Z(x) \leq \beta_0$.

(When $\alpha_0 = \beta_0$, we have $x \in \mathcal{E}_{N,2,r_N}$ and $\alpha_{N,2} \to \alpha_0 = \beta_0$.)

4.6. Lower bound for the spectrum of Z. Let $\alpha \in [\alpha_0, \beta_0]$, and consider the sequence of sets $\mathcal{F}_n(\alpha) = \bigcup_{k \in \mathcal{T}_{n,i_n}} I_{J_n,k}(\alpha_{n,i_n})$, where $i_n \in \{1, ..., n\}$ is the largest integer satisfying $\alpha_{n,i_n} < \alpha$. (In case of $\alpha = \alpha_0$, such an integer does not exist and we set $i_n = 1$ in this case.)

Consider the Cantor set

$$\mathcal{F}(\alpha) = \bigcap_{n \geq 2} \mathcal{F}_n(\alpha).$$

Obviously $\mathcal{F}(\alpha) \subset \mathcal{C} = \cap_{N=1}^{\infty} E'_{N,N+1}$, where $\mathcal{C}$ is the Cantor set defined by (40) on which the function Z has exponents between α_0 and β_0.

First remark that by (34) the sequence $(J_n)_{n \geq 2}$ is lacunary (for instance, we have $J_n \geq 2^{J_{n-1}}$). In addition, recall that by (37), one has

$$\#\mathcal{T}_{n,i_n} = 2^{J_n \frac{\gamma_{n,i_n}}{\alpha_{n,i_n}}(1-\varepsilon_{n,i_n})}.$$

From the lacunarity of (J_n), one deduces that

$$\log(\prod_{m=1}^{n} \#\mathcal{T}_{m,i_m}) \sim_{n \to +\infty} \log \ \#\mathcal{T}_{n,i_n}.$$

Finally, the intervals of I, which are all of length $2^{-J_n/\alpha_{n,i_n}}$ belonging to $\mathcal{F}_n(\alpha)$ are embedded in those of $\mathcal{F}_{n-1}(\alpha)$, and (remembering definition (36)), these intervals are separated by a distance at least equal to

$$2^{\left[J_n(1-\frac{\gamma_{n,i_n}}{\alpha_{n,i_n}})\right]} 2^{-J_n} \sim 2^{-J_n \frac{\gamma_{n,i_n}}{\alpha_{n,i_n}}}.$$

By a classical argument (see [7], examples 4.6 and 4.7 for instance) allowing to compute the Hausdorff dimension of this type of Cantor sets, we have

$$\begin{aligned} \dim_{\mathcal{H}} \mathcal{F}(\alpha) &\geq \liminf_{n \geq +\infty} \frac{-\log\left(\prod_{m=1}^{n} \#\mathcal{T}_{m,i_m}\right)}{\log 2^{-J_n/\alpha_{n,i_n}}} \\ &= \liminf_{n \geq +\infty} \frac{-\log \ \#\mathcal{T}_{n,i_n}}{\log 2^{-J_n/\alpha_{n,i_n}}} \\ &= \lim_{n \to +\infty} \gamma_{n,i_n}(1-\varepsilon_{n,i_n}) = d(1+\eta\alpha). \end{aligned}$$

Suppose $\alpha_0 < \alpha \leq \beta_0$ and $0 < r < 2^{-J_5-1}$ and choose N satisfying (41). Then $x \in \mathcal{E}_N(\alpha) = \bigcup_{k\in\mathcal{T}_{N,i_N}} I_{J_N,k}(\alpha_{N,i_N})$ and we have $\mathcal{E}_N(\alpha) \cap E''_{N,i_N,r} = \emptyset$. This implies $x \notin E_{N,i_N,r}(Z)$, that is,

$$\omega_{B(x,r)}(Z) < (2r)^{\alpha_{N,i_N}}.$$

If $\alpha = \alpha_0$ by (49) we have $\omega_{B(x,r)}(Z) < (2r)^{\alpha_0}$.

