Non-L^1 functions with rotation sets of Hausdorff dimension one

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November 6, 2008

Abstract

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a given measurable function, periodic by 1. For an $\alpha \in \mathbb{R}$ put $M_n^\alpha f(x) = \frac{1}{n+1} \sum_{k=0}^{n} f(x + k\alpha)$. Let Γ_f denote the set of those α's in $(0, 1)$ for which $M_n^\alpha f(x)$ converges for almost every $x \in \mathbb{R}$. We call Γ_f the rotation set of f. We proved earlier that from $|\Gamma_f| > 0$ it follows that f is integrable on $[0, 1]$, and hence, by Birkhoff's Ergodic Theorem all $\alpha \in [0, 1]$ belongs to Γ_f. However, $\Gamma_f \setminus \mathbb{Q}$ can be dense (even c-dense) for non-L^1 functions as well. In this paper we answer a more than a decade old question by showing that there are non-L^1 functions for which Γ_f is of Hausdorff dimension one.

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*Research supported by the Hungarian National Foundation for Scientific research K075242.

2000 Mathematics Subject Classification: Primary 37A30; Secondary 28A78, 28D05, 37E10.

Keywords: rotation, ergodic average, continued fraction, Hausdorff dimension

1 Introduction

I worked for a long time with Henstock-Kurzweil integrals and, as a Ph. D. student, got interested in ergodic averages of non-L^1 functions.

Answering one of my questions concerning non-L^1 functions and ergodic averages P. Major in [11] showed that there exists a function $f : X \to \mathbb{R}$, and $S, T : X \to X$ two ergodic transformations on a probability space (X, μ) such that

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} f(S^k x) = 0, \ \mu \text{ a.e.}$$

and

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} f(T^k x) = a \neq 0, \ \mu \text{ a.e.}$$

By Birkhoff's Ergodic Theorem the above f cannot belong to $L^1(X, \mu)$.

My thesis advisor M. Laczkovich raised the question whether in the above result the two transformations S and T can be irrational rotations of the unit circle, $\mathbb{T}$. In Major's construction the two transformations were conjugate and hence the answer to Laczkovich's question required a different approach.

In a slightly more general setting in [3] it was proved that if $S, T : X \to X$ are two μ-ergodic transformations which generate a free $\mathbb{Z}^2$ action on the finite non-atomic Lebesgue measure space $(X, \mathcal{S}, \mu)$ then for any $c_1, c_2 \in \mathbb{R}$ there exists a μ-measurable function $f : X \to \mathbb{R}$ such that

$$M_N^S f(x) = \frac{1}{N+1} \sum_{j=0}^{N} f(S^j x) \to c_1,$$

and

$$M_N^T f(x) = \frac{1}{N+1} \sum_{j=0}^{N} f(T^j x) \to c_2,$$

μ almost every x as $N \to \infty$.

Two different irrational rotations generate a free $\mathbb{Z}^2$ action on $\mathbb{T}$ and hence the above result from [3] answered Laczkovich's question.

There are some interesting recent results with respect to ergodic averages of non-L^1 functions and rotations by Ya. Sinai and C. Ulcigrai. In [15] the authors consider trigonometric sums

$$\frac{1}{N}\sum_{k=0}^{N-1}\frac{1}{1-e^{2\pi i(k\alpha+x)}}, \qquad (x,\alpha)\in(0,1)\times(0,1).$$

The product space $(0,1)\times(0,1)$ is endowed with the uniform probability distribution. It is proved that such trigonometric sums have a non-trivial joint limiting distribution in x and α as N tends to ∞. This result also applies to Birkhoff sums of a function with a singularity of type $1/x$ over a rotation. This limiting distribution is determined by results from [14].

Sinai's and Ulcigrai's paper, [15] gave me a very strong motivation to return to my question from ([1], [2]) concerning ergodic averages of rotations of non-L^1 functions. Theorem 1 of [2] is the following:

Theorem 1. *Let $f:\mathbb{R}\to\mathbb{R}$ be a given measurable function, periodic by* 1. *For an $\alpha\in\mathbb{R}$ put*

$$M_n^\alpha f(x)=\frac{1}{n+1}\sum_{k=0}^{n} f(x+k\alpha).$$

Let Γ_f denote the set of those α's in $(0,1)$ for which $M_n^\alpha f(x)$ converges for almost every $x\in\mathbb{R}$. Then from $|\Gamma_f|>0$ it follows that f is integrable on $[0,1]$.

In this paper $|A|$ denotes the Lebesgue measure of the measurable set $A\subset\mathbb{R}$. By Theorem 1 and by the Birkhoff Ergodic Theorem from $|\Gamma_f|>0$ it follows that $f\in L^1$ and for all $\alpha\in[0,1]\setminus\mathbb{Q}$ the limit of $M_n^\alpha f(x)$ equals $\int_0^1 f$. On the other hand, in [2] the following result is also verified:

Theorem 2. *For any sequence of independent irrationals $\{\alpha_j\}_{j=1}^\infty$ there exists a function $f:\mathbb{R}\to\mathbb{R}$, periodic by* 1 *such that $f\notin L^1[0,1]$ and $M_n^{\alpha_j}f(x)\to 0$ for almost every $x\in[0,1]$.*

This result implies that $\Gamma_f\setminus\mathbb{Q}$ can be dense for non-integrable functions. In [16] R. Svetic improved this result by showing that there exists a non-integrable $f:\mathbb{T}\to\mathbb{R}$ such that Γ_f is c-dense in $\mathbb{T}$. (A set $S\subset\mathbb{T}$ is *c-dense* if the cardinality of $S\cap I$ equals continuum for every nonempty open interval $I\subset\mathbb{T}$.)

It was not known whether Γ_f can be of Hausdorff dimension one for non-L^1 functions. This was my question which is answered in this paper. The new main result is the following:

Theorem 3. *There exist a measurable $f : \mathbb{R} \to \mathbb{R}$ periodic by one and a set $A \subset [0,1) \setminus \mathbb{Q}$ such that the Hausdorff dimension of A is one, for all $\alpha \in A$*

$$\lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} f(x+k\alpha) = 0 \tag{1}$$

for almost every $x \in [0,1)$ and

$$\int_{[0,1)} |f| = +\infty. \tag{2}$$

It is clear that if (1) holds for an x then $x \in \Gamma_f$ and hence $A \subset \Gamma_f$.

To help the reader going through the details of the proof of Theorem 3 here we give an outline.

First we define the sequences d_j and l_j converging to 0 and $K_j = 10j$ converging to ∞. Then we define a subset A of the irrationals in $(0,1)$. Suppose $\alpha \in A$ and its continued fraction development is $[a_{\alpha,1}, a_{\alpha,2}, ...] = \dfrac{1}{a_{\alpha,1} + \frac{1}{a_{\alpha,2}+\frac{1}{\ddots}}}$, and $p_{\alpha,n}/q_{\alpha,n}$ is its n'th convergent. We will define a sequence $n(j,\alpha) < n(j+1,\alpha)$. The $a_{\alpha,n(j,\alpha)}$ continued fraction partial denominators of α will be chosen in a very specific way so that $1/q_{\alpha,n(j,\alpha)}$ will be very close to l_j. If n is within a block determined by $n(j-1,\alpha)$ and $n(j,\alpha)$, that is $n(j-1,\alpha) < n < n(j,\alpha)$ then we only assume that $a_{\alpha,n}$ is bounded by K_j. In Proposition 5 we claim that the Hausdorff dimension of A equals one.

The function f is defined as the sum of the functions f_j. The functions f_j vanish outside a set B_j of length h_j. The set B_j is subdivided into an even number of intervals of length l_j and f_j equals $\pm 1/h_j$ alternately on these subintervals. This will provide us sufficient cancellation for the ergodic sums with respect to $\alpha \in A$ rotations. On the other hand, we have $\int |f_j| = 1$. Proposition 6 implies that the measure of those x's for which $\sup_{K>0} \left| \frac{1}{K} \sum_{k=1}^{K} f_j(x+k\alpha) \right| \geq 1/j^2$ is not greater than $1/j^2$. This weak maximal type inequality will imply the main result.

In Section 5 we compute the Hausdorff dimension of A. There are many papers related to computing Hausdorff dimension of sets obtained by restrictions on the continued fraction partial denominators $a_{\alpha,n}$ of the numbers $\alpha = [a_{\alpha,1}, a_{\alpha,2}, ...]$ belonging to these sets. See for example [4], [8], [9], [12] and further references in these papers. In the estimate of the Hausdorff dimension of our set A our bounds K_j on the $a_{\alpha,n}$ vary, and sometimes, for the $a_{\alpha,n(j,\alpha)}$'s there are very serious restrictions on the partial denominators. This is why we had to use a direct computation of the dimension, based on the estimate of the lower local dimension of a mass distribution on A.

Before turning to the details of the paper we mention a question which follows quite naturally from our original question and Sinai's and Ulcigrai's paper [15].

Question 1. Suppose $0 < t < 1$ and $f(x) = \frac{1}{x|\log|x||^t}$, when $|x| \leq 1/2$, $f(0) = 0$, and f is periodic by one. What can be said about the Hausdorff dimension of the rotation set Γ_f?

In case it is still zero for all $t \in (0,1)$ one could continue by asking the same question for functions defined as above, but for which we have $f(x) = \frac{1}{x \log|x|| \log|\log|x|||^t}$, when $|x| \leq 1/2$.

To prove Theorem 3 we found it much easier to work with functions which have a different type singularity and cancellation pattern than the ones mentioned in the above question.

2 Preliminaries

The open ball centered at α and of radius r is denoted by $B(\alpha, r)$.