On the other hand, $\mathcal{E}_N(\alpha) \subset E'_{N,i_N+1}$ and there exists $r' \leq 2^{-J_N/\alpha_{N,i_N+1}} \leq 2^{-J_N}$ such that

$$\omega_{B(x,r')}(Z) \geq (2r')^{\alpha_{N,i_N+1}}.$$

Since $\alpha_{N,i_N+1} - \alpha_{N,i_N} \to 0$ as $N \to \infty$ and $\alpha_{N,i_N} \leq \alpha < \alpha_{N,i_N+1}$ we have $h_Z(x) = \alpha$.

5. Multifractal spectrum prescription of a non-HM measure

Let $f \in \mathcal{F}$. We build a probability measure μ whose multifractal spectrum $d_\mu(h)$ is exactly $f(h)$ for the exponents $h < 1$. To get part (iii) of Theorem 1.11 (the spectrum for $h = 1$), it is enough to consider the measure $(\mu + \lambda)/2$, where λ is the Lebesgue measure on $[0, 1]$.

We call $(\tilde{f}_n)$ the sequence of functions associated with $f \in \mathcal{F}$, which are constant over a closed interval $I_n \subset [0, 1]$ and satisfy $\widetilde{f}_n(x) \leq f(x)$. We set

$$\alpha_0 = \inf_{n\geq 1} \min I_n > 0. \tag{51}$$

Recall that $\mathcal{F}^*$, introduced in Definition 4.2, was the set of functions suitable for Theorem 4.1. For each $\widetilde{f}_n$, there exists a countable sequence of affine functions $(f_{n,p})_{p\geq 1}$, $f_{n,p} \in \mathcal{F}^*$ such that $\text{Support}(f_{n,p}) = I_n$ for all p and $\widetilde{f}_n = \sup_p f_{n,p}$.

Hence, if we consider the countable family of functions $(f_{n,p})_{n\geq 1,\ p\geq 1}$, we still have $f = \sup_{n\geq 1,\ p\geq 1} f_{n,p}$. By adjusting our notation we call this new countable family $(f_p)_{p\geq 1}$, and we have $f = \sup_{p\geq 1} f_p$.

Remark 5.1. This procedure will be used in the next section also.

For every integer $p \geq 1$, by Theorem 4.1, one can find a surjective monotone function $Z_p : [0, 1] \to [0, 1]$ whose multifractal spectrum is exactly f_p. Let us call μ_p the measure defined as the integral of Z_n: $\mu_p([0, x]) = Z_p(x)$. We obtain the measure μ as follows: for every $p \geq 1$, the restriction of μ to the interval $[2^{-p}, 2^{-p+1}]$ coincides with $2^{-p}(\mu_p \circ \ell_p)$, where ℓ_p is the unique affine bijective increasing map from $[0, 1]$ to $[2^{-p}, 2^{-p+1}]$.

It is a trivial matter to see that

$$d_\mu = \sup_{p\geq 1} d_{\mu_p},$$

since affine mappings do not modify the multifractal spectrum and since the supports of the measures $\mu_p \circ \ell_p$ are disjoint. Problems may occur only at the concatenation points between the supports of the measure, i.e. at the rationals 2^{-p}, $p \geq 1$, and at 0. In reality, there is no problem at the rationals 2^{-p}, because of item (i) of Theorem 4.1 (and the fact that 0 and 1 are isolated points and do not belong to the support of the measures μ_p), which ensures that these points satisfy $h_\mu(2^{-p}) = +\infty$. Hence, only 0 may be a problem, but it is easy to modify the measure μ (for instance by performing a time subordination which is singular at 0) to ensure that $h_\mu(0) = +\infty$. This yields Theorem 1.11.

6. Multifractal spectrum prescription of a HM measure

Let f be a function belonging to the set of functions $\mathcal{F}$. We apply the same procedure as in the previous Section 5 to get a countable family of functions $(f_p)_{p\geq 1}$ all belonging to $\mathcal{F}^*$ and satisfying $f = \sup_{p\geq 1} f_p$.

For each function f_p, using Theorem 4.1, there exists a function Z_p whose spectrum on $[0, +\infty)$ is exactly f_p. The construction of Theorem 4.1 guarantees that the function Z_p has a particular form: the points x such that $h_{Z_p}(x) < +\infty$ are located on a Cantor set C_p, and the largest interval in the complementary set of C_p in $[0, 1]$ has length less than $1/10$. The functions Z_n are extended as continuous functions $\mathbb{R} \to \mathbb{R}$, by setting $Z_n(x) = 0$ if $x \leq 0$ and $Z_n(x) = 1$ if $x \geq 1$.

The idea behind our construction is to "insert" in each open interval complementary to the Cantor set C_p a copy of another function $Z_{p'}$, so that the new function will have a multifractal spectrum equal to the supremum of the spectra of f_p and $f_{p'}$, since the two functions have disjoint supports. We will repeat this a countable number of times, with a strong redundancy, and the function Z obtained as the uniform limit of continuous functions (Y_n) will have the desired homogeneous multifractal spectrum.

We will first construct the HM function Z as the uniform limit of a sequence of functions $(Y_n)_{n\geq 1}$, using a suitable subsequence of functions $(g_n)_{n\geq 1}$ which will be selected from the set of functions $(f_p)_{p\geq 1}$. For the moment, we do not explain how this choice is made, we will do it at the end of this section. By abuse of notation, we still denote by Z_n the function built in Theorem 4.1 whose spectrum equals g_n. Afterwards, we will explain how we choose each function g_n among the functions $(f_p)_{p\geq 1}$ in order to impose a homogeneous multifractal spectrum for Z.

We denote by Z_0 the function obtained from Proposition 1.16.

Set $Y_1 = Z_0 + Z_1$. Then for all $x \in [0, 1]$ we have $h_{Y_1}(x) \leq 1$ and the set of singularities $\widetilde{C}_1 = \{x \in [0, 1] : h_{Y_1}(x) < 1\}$ is located on a Cantor set.

For every $n \geq 2$, we assume that Y_n has been built, and that the set $\widetilde{C}_n$ of singularities of Y_n, i.e. $\widetilde{C}_n = \{x \in [0, 1] : h_{Y_n}(x) < 1\}$ has the structure of a Cantor set : there exists a sequence of sets $(C_{n,p})_{p\geq 1}$ satisfying the following:

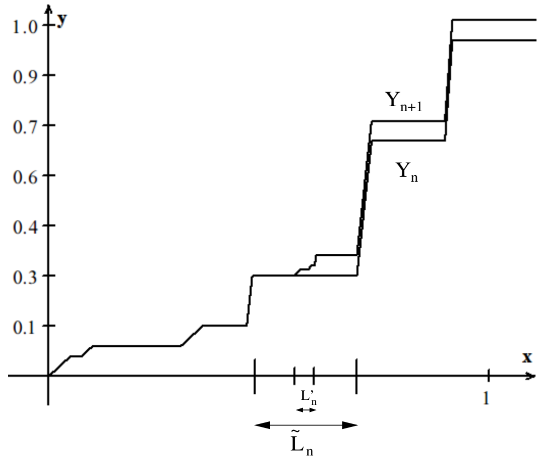

Figure 5. Definition of Y_n and Y_{n+1}

- the $C_{n,p}$ are nested, i.e. for every $p \geq 1$, $C_{n,p+1} \subset C_{n,p}$,
- $C_{n,p}$ is a finite union of pairwise disjoint closed intervals,
- the maximal length of the intervals in $C_{n,p}$ is strictly decreasing with p and tends to zero when p tends to infinity,
- $\widetilde{C}_n = \bigcap_{p \geq 1} C_{n,p}$.

We also assume that $h_{Y_n}(x) \geq \alpha_0$ for all $x \in [0,1]$.

Then we construct Y_{n+1} as follows: let $\widetilde{L}_n$ be one of the longest open intervals contiguous to $\widetilde{C}_n$ in $[0,1]$, and let L'_n be concentric with $\widetilde{L}_n$ but of length 2^{-n^2} times that of $\widetilde{L}_n$, i.e. $|L'_n| = 2^{-n^2}|\tilde{L}_n|$.