We recall some notation and properties concerning continued fractions. Suppose we have an irrational number $\alpha \in [0,1)$, then its continued fraction development is given by

$$\alpha = [a_{\alpha,1}, a_{\alpha,2}, ...] = \frac{1}{a_{\alpha,1} + \frac{1}{a_{\alpha,2} + \frac{1}{\dots}}}, \text{ with } a_{\alpha,n} \in \mathbb{N}.$$

The convergents of α are given by the rational numbers $p_{\alpha,n}/q_{\alpha,n}$ which have the terminating continued fraction development $p_{\alpha,n}/q_{\alpha,n} = [a_{\alpha,1}, a_{\alpha,2}, ..., a_{\alpha,n}]$. The numbers $p_{\alpha,n}$ and $q_{\alpha,n}$ can be defined by the following recursion:

$$p_{\alpha,-1} = q_{\alpha,0} = 1, \; q_{\alpha,-1} = p_{\alpha,0} = 0,$$

$$p_{\alpha,n} = a_{\alpha,n} p_{\alpha,n-1} + p_{\alpha,n-2}, \; q_{\alpha,n} = a_{\alpha,n} q_{\alpha,n-1} + q_{\alpha,n-2}, (n \in \mathbb{N}). \tag{3}$$

The Gauss map is given by $G(\alpha) = \{\frac{1}{\alpha}\}$, (see Figure 3 below) and $a_{\alpha,n} = \lfloor (G^{n-1}(\alpha))^{-1} \rfloor$. Set $\alpha_n = [a_{\alpha,n+1}, a_{\alpha,n+2}, ...] = G^n(\alpha)$.

We have (see for example [10] or [15])

$$\lambda_\alpha^{(n-1)} \stackrel{\text{def}}{=} |q_{\alpha,n-1}\alpha - p_{\alpha,n-1}| = \frac{1}{q_{\alpha,n} + q_{\alpha,n-1}\alpha_n}. \tag{4}$$

To be more precise,

$$\lambda_\alpha^{(n-1)} = (-1)^{n-1}(q_{\alpha,n-1}\alpha - p_{\alpha,n-1}) = \frac{1}{q_{\alpha,n} + q_{\alpha,n-1}G^n(\alpha)}. \tag{5}$$

We also have

$$\frac{\lambda_\alpha^{(n)}}{\lambda_\alpha^{(n-1)}} = [a_{\alpha,n+1}, a_{\alpha,n+2}, ...], \text{ and}$$

$$\frac{1}{a_{\alpha,n+1}+1} \leq \frac{\lambda_\alpha^{(n)}}{\lambda_\alpha^{(n-1)}} \leq \frac{1}{a_{\alpha,n+1}}. \tag{6}$$

One can recall some notation and properties of the distribution of $\{n\alpha\}$ by looking at Figure 1. According to (5) the intervals of length $\lambda_\alpha^{(n)}$ show up alternating on the sides of 0 modulo 1, to the right for even and to the left (close to 1 on the figure) for odd n's. The following properties are well-known.

Property 1. The points $k\alpha$, $k = 0, ..., q_{\alpha,n} - 1$ on the unit circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ are "almost equally spaced". Denote by $\mathcal{P}(n)$ the partition obtained by considering the points $k\alpha$, $k = 0, ..., q_{\alpha,n} - 1$. If $\mathcal{I} \subset \mathbb{T}$ is an arbitrary interval of length $\lambda_\alpha^{(n-2)}$ then there can be at most one $\mathcal{P}(n)$ partition subinterval $\mathcal{I}' \subset \mathcal{I}$ whose length is different from $\lambda_\alpha^{(n-1)}$. Moreover, the length of $\mathcal{I}'$ is larger than $\lambda_\alpha^{(n-1)}$ but less than $2 \cdot \lambda_\alpha^{(n-1)}$. (See the third or the seventh row on Figure 1.)

Property 2. If we add the point $q_{\alpha,n}\alpha$ to the partition points $k\alpha$, $k = 0, ..., q_{\alpha,n} - 1$ then one short interval of length $\lambda_\alpha^{(n)}$ shows up adjacent to 0 modulo 1. Denote now by $\mathcal{I}$ an interval belonging to the partition $\mathcal{P}(n)$. If $0 \leq k, k' < q_{\alpha,n+1}$, $k\alpha \in \mathcal{I}$ and $k'\alpha \in \mathcal{I}$ then $k - k'$ is an integer multiple of $q_{\alpha,n}$, that is, $k - k' = tq_{\alpha,n}$ for an integer t. If $k' = k + q_{\alpha,n} < q_{\alpha,n+1}$, $k\alpha, k'\alpha \in \mathcal{I}$, then the distance of $k\alpha$ and $k'\alpha$ equals $\lambda_\alpha^{(n)}$. Moreover, if $k'' = k + 2q_{\alpha,n} < q_{\alpha,n+1}$, $k''\alpha \in \mathcal{I}$ holds as well then $\{k''\alpha\} - \{k'\alpha\}$ and $\{k'\alpha\} - \{k\alpha\}$ are of the same sign.

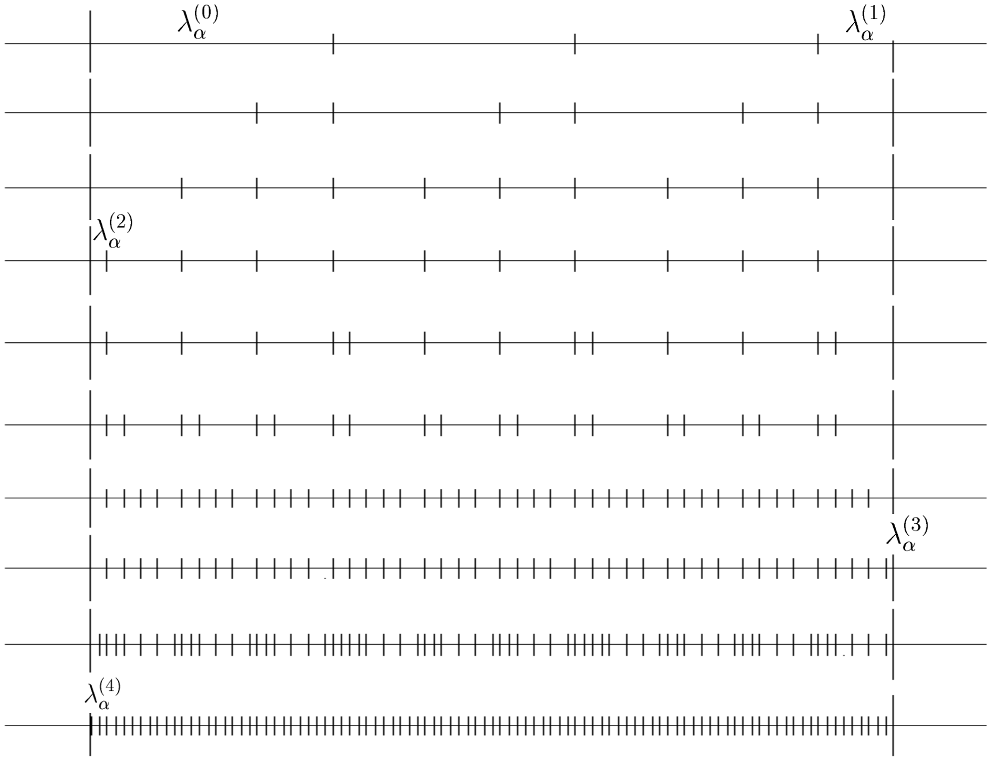

Figure 1: $\{n\alpha\}$ for some n's

Next we recall some results from fractal geometry, see for example Chapter 2 in K. Falconer's book [7]. Suppose that μ is a finite Borel measure, a mass distribution on $\mathbb{R}$, the lower local dimension of μ at $\alpha \in \mathbb{R}$ equals

$$\underline{\dim}_{\text{loc}}\,\mu(\alpha) = \liminf_{r\to 0+} \frac{\log_2 \mu(B(\alpha,r))}{\log_2 r}. \tag{7}$$

It does not matter which base we use for the logarithm in (7) since changing the base multiplies the numerator and the denominator by the same constant. For us it will be more convenient to use base two instead of e. We need Part (a) of Proposition 2.3 from p. 26 of [7].

Proposition 4. *Let $A \subset \mathbb{R}^n$ be a Borel set and let μ be a finite Borel measure. If $\underline{\dim}_{\text{loc}}\,\mu(\alpha) \geq s$ for all $\alpha \in A$ and $\mu(A) > 0$ then $\dim_{\text{H}} A \geq s$.*

3 Main Propositions and proof of Theorem 3

First we need to choose some constants. Set $d_0 = 1$, $l_0 = 1/100$ and $K_j = 10j$ for $j \in \mathbb{N}$. Suppose we have defined d_{j-1} and l_{j-1} and they satisfy the last inequality in (8). Next we select the constants d_j and l_j. In (9) and (10) we state some technical assumptions needed later. At this time one only needs to check that these technical assumptions can be satisfied by a suitable choice of the d_j's. Choose a sufficiently small

$$0 < d_j < l_{j-1}/3 < d_{j-1}/100 \tag{8}$$

such that

$$\frac{1}{3 \cdot (32 \cdot 10^4 \cdot K_j^3 j^6)^2} = \frac{1}{3 \cdot (32 \cdot 10^7 j^9)^2} > \left(1 - \frac{4}{10j}\right)^{\log_2(8K_j^2/d_j^2)} \tag{9}$$

and for $j \geq 2$ we also have

$$\left(1 - \frac{4}{10(j-1)}\right)^{-3\log_2(8K_{j-1}^2/d_{j-1}^2)} < \left(1 - \frac{4}{10j}\right)^{-\log_2(8K_j^2/d_j^2)}. \tag{10}$$

Set

$$l_j = \frac{d_j}{16 \cdot 10^4 \cdot K_j^3 j^6}. \tag{11}$$

Clearly, the last inequality in (8) holds with $j-1$ replaced by j. Set $n(0,\alpha) = 0$ and suppose $j \geq 1$. For any $\alpha \in [0,1) \setminus \mathbb{Q}$ choose $n(j,\alpha)$ so that

$$\frac{1}{q_{\alpha,n(j,\alpha)-2}} > d_j, \text{ but } \frac{1}{q_{\alpha,n(j,\alpha)-1}} \leq d_j. \tag{12}$$

By (3), (4) and $\alpha_{n(j,\alpha)-1}, \alpha_{n(j,\alpha)-2} \in (0,1)$ we have

$$\lambda_\alpha^{(n(j,\alpha)-2)} = \frac{1}{q_{\alpha,n(j,\alpha)-1} + q_{\alpha,n(j,\alpha)-2}\alpha_{n(j,\alpha)-1}} < \frac{1}{q_{\alpha,n(j,\alpha)-1}} \leq d_j \tag{13}$$

and

$$2\lambda_\alpha^{(n(j,\alpha)-3)} = \frac{2}{q_{\alpha,n(j,\alpha)-2} + q_{\alpha,n(j,\alpha)-3}\alpha_{n(j,\alpha)-2}} > \frac{2}{2q_{\alpha,n(j,\alpha)-2}} > d_j. \tag{14}$$

The choice of d_j implies that $n(j-1,\alpha) \leq n(j,\alpha)$.

We denote by A the set of those $\alpha = [a_{\alpha,1}, a_{\alpha,2}, ...] \in [0,1) \setminus \mathbb{Q}$ for which

$$a_{\alpha,n} \leq K_j \text{ holds for } n(j-1,\alpha) < n < n(j,\alpha), \text{ and} \tag{15}$$

$$\frac{1}{q_{\alpha,n(j,\alpha)}} < l_j \leq \frac{1}{q_{\alpha,n(j,\alpha)} - q_{\alpha,n(j,\alpha)-1}}. \tag{16}$$

Property (16) can be rephrased by using (3) as

$$\frac{1}{a_{\alpha,n(j,\alpha)} \cdot q_{\alpha,n(j,\alpha)-1} + q_{\alpha,n(j,\alpha)-2}} < l_j \leq \frac{1}{(a_{\alpha,n(j,\alpha)} - 1)q_{\alpha,n(j,\alpha)-1} + q_{\alpha,n(j,\alpha)-2}}. \tag{17}$$

The Hausdorff dimension estimate of Theorem 3 is based on:

Proposition 5. $\dim_{\mathrm{H}} A = 1$.

We postpone the proof of this proposition to Section 5.

Set

$$h_j = \frac{d_j}{100 \cdot K_j j^2} \text{ and } B_j = [\frac{1}{j} - 2h_j, \frac{1}{j} - h_j) \subset [0,1). \tag{18}$$

Clearly, the intervals B_j are disjoint for $j = 1, 2, ...$ and br (11)

$$\frac{h_j}{l_j} = \frac{16 \cdot 10^4 \cdot K_j^3 j^6}{10^2 K_j j^2} = 16 \cdot 10^2 K_j^2 j^4 \tag{19}$$

is an even integer.

Set $f_j(x) = 0$ if $x \in [0,1) \setminus B_j$.

For $t = 1, 2, ..., (h_j/l_j)$ set

$$f_j(x) = \frac{(-1)^t}{h_j} \text{ if } x \in [\frac{1}{j} - 2h_j + (t-1)l_j, \frac{1}{j} - 2h_j + t l_j). \tag{20}$$

This way we have defined f_j on $[0,1)$ and we can extend its definition to $\mathbb{R}$ by making it periodic by one. Clearly, $|f_j(x)| = 1/h_j$ for $x \in B_j$,

$$\int_{[0,1)} |f_j| = 1 \text{ and } \int_{[0,1)} f_j = 0. \tag{21}$$

Set

$$M^*(f_j, x, \alpha) = \sup_{K>0} \left| \frac{1}{K} \sum_{k=1}^{K} f_j(x + k\alpha) \right|. \tag{22}$$

Denote by $X^*(f_j,\alpha)$ the set of those $x \in [0,1)$ for which

$$M^*(f_j,x,\alpha) \geq \epsilon_j \stackrel{\text{def}}{=} \frac{1}{j^2}. \tag{23}$$

The next proposition establishes a weak maximal type inequality:

Proposition 6. *If $\alpha \in A$ then*

$$|X^*(f_j,\alpha)| \leq \frac{1}{j^2}. \tag{24}$$

We will prove this proposition in Section 4. Now we turn to the proof of Theorem 3.

Proof of Theorem 3. Put $f = \sum_{j=1}^{\infty} f_j$ and let A be the set which appears in Proposition 5. By (21) and the disjointness of the sets B_j we have (2). Suppose $\alpha \in A$ is fixed.

Since the functions f_j are bounded by Birkhoff's Ergodic Theorem we can choose $X_1 \subset [0,1)$ such that $|X_1| = 1$ and for all $j \in \mathbb{N}$ and $x \in X_1$ we have

$$\lim_{K\to\infty} \frac{1}{K}\sum_{k=1}^{K} f_j(x+k\alpha) = \int_{[0,1)} f_j = 0. \tag{25}$$

By (24) and by the Borel–Cantelli lemma almost every $x \in [0,1)$ belongs to only finitely many sets $X^*(f_j,\alpha)$. Choose $X_2 \subset X_1$ such that $|X_2| = 1$ and for all $x \in X_2$ there exists $J_{1,x}$ such that $x \notin X^*(f_j,\alpha)$ for all $j \geq J_{1,x}$, that is,

$$M^*(f_j,x,\alpha) < \epsilon_j = \frac{1}{j^2} \text{ for all } j \geq J_{1,x}. \tag{26}$$

Suppose $x \in X_2$ is fixed and $\epsilon > 0$ is given.

Choose $J \geq J_{1,x}$ such that

$$\sum_{j\geq J}^{\infty} \frac{1}{j^2} < \epsilon. \tag{27}$$

From (26) and (27) it follows that

$$\sum_{j=J}^{\infty} \sup_{K>0} \left| \frac{1}{K}\sum_{k=1}^{K} f_j(x+k\alpha) \right| < \epsilon. \tag{28}$$

Using (25) for $j = 1, ..., J-1$ and (28) we can deduce that

$$\limsup_{K\to\infty} \left| \frac{1}{K} \sum_{k=1}^{K} f(x+k\alpha) \right| \leq$$

$$\limsup_{K\to\infty} \left| \frac{1}{K} \sum_{k=1}^{K} \sum_{j=1}^{J-1} f_j(x+k\alpha) \right| + \limsup_{K\to\infty} \left| \frac{1}{K} \sum_{k=1}^{K} \sum_{j=J}^{\infty} f_j(x+k\alpha) \right| < \epsilon.$$

□

4 Proof of Proposition 6

We start this section with some estimates which will be useful later as well.

Suppose $\alpha \in A$ is fixed. For ease of notation in this section we do not emphasize the dependence of some parameters on α and write for example a_n, p_n, q_n, $\lambda^{(n)}$, instead of $a_{\alpha,n}$, $p_{\alpha,n}$, $q_{\alpha,n}$, $\lambda_\alpha^{(n)}$.

Since $\frac{1}{q_0} = 1 > \frac{1}{100} > d_1$ by (12) we have $n(1,\alpha) \geq 2 > n(0,\alpha)+1$.

Suppose $j \geq 1$ and

$$n(j,\alpha) > n(j-1,\alpha)+1. \tag{29}$$

Then by (15) we have $a_{n(j,\alpha)-1} \leq K_j$. By (3) and (12)

$$\frac{d_j}{2K_j} < \frac{d_j}{K_j+1} < \frac{1}{a_{n(j,\alpha)-1}q_{n(j,\alpha)-2} + q_{n(j,\alpha)-3}} = \tag{30}$$

$$\frac{1}{q_{n(j,\alpha)-1}} \leq d_j.$$

Using (3), (4), (30) and $\alpha_{n(j,\alpha)-1} \in (0,1)$ we obtain

$$d_j > \lambda^{(n(j,\alpha)-2)} = \frac{1}{q_{n(j,\alpha)-1} + q_{n(j,\alpha)-2}\alpha_{n(j,\alpha)-1}} > \frac{1}{2q_{n(j,\alpha)-1}} > \frac{d_j}{4K_j}. \tag{31}$$

Next we verify that l_j approximately equals

$$\lambda^{(n(j,\alpha)-1)} = \frac{1}{q_{n(j,\alpha)} + q_{n(j,\alpha)-1}\alpha_{n(j,\alpha)}} = \tag{32}$$

$$\frac{1}{(a_{n(j,\alpha)}q_{n(j,\alpha)-1} + q_{n(j,\alpha)-2}) + q_{n(j,\alpha)-1}\alpha_{n(j,\alpha)}}$$

where we used (3) and (4). Looking at (17) and (32) and recalling that $\alpha_{n(j,\alpha)} \in (0,1)$ we can see that $l_j \approx \lambda^{(n(j,\alpha)-1)}$. It is obvious that

$$\lambda^{(n(j,\alpha)-1)} < l_j. \tag{33}$$

We need estimates of $a_{n(j,\alpha)}$ which we deduce by using (11), (17) and (30). We have

$$a_{n(j,\alpha)} > (\frac{1}{l_j} - q_{\alpha,n(j,\alpha)-2})\frac{1}{q_{n(j,\alpha)-1}} > \frac{16 \cdot 10^4 \cdot K_j^3 j^6}{d_j q_{n(j,\alpha)-1}} - 1 > \tag{34}$$

$$\frac{16 \cdot 10^4 \cdot K_j^3 j^6}{2K_j} - 1 > 4 \cdot 10^4 \cdot K_j^2 j^6 > 10$$

and

$$a_{n(j,\alpha)} \leq (\frac{1}{l_j} - q_{n(j,\alpha)-2})\frac{1}{q_{n(j,\alpha)-1}} + 1 < \frac{16 \cdot 10^4 \cdot K_j^3 j^6}{d_j q_{n(j,\alpha)-1}} + 1 \leq \tag{35}$$

$$16 \cdot 10^4 \cdot K_j^3 j^6 + 1 < 32 \cdot 10^4 \cdot K_j^3 j^6.$$

Now recalling again (17) we estimate l_j from above by using $\lambda^{(n(j,\alpha)-1)}$:

$$l_j \leq \frac{1}{(a_{n(j,\alpha)} - 1)q_{n(j,\alpha)-1} + q_{n(j,\alpha)-2}} = \tag{36}$$

$$\frac{1}{(a_{n(j,\alpha)} - 2)q_{n(j,\alpha)-1} + q_{n(j,\alpha)-1} + q_{n(j,\alpha)-2}} =$$

$$\frac{a_{n(j,\alpha)}}{a_{n(j,\alpha)} - 2} \cdot \frac{1}{a_{n(j,\alpha)} q_{n(j,\alpha)-1} + \frac{a_{n(j,\alpha)}}{a_{n(j,\alpha)}-2}(q_{n(j,\alpha)-1} + q_{n(j,\alpha)-2})} <$$