We set (see Figure 5)

$$\forall\, x \in [0,1], \quad Y_{n+1}(x) = Y_n(x) + 2^{-n^2/|\widetilde{L}_n|} \cdot Z_{n+1}(S_n(x)), \tag{52}$$

where S_n on L'_n is the unique increasing affine function mapping L'_n to $[0,1]$, otherwise S_n is continuous and constant on the components of $[0,1] \setminus L'_n$.

By (27) of Theorem 4.1 we have $h_{Y_{n+1}}(x) \geq \alpha_0$ for all $x \in [0,1]$.

Then Y_{n+1} obviously satisfies the same properties as Y_n: its set of singularities $\widetilde{C}_{n+1} = \{x \in [0,1] : h_{Y_{n+1}}(x) < 1\}$ has the structure of a Cantor set, since it is the union of two Cantor sets ($\widetilde{C}_n$ and the image of C_{n+1} by S_n^{-1}) which "do not cross", i.e. between any two points of the image of C_{n+1} by S_n^{-1}, there is no point of $\widetilde{C}_n$. Moreover, in case $\widetilde{C}_n$ had only one contiguous interval of maximal size, $\widetilde{L}_n$ then the size of one of the largest open interval in the complement of $\widetilde{C}_{n+1}$ is less than that of the largest open intervals in the complement of $\widetilde{C}_n$, since the interval $\widetilde{L}_n$ has been cut in many parts (at least 2). Otherwise the (finite) number of contiguous intervals to $\widetilde{C}_{n+1}$ of size $\widetilde{L}_n$ is one less than that for $\widetilde{C}_n$.

We iterate this construction.

Proposition 6.1. *The sequence of functions $(Y_n)_{n\geq 1}$ converges uniformly to a continuous function $Z : [0,1] \to \mathbb{R}$.*

Proof. Using the definition of Y_n in (52), this follows from the fact that $0 < |\widetilde{L}_n| < 1$ is non-increasing, and that the series $\sum_{n\geq 1} 2^{-n^2/|\widetilde{L}_n|}$ converges. □

In addition, the sequence of lengths $(|\widetilde{L}_n|)_{n\geq 1}$ is not only non-increasing, but it also tends to zero. Indeed, at a fixed step n, the set of intervals in the complementary set of $\widetilde{C}_n$ whose size is between $|\widetilde{L}_n|$ and $|\widetilde{L}_n|/2$ is finite. Since any further step divides by more than two the size of (at least one of) the maximal interval(s), after a finite number of steps, the size of the maximal interval(s) in the complement of $\widetilde{C}_{n+n'}$ will be less than $|\widetilde{L}_n|/2$. Hence, $(|\widetilde{L}_n|)_{n\geq 1}$ converges to zero when n tends to infinity.

Since we are adding monotone functions the oscillation of Y_n on a given interval I can only increase when n increases. One consequence is that for each $x \in \widetilde{C}_N$, the Hölder exponent of Z at x is not larger than the Hölder exponent of Y_N at x.

Moreover, using (27), all the Hölder exponents of Z are larger than α_0.

Hence the multifractal spectrum of Z has a support included in $[\alpha_0, +\infty]$.

We now prove the key proposition to obtain Theorem 1.15: it asserts that the set of those points where the Hölder exponent can be altered during the iterative construction of the functions Y_n has Hausdorff dimension 0.

Proposition 6.2. *For every $N \geq 2$, the Hausdorff dimension of the set*

$$\widetilde{\mathcal{F}}_N = \{x \in [0,1] : h_Z(x) < h_{Y_N}(x)\}$$

is zero.

Proof. Let us choose an $x \in [0,1]$ with $h_Z(x) < h_{Y_N}(x) \leq 1$. This means that $h_{Y_N}(x)$ has a value, greater than α_0 and not exceeding 1. Let $n > N$. The maximal value of the oscillation of the contribution of Y_n is $2^{-n^2/|\widetilde{L}_n|}$. Moreover, all the further contributions of the $Y_{n'}$, for $n' \geq n$, are all of magnitudes less than $2^{-(n')^2/|\widetilde{L}_{n'}|}$ (which itself is less than $2^{-(n')^2/|\widetilde{L}_n|}$), so the sum of all the maximal oscillations is less than $2 \cdot 2^{-n^2/|\widetilde{L}_n|}$.