(using (32) and $\alpha_{n(j,\alpha)-1} \in (0,1)$)

$$\frac{a_{n(j,\alpha)}}{a_{n(j,\alpha)} - 2} \cdot \frac{1}{a_{n(j,\alpha)} q_{n(j,\alpha)-1} + q_{n(j,\alpha)-2} + q_{n(j,\alpha)-1}} < \frac{a_{n(j,\alpha)}}{a_{n(j,\alpha)} - 2}\lambda^{(n(j,\alpha)-1)}.$$

We can continue the estimation started at (36) by using $j \geq 2$, the monotonicity of the function $\frac{a}{a-2} = 1 + \frac{2}{a-2}$ when $a > 2$, and (34) to obtain

$$l_j < \frac{a_{n(j,\alpha)}}{a_{n(j,\alpha)} - 2}\lambda^{(n(j,\alpha)-1)} < \frac{10}{8}\lambda^{(n(j,\alpha)-1)} < 2\lambda^{(n(j,\alpha)-1)}. \tag{37}$$

Arguing as we obtained the first inequality in (13) we deduce $\frac{1}{q_{n(j,\alpha)}} > \lambda^{(n(j,\alpha)-1)}$ and from (8) and (37) it follows that $\frac{1}{q_{n(j,\alpha)}} > \frac{l_j}{2} > d_{j+1}$. Therefore, by (12) we obtain $n(j+1,\alpha) > n(j,\alpha)+1$. Hence by mathematical induction we see that (29) holds for all $j \in \mathbb{N}$.

When $j \geq 2$ using (19) and (34) we will need a refined version of the rough estimate in (37):

$$l_j < \frac{a_{n(j,\alpha)}}{a_{n(j,\alpha)}-2}\lambda^{(n(j,\alpha)-1)} < \frac{4 \cdot 10^4 \cdot K_j^2 j^6}{4 \cdot 10^4 \cdot K_j^2 j^6 - 2}\lambda^{(n(j,\alpha)-1)} < \tag{38}$$

$$(1+\frac{2}{2 \cdot 10^4 \cdot K_j^2 j^6})\lambda^{(n(j,\alpha)-1)} < (1+\frac{l_j}{10h_j})\lambda^{(n(j,\alpha)-1)} < 1.001 \cdot \lambda^{(n(j,\alpha)-1)}.$$

Now we are ready to prove Proposition 6.

Proof of Proposition 6. If $j=1$ then $|X^*(f_1,\alpha)| \subset |[0,1)| = 1$ is obvious. Hence we can suppose that $j \geq 2$. Put $\overline{B}_j = [\frac{1}{j}-3h_j, \frac{1}{j}]$. Denote by $X^{**}(f_j,\alpha)$ the set of those $x \in [0,1)$ for which $\{x+k\alpha\} \notin \overline{B}_j$ holds for $k=1,...,q_{n(j,\alpha)-1}$. Using (18) and (30) we can estimate the measure of $X^{**}(f_j,\alpha)$ from below:

$$|X^{**}(f_j,\alpha)| \geq 1 - q_{n(j,\alpha)-1}3h_j = 1 - q_{n(j,\alpha)-1}3\frac{d_j}{100K_j j^2} > \tag{39}$$

$$> 1 - \frac{3}{100K_j j^2}2K_j > 1 - \frac{1}{j^2}.$$

Claim 7. *If $x \in X^{**}(f_j,\alpha)$ then*

$$M^*(f_j,x,\alpha) < \epsilon_j = \frac{1}{j^2}. \tag{40}$$

Once we have verified Claim 7 we are done with the proof of Proposition 6 since (40) implies that $X^*(f_j,\alpha) \subset [0,1) \setminus X^{**}(f_j,\alpha)$ and hence by (39) $|X^*(f_j,\alpha)| \leq 1/j^2$, as stated in (24). □

Therefore, next we prove Claim 7.

Proof of Claim 7. Suppose $x \in X^{**}(f_j,\alpha)$ is fixed and denote by k_1 the least $k_1 \in \mathbb{N}$ for which $\{x+k_1\alpha\} \in \overline{B}_j$. Then $k_1 > q_{n(j,\alpha)-1}$ and $\{x+(k_1-q_{n(j,\alpha)-1})\alpha\} \notin \overline{B}_j$. The distance between $\{x+k_1\alpha\}$ and $\{x+(k_1-q_{n(j,\alpha)-1})\alpha\}$ equals $\lambda^{(n(j,\alpha)-1)}$. Hence, the distance of $\{x+k_1\alpha\}$ from the boundary of $\overline{B}_j$

is not greater than $\lambda^{(n(j,\alpha)-1)}$. By Property 2 the points $Q(t)\stackrel{\text{def}}{=}\{x+(k_1+tq_{n(j,\alpha)-1})\alpha\}$ will belong to $\overline{B}_j$ for $t<(3h_j/\lambda^{(n(j,\alpha)-1)})-1$, $t\in\mathbb{N}\cup\{0\}$. For these t's $|Q(t+1)-Q(t)|=\lambda^{(n(j,\alpha)-1)}$, the differences $Q(t+1)-Q(t)$ and $Q(t)-Q(t-1)$ are of the same sign provided $Q(t-1),Q(t),Q(t+1)\in\overline{B}_j$. Moreover, if $0\le t-1<t<(3h_j/\lambda^{(n(j,\alpha)-1)})-1$ and

$$k_1+(t-1)q_{n(j,\alpha)-1}<k<k_1+tq_{n(j,\alpha)-1},$$

then $\{x+k\alpha\}\notin\overline{B}_j$. Denote by k_2 the least $k_2\in\mathbb{N}$ for which $\{x+k_2\alpha\}\in B_j$. (See Figure 2.) By the previous remarks there is $t_2\in\mathbb{N}$ such that

$$k_2-k_1=t_2q_{n(j,\alpha)-1} \tag{41}$$

and $(t_2+1)\lambda^{(n(j,\alpha)-1)}>h_j$, that is, by (19) and (33)

$$t_2+1>\frac{h_j}{\lambda^{(n(j,\alpha)-1)}}>\frac{h_j}{l_j}=16\cdot 10^2K_j^2j^4$$

and

$$t_2>8\cdot 10^2K_j^2j^4. \tag{42}$$

Denote by t_3 the least $t_3\ge t_2$ for which $Q(t_3-1)=\{x+(k_1+(t_3-1)q_{n(j,\alpha)-1})\alpha\}\in B_j$ but $Q(t_3)=\{x+(k_1+t_3q_{n(j,\alpha)-1})\alpha\}\notin B_j$. Set $k_3=k_1+t_3q_{n(j,\alpha)-1}$. Then by Property 2 (see again Figure 2) for $k\le k_3$ we have $\{x+k\alpha\}\in B_j$ if and only if

$$k=k_1+tq_{n(j,\alpha)-1} \text{ for } t_2\le t<t_3. \tag{43}$$

Moreover, for all $1\le k\le q_{n(j,\alpha)}$ we have $\{x+k\alpha\}\in B_j$ if and only if (43) holds. Since $|Q(t+1)-Q(t)|=\lambda^{(n(j,\alpha)-1)}$ we have $(t_3-t_2+1)\lambda^{(n(j,\alpha)-1)}>h_j$ but $(t_3-t_2-1)\lambda^{(n(j,\alpha)-1)}<h_j$, that is, by using (33)

$$t_3-t_2>\frac{h_j}{l_j}\cdot\frac{l_j}{\lambda^{(n(j,\alpha)-1)}}-1>\frac{h_j}{l_j}-1.$$

By (38) we obtain an opposite estimate

$$t_3-t_2<\frac{h_j}{l_j}\cdot\frac{l_j}{\lambda^{(n(j,\alpha)-1)}}+1<\frac{h_j}{l_j}(1+\frac{l_j}{10h_j})+1<\frac{h_j}{l_j}+2.$$

Using the above properties and the definition of f_j we can see that there can be less than four different $t\in[t_2,t_3-1)\cap\mathbb{Z}$ for which $f_j(Q(t+1))+f_j(Q(t))\neq$

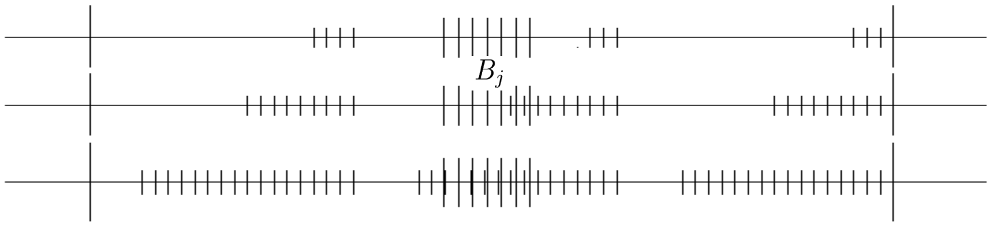

Figure 2: The points $Q(t)$ as they "go through" B_j

0. Indeed, $|Q(t+1)-Q(t)| = \lambda^{(n(j,\alpha)-1)} \approx l_j$ and if $Q(t+1), Q(t) \in B_j$, $Q(t) \in [\frac{1}{j}-2h_j+(t'-1)l_j, \frac{1}{j}-2h_j+t'l_j)$ and $Q(t+1) \in [\frac{1}{j}-2h_j+t'l_j, \frac{1}{j}-2h_j+(t'+1)l_j)$ then $|f_j(Q(t))| = |f_j(Q(t+1))|$ but by (20), $f_j(Q(t))$ and $f_j(Q(t+1))$ are of opposite sign.