One knows that for every r small enough,

$$\omega_{B(x,r)}(Y_N) \leq r^{h_{Y_N}(x)-\varepsilon}. \tag{53}$$

We assumed that $h_Z(x) < h_{Y_N}(x)$, and let $\varepsilon > 0$ be such that $h_Z(x)+3\varepsilon < h_{Y_N}(x)$. Necessarily, for some small values for r, one must have

$$\omega_{B(x,r)}(Z) \geq r^{h_Z(x)+\varepsilon} > r^{h_{Y_N}(x)-2\varepsilon} > 2r^{h_{Y_N}(x)-\varepsilon} \text{ and } r^{h_{Y_N}(x)+\varepsilon}/2 \geq r^{h_{Y_N}(x)+2\varepsilon}.$$

This and (53) imply that for some small values of $r > 0$

$$\omega_{B(x,r)}(Z - Y_N) \geq r^{h_Z(x)+\varepsilon}/2 \geq r^{h_Z(x)+2\varepsilon} > r. \tag{54}$$

In order to modify the oscillation of Y_N on $B(x,r)$, the ball $B(x,r)$ should intersect at least one of the intervals L'_n for an $n \geq N+1$. Let $n \geq N+1$ be the minimal integer such that $B(x,r) \cap L'_n \neq \emptyset$. The maximal possible value of the oscillation of the function $2^{-n^2/|\widetilde{L}_n|} \cdot Z_{n+1}(S_n(x))$ (which is added at step $n+1$ to construct Y_{n+1} from Y_n) equals $2^{-n^2/|\widetilde{L}_n|}$. This oscillation is obtained on the interval L'_n of length $2^{-n^2}|\widetilde{L}_n|$. The difference between $\omega_{B(x,r)}(Y_n)$ and $\omega_{B(x,r)}(Y_{n+1})$ is at most $2^{-n^2/|\widetilde{L}_n|}$. Using the remark above, summing all the further oscillations (for $n' \geq n+1$) does not change much the size of $\omega_{B(x,r)}(Y_n)$.

Assume that $r > |L'_n| = 2^{-n^2}|\widetilde{L}_n|$. Also recall that $|\widetilde{L}_n| \leq 1/10$. The difference between $\omega_{B(x,r)}(Y_n)$ and $\omega_{B(x,r)}(Z)$ is at most

$$2 \cdot 2^{-n^2/|\widetilde{L}_n|} \ll 2^{-n^2+\log_2|\widetilde{L}_n|} = 2^{-n^2}|\widetilde{L}_n| = |L'_n| \leq r.$$

Therefore, we have $|\omega_{B(x,r)}(Z) - \omega_{B(x,r)}(Y_n)| \leq r$, $|\omega_{B(x,r)}(Y_n) - \omega_{B(x,r)}(Y_N)| = 0$. This contradicts (54).

Hence, we need to consider the case $r \leq |L'_n|$ and $B(x,r) \cap L'_n \neq \emptyset$. This implies that x must belong to the interval concentric with L'_n, but of length $3|L'_n|$. Let us call these intervals L''_n.

In order to change at x the oscillation of Z compared to that of Y_N, the point x must belong to an infinite number of intervals L''_n. Let us now find an upper-bound for the dimension of the set $\mathcal{S}$ of such points. For any $s > 0$, if $\eta_n = |L''_n|$, the union $\bigcup_{n' \geq n} L''_{n'}$ forms an η_n-covering of $\mathcal{S}$, thus we have

$$\mathcal{H}^s_{\eta_n}(\mathcal{S}) \leq \sum_{n' \geq n} |L''_{n'}|^s \leq 3^s \sum_{n' \geq n} |2^{-n'^2/|\widetilde{L}_{n'}|}|^s \leq 3^s \sum_{n' \geq n} |2^{-n'^2/|\widetilde{L}_N|}|^s.$$