By (18), (20) and by our previous observation for any $K \le k_3$ we have

$$|\sum_{k=1}^{K} f_j(x+k\alpha)| \le \frac{4}{h_j} = 4\frac{100K_j j^2}{d_j} \tag{44}$$

and for $K < k_2$

$$|\sum_{k=1}^{K} f_j(x+k\alpha)| = 0. \tag{45}$$

Before continuing our estimates in (44) we need to obtain a lower estimate for $h_j q_{n(j,\alpha)-1}$ by using (12) and (18):

$$h_j q_{n(j,\alpha)-1} = \frac{d_j}{100K_j j^2} q_{n(j,\alpha)-1} \ge \frac{1}{100K_j j^2}. \tag{46}$$

Therefore, by $K_j = 10j$, (41), (42), (44), (45), and (46)

$$\sup_{0<K\le k_3} \left| \frac{1}{K} \sum_{k=1}^{K} f_j(x+k\alpha) \right| \le \frac{1}{k_2} \cdot \frac{4}{h_j} < \frac{1}{t_2 q_{n(j,\alpha)-1}} \cdot \frac{4}{h_j} \le \tag{47}$$

$$\frac{4}{t_2} 100K_j j^2 < \frac{4}{8 \cdot 10^2 K_j^2 j^4} 200K_j j^2 = \frac{1}{K_j j^2} \le \frac{1}{10j^2}.$$

As we remarked earlier after (43) for $k_3 < k < q_{n(j,\alpha)}$, $\{x+k\alpha\} \notin B_j$ and hence (47) can be strengthened to

$$\sup_{0<K<q_{n(j,\alpha)}} \left| \frac{1}{K} \sum_{k=1}^{K} f_j(x+k\alpha) \right| < \frac{1}{10j^2}. \tag{48}$$

If $y \in [0,1)$ is arbitrary then modulo one the points $y+k\alpha$ for $k=1,...,q_{n(j,\alpha)}$ are translated copies of $k\alpha$ for $k=0,...,q_{n(j,\alpha)}-1$ by $y+\alpha$. Hence, by Property 1 they are "almost equally spaced" with distance $\lambda^{(n(j,\alpha)-1)}$ apart. Since by (18) and (31), $3h_j$, the length of $\overline{B}_j$ is less than $d_j/4K_j < \lambda^{(n(j,\alpha)-2)}$ there can be at most one subinterval in $\overline{B}_j$ of the partition $y+k\alpha$, $(k=1,...,q_{n(j,\alpha)})$ which is not of length $\lambda^{(n(j,\alpha)-1)}$ and the length of this exceptional interval is larger than $\lambda^{(n(j,\alpha)-1)}$ but smaller than $2\lambda^{(n(j,\alpha)-1)}$. By arguing the way we obtained (44) for any $y \in [0,1)$ we obtain

$$|\sum_{k=1}^{q_{n(j,\alpha)}} f_j(y+k\alpha)| \leq \frac{5}{h_j}. \tag{49}$$

For arbitrary $K \geq q_{n(j,\alpha)}$ letting $M = \lfloor K/q_{n(j,\alpha)} \rfloor$ and $K_0 = K - Mq_{n(j,\alpha)}$ we have

$$|\sum_{k=1}^{K} f_j(x+k\alpha)| \leq |\sum_{k=1}^{K_0} f_j(x+k\alpha)| + \sum_{l=0}^{M-1} |\sum_{k=1}^{q_{n(j,\alpha)}} f_j(x+(K_0+lq_{n(j,\alpha)}+k)\alpha)| \leq$$

$$(M+1)\frac{5}{h_j} \leq \frac{2K}{q_{n(j,\alpha)}} \cdot \frac{5}{h_j}.$$

Therefore, by (3), (34) and (46) we infer

$$\sup_{q_{n(j,\alpha)} \leq K} \left| \frac{1}{K} \sum_{k=1}^{K} f_j(x+k\alpha) \right| \leq \frac{10}{q_{n(j,\alpha)} h_j} < \tag{50}$$

$$\frac{10}{a_{n(j,\alpha)} q_{n(j,\alpha)-1} h_j} < \frac{10 \cdot 100 K_j j^2}{4 \cdot 10^4 K_j^2 j^6} = \frac{1}{40 K_j j^4}.$$

From (48) and (50) it follows that (40) holds for $x \in X^{**}(f_j, \alpha)$. This completes the proof of Proposition 6. □

5 Proof of Proposition 5

Before turning to the proof of Proposition 5 we need some lemmas. Some of them almost surely exist (sometimes in an implicit form) in the numerous papers about diophantine approximation but for the sake of completeness we provide the details of their simple proof. In the proof of Proposition 6 we had a fixed $\alpha \in A$ and we wanted to obtain estimates of objects in the "phase space" of the dynamical system $x \mapsto \{x+\alpha\}$. Next we need some estimates concerning the parameter space, consisting of the α's, of our dynamical systems. This means that in the sequel in our notation we will emphasize again the dependence of some parameters on the α's. Using some well-known concepts and estimates related to continued fractions first we recall the concept of fundamental intervals see for example [4] and [8]. Suppose $\alpha \in [0,1) \setminus \mathbb{Q}$, $\alpha = [a_{\alpha,1}, a_{\alpha,2}, ...]$. The fundamental interval $I(n,\alpha)$ denotes the closed interval with endpoints $\frac{p_{\alpha,n}}{q_{\alpha,n}} = [a_{\alpha,1}, ..., a_{\alpha,n}]$ and $\frac{p_{\alpha,n}+p_{\alpha,n-1}}{q_{\alpha,n}+q_{\alpha,n-1}} = [a_{\alpha,1}, ..., a_{\alpha,n}+1]$. We also put $I(0,\alpha) = [0,1]$. The intervals $I(1,\alpha)$ are determined by the dicontinuities of the first iterate of the Gauss map (see the left side of Figure 3), and the intervals $I(2,\alpha)$ are determined by the discontinuities of the second iterate of the Gauss map (see the right side of Figure 3 where both G^1 and G^2 are pictured).

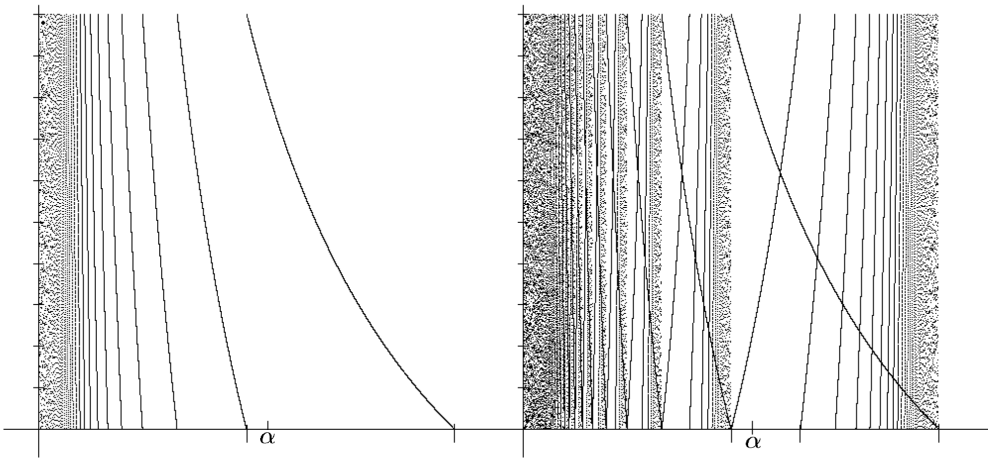

Figure 3: The Gauss map and its second iterate

By formula (7) on p. 279 of [4], or by Section 12 of [10] for $n \geq 1$

$$|I(n,\alpha)| = \frac{1}{q_{\alpha,n}(q_{\alpha,n}+q_{\alpha,n-1})}. \tag{51}$$

The n'th iterate of the Gauss map, $G^n(\alpha)$ maps $I(n,\alpha)$ onto $[0,1)$. Suppose $\alpha_0 = [a_{\alpha_0,1}, a_{\alpha_0,2}, ...]$, then G^n maps in a strict monotone way $I(n,\alpha_0)$ onto $[0,1)$. Denote by F_{n,α_0} the inverse of G^n restricted to $I(n,\alpha_0)$ and extend its definition to $[0,1]$ so that it is still continuous and strictly monotone. Then by (5)

$$(-1)^{n-1}(q_{\alpha_0,n-1}F_{n,\alpha_0}(\alpha) - p_{\alpha_0,n-1}) = \frac{1}{q_{\alpha_0,n} + q_{\alpha_0,n-1}\alpha} \tag{52}$$

and hence an easy computation shows that

$$F'_{n,\alpha_0}(\alpha) = \frac{(-1)^n}{(q_{\alpha_0,n} + q_{\alpha_0,n-1}\alpha)^2}, \tag{53}$$

(see also [12] page 3). This implies that F_{n,α_0}, and hence its inverse $G^n|_{I(n,\alpha_0)}$ both satisfy a bounded distortion property: for all $n \in \mathbb{N}$

$$\frac{F'_{n,\alpha_0}(\alpha)}{F'_{n,\alpha_0}(\beta)} \leq 4,\ \forall\, \alpha, \beta \in [0,1] \text{ and } \frac{(G^n)'(\alpha)}{(G^n)'(\beta)} \leq 4,\ \forall\, \alpha, \beta \in \operatorname{int}(I(n,\alpha_0)). \tag{54}$$

Lemma 8. *Suppose* $\alpha \in [0,1) \setminus \mathbb{Q}$*. Then for* $n \geq 1$

$$\frac{1}{2} > \frac{|I(n+1,\alpha)|}{|I(n,\alpha)|} > \frac{1}{3a_{\alpha,n+1}^2}. \tag{55}$$

Proof. Using $a_{\alpha,n+1} \geq 1$, (3) and (51) we have

$$\frac{|I(n+1,\alpha)|}{|I(n,\alpha)|} = \frac{q_{\alpha,n}(q_{\alpha,n} + q_{\alpha,n-1})}{q_{\alpha,n+1}(q_{\alpha,n+1} + q_{\alpha,n})} =$$

$$\frac{q_{\alpha,n}(q_{\alpha,n} + q_{\alpha,n-1})}{(a_{\alpha,n+1}q_{\alpha,n} + q_{\alpha,n-1})(a_{\alpha,n+1}q_{\alpha,n} + q_{\alpha,n-1} + q_{\alpha,n})} \leq$$

$$\frac{q_{\alpha,n}(q_{\alpha,n} + q_{\alpha,n-1})}{(q_{\alpha,n} + q_{\alpha,n-1})(2q_{\alpha,n} + q_{\alpha,n-1})} < \frac{q_{\alpha,n}^2 + q_{\alpha,n}q_{\alpha,n-1}}{2q_{\alpha,n}^2 + 2q_{\alpha,n}q_{\alpha,n-1}} = \frac{1}{2}.$$

On the other hand, using $a_{\alpha,n+1} \geq 1$, $q_{\alpha,n-1} < q_{\alpha,n}$ and (3) we also have

$$\frac{|I(n+1,\alpha)|}{|I(n,\alpha)|} = \frac{q_{\alpha,n}(q_{\alpha,n} + q_{\alpha,n-1})}{q_{\alpha,n+1}(q_{\alpha,n+1} + q_{\alpha,n})} =$$

$$\frac{q_{\alpha,n}^2 + q_{\alpha,n}q_{\alpha,n-1}}{(a_{\alpha,n+1}^2 + a_{\alpha,n+1})q_{\alpha,n}^2 + (2a_{\alpha,n+1} + 1)q_{\alpha,n}q_{\alpha,n-1} + q_{\alpha,n-1}^2} > \frac{1}{3a_{\alpha,n+1}^2}.$$

□

Remark 1. For $n = 0$ we have $|I(0,\alpha)| = 1$ and $|I(1,\alpha)| = \frac{1}{a_{\alpha,1}} - \frac{1}{a_{\alpha,1}+1} = \frac{1}{a_{\alpha,1}(a_{\alpha,1}+1)}$. Hence for $n \geq 0$

$$\frac{1}{2} \geq \frac{|I(n+1,\alpha)|}{|I(n,\alpha)|} > \frac{1}{3a_{\alpha,n+1}^2} \tag{56}$$

holds.

In the next lemma we compare phase space and parameter space distances.

Lemma 9. *For $j \geq 1$ we have*

$$1 > \frac{|I(n(j,\alpha)-1,\alpha)|}{d_j^2} > \frac{1}{8K_j^2}. \tag{57}$$

Proof. Using (51)

$$|I(n(j,\alpha)-1,\alpha)| = \frac{1}{q_{\alpha,n(j,\alpha)-1}(q_{\alpha,n(j,\alpha)-1} + q_{\alpha,n(j,\alpha)-2})} \tag{58}$$

and $0 < q_{\alpha,n(j,\alpha)-2} < q_{\alpha,n(j,\alpha)-1}$ imply

$$\frac{1}{2q_{\alpha,n(j,\alpha)-1}^2} < |I(n(j,\alpha)-1,\alpha)| < \frac{1}{q_{\alpha,n(j,\alpha)-1}^2}. \tag{59}$$

Recalling (30) we have

$$\frac{1}{q_{\alpha,n(j,\alpha)-1}} \leq d_j < 2K_j \frac{1}{q_{\alpha,n(j,\alpha)-1}}.$$

By (59) we obtain (57). □

Lemma 10. *Suppose $\alpha_0 \in [0,1) \setminus \mathbb{Q}$, $\alpha_0 = [a_{\alpha_0,1}, a_{\alpha_0,2}, ...]$. If $a_{\alpha_0,n} \geq 2$ then for all $\alpha \in I(n,\alpha_0)$ we have*

$$B(\alpha, |I(n,\alpha_0)|) \subset I(n-1,\alpha_0). \tag{60}$$

Proof. Denote by

$$I_k(n,\alpha_0) \text{ the closed interval with endpoints} \tag{61}$$

$$[a_{\alpha_0,1}, ..., a_{\alpha_0,n-1}, k] \text{ and } [a_{\alpha_0,1}, ..., a_{\alpha_0,n-1}, k+1].$$

Then $I(n, \alpha_0) = I_{a_{\alpha_0,n}}(n, \alpha_0)$ and by a suitable modification of (51) and using (3) we obtain

$$|I_k(n, \alpha_0)| = \frac{1}{(kq_{\alpha_0,n-1} + q_{\alpha_0,n-2})((k+1)q_{\alpha_0,n-1} + q_{\alpha_0,n-2})}. \qquad (62)$$

It is obvious that $|I_{k+1}(n, \alpha_0)| < |I_k(n, \alpha_0)|$ for all $k \in \mathbb{N}$ and

$$|I(n, \alpha_0)| = |I_{a_{\alpha_0,n}}(n, \alpha_0)| \leq |I_1(n, \alpha_0)|. \qquad (63)$$

By direct computation we will verify later that

$$|I_{a_{\alpha_0,n}+j}(n, \alpha_0)| \geq \frac{1}{(1 + \frac{j}{a_{\alpha_0,n}})(1 + \frac{j}{a_{\alpha_0,n}+1})}|I(n, \alpha_0)| \geq \qquad (64)$$

$$\frac{1}{(1 + \frac{j}{2})(1 + \frac{j}{3})}|I(n, \alpha_0)|.$$

Before verifying (64) we remark that $a_{\alpha_0,n} \geq 2$ and (64) imply:

$$|I_{a_{\alpha_0,n}+1}(n, \alpha_0)| \geq \frac{1}{(1 + \frac{1}{2})(1 + \frac{1}{3})}|I(n, \alpha_0)| = \frac{1}{2}|I(n, \alpha_0)|$$

$$|I_{a_{\alpha_0,n}+2}(n, \alpha_0)| \geq \frac{1}{(1 + \frac{2}{2})(1 + \frac{2}{3})}|I(n, \alpha_0)| = \frac{3}{10}|I(n, \alpha_0)|$$

$$|I_{a_{\alpha_0,n}+3}(n, \alpha_0)| \geq \frac{1}{(1 + \frac{3}{2})(1 + \frac{3}{3})}|I(n, \alpha_0)| = \frac{1}{5}|I(n, \alpha_0)|,$$

and hence

$$|\bigcup_{j=1}^{3} I_{a_{\alpha_0,n}+j}(n, \alpha_0)| \geq (\frac{1}{2} + \frac{3}{10} + \frac{1}{5})|I(n, \alpha_0)| = |I(n, \alpha_0)|. \qquad (65)$$

Since $I(n, \alpha_0) \subset I(n-1, \alpha_0)$, on one side of $I(n, \alpha_0)$ there is $\cup_{j=1}^{3} I_{a_{\alpha_0,n}+j}(n, \alpha_0) \subset I(n-1, \alpha_0)$ and on the other side there is $I_1(n, \alpha_0)$ from (63) and (65) it follows (60), by using $B(\alpha, |I(n, \alpha_0)|) \subset I_1(n, \alpha_0) \cup ... \cup I_{a_{\alpha_0,n}-1}(n, \alpha_0) \cup I(n, \alpha_0) \cup \cup_{j=1}^{3} I_{a_{\alpha_0,n}+j}(n, \alpha_0) \subset I(n-1, \alpha_0)$.

To verify (64) use (3), (51) and (62):

$$|I_{a_{\alpha_0,n}+j}(n, \alpha_0)| =$$

$$\frac{1}{((a_{\alpha_0,n}+j)q_{\alpha_0,n-1}+q_{\alpha_0,n-2})((a_{\alpha_0,n}+j+1)q_{\alpha_0,n-1}+q_{\alpha_0,n-2})} =$$

$$\frac{1}{\frac{a_{\alpha_0,n}+j}{a_{\alpha_0,n}}(a_{\alpha_0,n}q_{\alpha_0,n-1}+\frac{a_{\alpha_0,n}}{a_{\alpha_0,n}+j}q_{\alpha_0,n-2})\frac{a_{\alpha_0,n}+j+1}{a_{\alpha_0,n}+1}((a_{\alpha_0,n}+1)q_{\alpha_0,n-1}+\frac{a_{\alpha_0,n}+1}{a_{\alpha_0,n}+j+1}q_{\alpha_0,n-2})} \geq$$

$$\frac{1}{(1+\frac{j}{a_{\alpha_0,n}})(1+\frac{j}{a_{\alpha_0,n}+1})(a_{\alpha_0,n}q_{\alpha_0,n-1}+q_{\alpha_0,n-2})((a_{\alpha_0,n}+1)q_{\alpha_0,n-1}+q_{\alpha_0,n-2})} =$$

$$\frac{1}{(1+\frac{j}{a_{\alpha_0,n}})(1+\frac{j}{a_{\alpha_0,n}+1})}|I(n,\alpha_0)|.$$

□

Lemma 11. *Suppose* $\alpha_0 \in [0,1) \setminus \mathbb{Q}$, $n \geq 2$ *and* $\alpha \in I(n,\alpha_0)$ *then*

$$B(\alpha, |I(n,\alpha_0)|) \subset I(n-2,\alpha_0). \tag{66}$$

Proof. If $a_{\alpha_0,n} \geq 2$ then (60) and $I(n-1,\alpha_0) \subset I(n-2,\alpha_0)$ implies (66). Similarly, if $a_{\alpha_0,n} = 1$, but $a_{\alpha_0,n-1} \geq 2$ by Lemma 10 used with $n-1$ we obtain

$$B(\alpha, |I(n,\alpha_0)|) \subset B(\alpha, |I(n-1,\alpha_0)|) \subset B(\alpha, |I(n-2,\alpha_0)|).$$

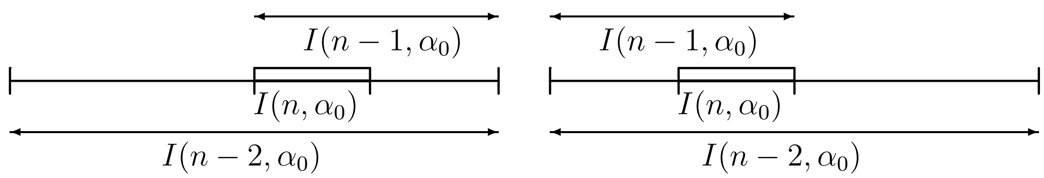

Figure 4: The location of the intervals $I(n,\alpha_0)$, $I(n-1,\alpha_0)$ and $I(n-2,\alpha_0)$

Finally, if $a_{\alpha_0,n} = 1$ and $a_{\alpha_0,n-1} = 1$ then by (55) and (56) we have $|I(n-2,\alpha_0)|/2 \geq |I(n-1,\alpha_0)|$, $|I(n-1,\alpha_0)|/2 > |I(n,\alpha_0)|$ and the intervals are situated as either on the left, or on the right side of Figure 4. This implies that the distance of $I(n,\alpha_0)$ from the boundary of $I(n-2,\alpha_0)$ exceeds $|I(n-1,\alpha_0)|/2$. This concludes the proof of (66). □

Now we are ready to prove Proposition 5.