This last sum obviously converges for any value of $s > 0$. Hence $\mathcal{H}^s(\mathcal{S}) = 0$ and $\dim(\mathcal{S}) \leq 0$. Since $\widetilde{\mathcal{F}}_N \subset \mathcal{S}$ we proved the proposition. □

By taking a countable union of sets of dimension zero, we obtain:

Corollary 6.3. *Let $\mathcal{F} = \cup_{N=2}^{\infty} \widetilde{\mathcal{F}}_N$. Then $\mathcal{F}$ is the subset of $[0,1]$ for which the exponent of Z at x is modified due to our "scheme of iteration" and $\dim \mathcal{F} = 0$.*

We finish by explaining the choice of the functions $(f_n)_{n \geq 1}$ in the construction of the function Z. This sequence is obtained recursively:

- **Step 1:** We use $g_n = f_1$ until each dyadic interval $I_{1,k}$, for $k = 0, 1$ contains a copy of Z_1.
- **Step 2:** We use $g_n = f_1$ until each dyadic interval $I_{2,k}$, for $k = 0, \cdots, 2^2 - 1$ contains a copy of Z_1. Then we use $f_n = g_2$ until each dyadic interval $I_{2,k}$, for $k = 0, \cdots, 2^2 - 1$ contains a copy of Z_2.

- ...
- **Step** p**:** We use $g_n = f_1$ until each dyadic interval $I_{p,k}$, for $k = 0, \cdots, 2^p - 1$ contains a copy of Z_1. Then we use $g_n = f_2$ until each dyadic interval $I_{p,k}$, for $k = 0, \cdots, 2^p - 1$ contains a copy of Z_2 .... Finally we use $g_n = f_p$ until each dyadic interval $I_{p,k}$, for $k = 0, \cdots, 2^p - 1$ contains a copy of Z_p.
- ...

At the end of the construction, we obviously have the following property: Any non-trivial interval $I \subset [0,1]$ contains a copy of any function f_p. If $h \in \text{Support}(f)$ and $f(h) > 0$ then from $\dim \mathcal{F} = 0$ it follows that $d_Z(h) = f(h)$. If $f(h) = 0$ and $h \geq \alpha_0$ then Theorem 3.3 can be used to verify that $d_Z(h) = f(h) = 0$. Indeed, $h_Z(x) \leq h_{Y_N}(x)$ for any $x \in [0,1]$ and $N \in \mathbb{N}$. By the choice of α_0 in (51) and by construction $\{x : h_Z(x) \leq \alpha_0 + \varepsilon\}$ is dense in $[0,1]$ for any $\varepsilon > 0$. Moreover, $h_Z(x) \geq \alpha_0$ for any $x \in [0,1]$. The existence and the value of the rest of the spectrum (i.e. the exponents h for which $d_Z(h) = 0$) are obtained by combining the results of the previous section.

7. Multifractal spectrum prescription of a HM (non-monotone) function

Theorem 1.17 is simply the consequence of the following Theorem proved in [2]. Let $(\psi_{j,k})_{j\in\mathbb{Z},k\in\mathbb{Z}}$ be any orthogonal wavelet basis of $L^2(\mathbb{R})$ (see for instance [15] for the existence and the construction of wavelet bases).

Theorem 7.1. *Let μ be a measure on $[0,1]$, $0 < \alpha < \beta$. Consider the wavelet series*

$$F_\mu(x) = \sum_{j\geq 1} \sum_{k=0}^{2^j-1} d_{j,k} \ \psi_{j,k}(x),$$

where the wavelet coefficients of F_μ are defined by $d_{j,k} = 2^{-j\alpha}\mu\left(I_{j,k}\right)^{\beta-\alpha}$.

Then for every $x \in [0,1]$, $h_{F_\mu}(x) = \alpha + (\beta - \alpha) \cdot h_\mu(x)$.

This implies that for every exponent h such that $\frac{h-\alpha}{\beta-\alpha}$ belongs to Support (d_μ), one has $d_{F_\mu}(h) = d_\mu\left(\frac{h-\alpha}{\beta-\alpha}\right)$.

Theorem 7.1 combined with Theorem 1.12 yields Theorem 1.17.

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