Proof of Proposition 5. In order to apply Proposition 4 we have to define a mass distribution, a measure μ on A. To define μ as a mass distribution it is sufficient to define it on the fundamental intervals of the form $I(n, \alpha)$, $n \in \mathbb{N}$, $\alpha \in [0, 1) \setminus \mathbb{Q}$. For any α we put

$$\mu(I(0, \alpha)) = \mu([0, 1]) = 1. \tag{67}$$

If for an $\alpha_0 \in [0, 1) \setminus \mathbb{Q}$ and an $n \in \mathbb{N}$ we have $\text{int}(I(n, \alpha_0)) \cap A = \emptyset$ then we set $\mu(I(n, \alpha_0)) = 0$. We remark that all $\alpha \in A$ are irrational and the endpoints of $I(n, \alpha_0)$ are rational, so $\text{int}(I(n, \alpha_0)) \cap A = I(n, \alpha_0) \cap A$. Suppose $\alpha_0 \in A$, $\alpha_0 = [a_{\alpha_0,1}, a_{\alpha_0,2}, ...]$. We need to define $\mu(I(n, \alpha_0))$ for all $n \in \mathbb{N}$. Suppose that $\mu(I(n-1, \alpha_0))$ is defined and

$$\Gamma(n-1, \alpha_0) \overset{\text{def}}{=} \frac{\mu(I(n-1, \alpha_0))}{|I(n-1, \alpha_0)|}. \tag{68}$$

We want to define $\mu(I(n, \alpha_0))$. First suppose that we can find $j \in \mathbb{N}$ such that

$$n(j-1, \alpha_0) < n < n(j, \alpha_0). \tag{69}$$

We assume that for $j > 1$ after the previous step of our induction we have

$$\Gamma(n(j-1, \alpha_0), \alpha_0) \le \left(1 - \frac{4}{10(j-1)}\right)^{-3\log_2(8K_{j-1}^2/d_{j-1}^2)}. \tag{70}$$

When $j = 1$ we have $\Gamma(n(0, \alpha_0), \alpha_0) = \Gamma(0, \alpha_0) = 1$.

If $\alpha_1 \in I(n-1, \alpha_0) \setminus \mathbb{Q}$, $\alpha_1 = [a_{\alpha_1,1}, a_{\alpha_1,2}, ...]$ then

$$a_{\alpha_0,\nu} = a_{\alpha_1,\nu} \text{ for } \nu = 1, ..., n-1. \tag{71}$$

This also implies by (3) that

$$q_{\alpha_0,\nu} = q_{\alpha_1,\nu} \text{ for } \nu = 1, ..., n-1, \tag{72}$$

and hence by (12)

$$n(j', \alpha_0) = n(j', \alpha_1) \text{ for } j' = 1, ..., j-1. \tag{73}$$

From (12), (69) and (72) it also follows that

$$\frac{1}{q_{\alpha_0,n-1}} = \frac{1}{q_{\alpha_1,n-1}} > d_j \text{ and hence } n < n(j, \alpha_1). \tag{74}$$

Suppose, in addition, that $\alpha_1 \in I(n-1,\alpha_0) \cap A$. Then (15), (69), (73) and (74) imply

$$a_{\alpha_0,n} \le K_j,\ a_{\alpha_1,n} \le K_j. \tag{75}$$

Therefore, by using notation from (52) and (61) we have

$$A \cap I(n-1,\alpha_0) \subset \bigcup_{k=1}^{K_j} I_k(n,\alpha_0) \stackrel{\text{def}}{=} I'(n,\alpha_0) = \tag{76}$$

$$F_{n-1,\alpha_0}([\frac{1}{K_j},1]) = F_{n-1,\alpha_0}([\frac{1}{10j},1]).$$

By (54), that is, by the bounded distortion property of F_{n-1,α_0} and by its strict monotonicity

$$\frac{|I(n-1,\alpha_0) \setminus I'(n,\alpha_0)|}{|I(n-1,\alpha_0)|} = \frac{|F_{n-1,\alpha_0}([0,\frac{1}{10j}])|}{|F_{n-1,\alpha_0}([0,1])|} \le \frac{4}{10j}.$$

Therefore,

$$|I'(n,\alpha_0)| \ge (1-\frac{4}{10j})|I(n-1,\alpha_0)|. \tag{77}$$

We put

$$\mu(I(n,\alpha_0)) \stackrel{\text{def}}{=} \frac{|I(n,\alpha_0)|}{|I'(n,\alpha_0)|}\mu(I(n-1,\alpha_0)). \tag{78}$$

Next we see that for all $k = 1,...,K_j$ there exists $\alpha_1 \in \text{int}(I_k(n,\alpha_0)) \cap A$. For $\nu \le n-1$ set $a_{\alpha_1,\nu} = a_{\alpha_0,\nu}$. Put $a_{\alpha_1,n} = k$. This will imply that $\alpha_1 \in \text{int}(I_k(n,\alpha_0))$. If $a_{\alpha_1,\nu-1}$ is defined for $n < \nu$ and $\frac{1}{q_{\alpha_1,\nu}} > d_j$ then put $a_{\alpha_1,\nu} = 1$. In finitely many steps we reach a $\nu = n(j,\alpha_1)-1$ for which $\frac{1}{q_{\alpha_1,\nu}} \le d_j$ but $\frac{1}{q_{\alpha_1,\nu-1}} > d_j$. At the next step we use (17) for the definition of $a_{\alpha_1,n(j,\alpha_1)}$. By (11) and (12) one can see that (17) yields an $a_{\alpha_1,n(j,\alpha_1)} \in \mathbb{N}$. Then this procedure can be repeated. If $a_{\alpha_1,\nu-1}$ is defined for a $\nu > n(j,\alpha_1)$ and $\frac{1}{q_{\alpha_1,\nu}} > d_{j+1}$ then put $a_{\alpha_1,\nu} = 1$. In finitely many steps we reach a $\nu = n(j+1,\alpha_1)-1$ for which $\frac{1}{q_{\alpha_1,\nu}} \le d_{j+1}$ but $\frac{1}{q_{\alpha_1,\nu-1}} > d_{j+1}$. At the next step we use (17) with j replaced by $j+1$ for the definition of $a_{\alpha_1,n(j+1,\alpha_1)}$. Continuing this procedure one can define an $\alpha_1 = [a_{\alpha_1,1}, a_{\alpha_1,2},...] \in \text{int}(I_k(n,\alpha_0)) \cap A$.

This way

$$\mu(I'(n,\alpha_0)) = \mu(\bigcup_{\alpha \in A \cap I(n-1,\alpha_0)} I(n,\alpha)) = \sum_{k=1}^{K_j} \mu(I_k(n,\alpha_0)) = \tag{79}$$

$$\frac{\sum_{k=1}^{K_j} |I_k(n,\alpha_0)|}{|I'(n,\alpha_0)|}\mu(I(n-1,\alpha_0)) = \mu(I(n-1,\alpha_0)).$$

We also have by (77) and (78)

$$\Gamma(n,\alpha_0) \stackrel{\text{def}}{=} \frac{\mu(I(n,\alpha_0))}{|I(n,\alpha_0)|} = \frac{\mu(I(n-1,\alpha_0))}{|I'(n,\alpha_0)|} = \tag{80}$$

$$\frac{\mu(I(n-1,\alpha_0))}{|I(n-1,\alpha_0)|}\cdot\frac{|I(n-1,\alpha_0)|}{|I'(n,\alpha_0)|} \le \Gamma(n-1,\alpha_0)\left(1-\frac{4}{10j}\right)^{-1}.$$

By (55) and (56) we also know that $|I(n,\alpha_0)| \le \frac{1}{2}|I(n-1,\alpha_0)|$.

The cases which are not covered by (69) are the ones when there exists a $j \in \mathbb{N}$ for which $n = n(j,\alpha_0)$, that is, $\mu(I(n(j,\alpha_0)-1,\alpha_0))$ is defined and we want to define $\mu(I(n(j,\alpha_0),\alpha_0))$. In this case we simply put

$$\mu(I(n(j,\alpha_0),\alpha_0)) \stackrel{\text{def}}{=} \mu(I(n(j,\alpha_0)-1,\alpha_0)).$$

Again, as in (71) and (72), for any $\alpha_1 \in I(n(j,\alpha_0)-1,\alpha_0)\setminus\mathbb{Q}$ we have $a_{\alpha_0,\nu} = a_{\alpha_1,\nu}$ and $q_{\alpha_0,\nu} = q_{\alpha_1,\nu}$ when $\nu = 1,...,n(j,\alpha_0)-1$. By (12) this implies that $n(j,\alpha_0) = n(j,\alpha_1)$. If, in addition, $\alpha_1 \in I(n(j,\alpha_0)-1,\alpha_0)\cap A$ then (16) and (17) uniquely determine $a_{\alpha_1,n(j,\alpha_1)} = a_{\alpha_0,n(j,\alpha_0)}$, $q_{n(j,\alpha_1)} = q_{n(j,\alpha_0)}$ and hence $A \cap I(n(j,\alpha_0)-1,\alpha_0) \subset I(n(j,\alpha_0),\alpha_0) = I(n(j,\alpha_1),\alpha_1)$. By (9), (35) and (55)

$$\frac{|I(n(j,\alpha_0),\alpha_0)|}{|I(n(j,\alpha_0)-1,\alpha_0)|} > \frac{1}{3a^2_{\alpha_0,n(j,\alpha_0)}} > \tag{81}$$

$$\frac{1}{3\cdot(32\cdot 10^4 K_j^3 j^6)^2} > \left(1-\frac{4}{10j}\right)^{\log_2(8K_j^2/d_j^2)}.$$

Hence

$$\Gamma(n(j,\alpha_0),\alpha_0) < \left(1-\frac{4}{10j}\right)^{-\log_2(8K_j^2/d_j^2)}\Gamma(n(j,\alpha_0)-1,\alpha_0). \tag{82}$$

By our choice of d_j in (10) we can estimate from above the right-hand side of (70) to obtain

$$\Gamma(n(j-1,\alpha_0),\alpha_0) < \left(1-\frac{4}{10j}\right)^{-\log_2(8K_j^2/d_j^2)}. \tag{83}$$

Since

$$\Gamma(n(0,\alpha_0),\alpha_0) = \frac{\mu(I(0,\alpha_0))}{|I(0,\alpha_0)|} = 1 < \left(1 - \frac{4}{10}\right)^{-\log_2(8K_1^2)/d_1^2} \tag{84}$$

we have (83) even for $j = 1$. Next we need an estimate of $n(j,\alpha_0) - 1$. By (57)

$$\frac{d_j^2}{8K_j^2} < |I(n(j,\alpha_0) - 1, \alpha_0)|. \tag{85}$$

On the other hand, $|I(0,\alpha_0)| = 1$ and by (55) and (56) we have $|I(n+1,\alpha_0)| \leq \frac{1}{2}|I(n,\alpha_0)|$ with strict inequality when $n > 1$. Thus

$$|I(n(j,\alpha_0) - 1, \alpha_0)| < \frac{1}{2^{n(j,\alpha_0)-1}}. \tag{86}$$

From (85) and (86) we infer $2^{n(j,\alpha_0)-1} < \frac{8K_j^2}{d_j^2}$ which implies

$$n(j,\alpha_0) - 1 < \log_2(8K_j^2/d_j^2). \tag{87}$$

By (80) we have

$$\frac{\Gamma(n,\alpha_0)}{\Gamma(n-1,\alpha_0)} \leq \left(1 - \frac{4}{10j}\right)^{-1} \text{ for } n(j-1,\alpha_0) < n \leq n(j,\alpha_0) - 1. \tag{88}$$

Using (82), (83), (87) and (88) we obtain

$$\Gamma(n(j,\alpha_0),\alpha_0) \leq \Gamma(n(j-1,\alpha_0),\alpha_0) \frac{\Gamma(n(j-1,\alpha_0)+1,\alpha_0)}{\Gamma(n(j-1,\alpha_0),\alpha_0)} \cdots \tag{89}$$

$$\frac{\Gamma(n(j,\alpha_0)-1,\alpha_0)}{\Gamma(n(j,\alpha_0)-2,\alpha_0)} \cdot \frac{\Gamma(n(j,\alpha_0),\alpha_0)}{\Gamma(n(j,\alpha_0)-1,\alpha_0)} < \left(1 - \frac{4}{10j}\right)^{-3\log_2(8K_j^2/d_j^2)}$$

which establishes (70) with j instead of $j-1$.

Next we need to estimate from below the right-hand side of (7). Suppose $\alpha_0 \in A$ and $0 < r < |I(n(2,\alpha_0),\alpha_0)|$. Choose n such that

$$|I(n+1,\alpha_0)| \leq r < |I(n,\alpha_0)| \tag{90}$$

and $j > 1$ such that $n(j-1,\alpha_0) < n \leq n(j,\alpha_0)$. By (15), (35) and (55) we have

$$\frac{|I(n+1,\alpha_0)|}{|I(n,\alpha_0)|} > \frac{1}{3a_{\alpha_0,n+1}^2} \geq \tag{91}$$

$$\min\left\{\frac{1}{3K_j^2}, \frac{1}{3K_{j+1}^2}, \frac{1}{3\cdot(32\cdot 10^4 K_j^3 j^6)^2}\right\} > \frac{1}{10^{13}K_j^6 j^{12}}.$$

Using $K_{j-1} = 10(j-1) < K_j = 10j$ for $n(j-1,\alpha_0) < n \le n(j,\alpha_0)$ we obtain

$$\frac{|I(n,\alpha_0)|}{|I(n-1,\alpha_0)|} > \frac{1}{10^{13}K_j^6 j^{12}} \text{ and } \frac{|I(n-1,\alpha_0)|}{|I(n-2,\alpha_0)|} > \frac{1}{10^{13}K_j^6 j^{12}}. \tag{92}$$

Arguing as we obtained (89) we can deduce for $n(j-1,\alpha_0) < n < n(j,\alpha_0)$

$$\Gamma(n,\alpha_0) = \tag{93}$$

$$\Gamma(n(j-1,\alpha_0),\alpha_0)\frac{\Gamma(n(j-1,\alpha_0)+1,\alpha_0)}{\Gamma(n(j-1,\alpha_0),\alpha_0)}\cdots\frac{\Gamma(n,\alpha_0)}{\Gamma(n-1,\alpha_0)} <$$

$$\left(1-\frac{4}{10(j-1)}\right)^{-3\log_2(8K_{j-1}^2/d_{j-1}^2)}\cdot\left(1-\frac{4}{10j}\right)^{\log_2|I(n,\alpha_0)|} <$$

$$\left(1-\frac{4}{10(j-1)}\right)^{-3\log_2(8K_{j-1}^2/d_{j-1}^2)+\log_2|I(n,\alpha_0)|}. \tag{94}$$

To continue our estimates observe that by $K_{j-1} = 10(j-1)$ and by (8) we have

$$d_{j-1} < 100^{-1}d_{j-2} < \ldots < 100^{-(j-1)}d_0 = 100^{-(j-1)}$$

and hence

$$8K_{j-1}^2 < d_{j-1}^{-2}. \tag{95}$$

By (11), (16), (51) and (90)

$$r < |I(n,\alpha_0)| < |I(n(j-1,\alpha_0),\alpha_0)| < \frac{1}{q^2_{\alpha_0,n(j-1,\alpha_0)}} < l_{j-1}^2 < d_{j-1}^2 < 100^{-2(j-1)}. \tag{96}$$

From (95) and (96) we infer

$$-3\log_2(8K_{j-1}^2/d_{j-1}^2) > -3\log_2(d_{j-1}^{-4}) > 6\log_2 r. \tag{97}$$

Using (95), (96) and (97) we can estimate from above (94) and obtain

$$\Gamma(n,\alpha_0) < \left(1-\frac{4}{10(j-1)}\right)^{7\log_2 r}. \tag{98}$$

We also need to consider the case when $n = n(j, \alpha_0)$. Then, as above, $8K_j^2 < d_j^{-2}$ and by (57) we have

$$r < |I(n, \alpha_0)| = |I(n(j, \alpha_0), \alpha_0)| < |I(n(j, \alpha_0) - 1, \alpha_0)| < d_j^2.$$

This implies $\log_2 r < \log_2 d_j^2$. Using (89) we infer

$$\Gamma(n, \alpha_0) < \left(1 - \frac{4}{10j}\right)^{-3\log_2(8K_j^2)+3\log_2(d_j^2)} < \tag{99}$$

$$\left(1 - \frac{4}{10j}\right)^{6\log_2 r} < \left(1 - \frac{4}{10(j-1)}\right)^{7\log_2 r}.$$

Hence (98) holds for any $n(j-1, \alpha_0) < n \le n(j, \alpha_0)$.

Elementary calculus computation shows that for large j we have

$$\log_2\left(1 - \frac{4}{10(j-1)}\right) > -\frac{8}{10(j-1)}. \tag{100}$$

We can assume that r is so small that the corresponding n and j satisfying $n(j-1, \alpha_0) < n \le n(j, \alpha_0)$ are so large that (100) is applicable and we need to keep in mind that $\log_2 r < 0$ and hence multiplying (100) by $7\log_2 r$ reverses the inequality we obtain by taking $\log_2$ of both sides of (99). For these r's and n's

$$\log_2 \Gamma(n, \alpha_0) < -\frac{8 \cdot 7\log_2 r}{10(j-1)} < -\frac{6\log_2 r}{j-1}. \tag{101}$$

By Lemma 11, $B(\alpha_0, r) \subset B(\alpha_0, |I(n, \alpha_0)|) \subset I(n-2, \alpha_0)$. From (90), (91) and (92) we also have

$$|I(n-2, \alpha_0)| < (10^{13}K_j^6 j^{12})^3 |I(n+1), \alpha_0| \le r(10^{13}K_j^6 j^{12})^3,$$

and

$$\mu(B(\alpha_0, r)) \le \mu(I(n-2, \alpha_0)) = \Gamma(n-2, \alpha_0)|I(n-2, \alpha_0)| < \Gamma(n, \alpha_0)(10^{13}K_j^6 j^{12})^3 r.$$

Thus by using $K_j = 10j$, (100) and (101)

$$\frac{\log_2 \mu(B(\alpha_0, r))}{\log_2 r} \ge \frac{\log_2(\Gamma(n, \alpha_0)) + 3\log_2(10^{19}j^{18}) + \log_2 r}{\log_2 r} > \tag{102}$$

$$1 - \frac{6}{j-1} - \frac{3\log_2(10^{19}j^{18})}{-\log_2 r}.$$

By (96), $\log_2 r < -2(j-1)\log_2 100$. This implies $-\log_2 r > 2(j-1)\log_2 100$, and hence

$$0 > -\frac{3\log_2(10^{19}j^{18})}{-\log_2 r} > -\frac{3\log_2(10^{19}j^{18})}{2(j-1)\log_2 100} \to 0 \text{ as } j \to \infty.$$

From $r \to 0+$, it follows that $n \to \infty$, which implies $j \to \infty$ and we infer from (102) that

$$\liminf_{r\to 0+} \frac{\log_2 \mu(B(\alpha_0, r))}{\log_2 r} \geq 1.$$

From this by Proposition 4 it follows that $\dim_{\mathrm{H}} A \geq 1$. □

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