Singularity spectrum of generic α-Hölder regular functions after time subordination

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Abstract

Related to a question by Yves Meyer, there was some research interest concerning "time" subordinations of real functions. Given a function $g \in \mathrm{Lip}^\alpha$, one obtains a time subordination of g simply by considering the map $Z = g \circ f$, where $f \in \mathcal{M} := \big\{f : f(0) = 0,\ f(1) = 1$ and f is a continuous nondecreasing function on $[0,1]\big\}$. The space $\mathcal{E}^\alpha = \mathcal{M} \times \mathrm{Lip}^\alpha$ equipped with the L^∞ (supremum) norm is a complete metric space. In this paper, multifractal properties of the map $g \circ f$ for a generic (typical) element $(f, g) \in \mathcal{E}^\alpha$ are investigated, for all $\alpha \in [0,1)$. In particular we determine the generic Hölder singularity spectrum of $g \circ f$.

Key words: Hölder regularity, Time subordination, Hausdorff measure, Hausdorff dimension
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1. Introduction

For $\alpha \in (0,1)$, let us denote by Lip^α the space of (continuous) functions f defined on the unit interval $[0,1]$ satisfying

$$\text{for every } (x,y) \in [0,1]^2, \quad |f(x) - f(y)| \le |x-y|^\alpha. \tag{1}$$

By $\mathcal{M}$ we denote the set of continuous non-decreasing functions f defined on $[0,1]$, satisfying $f(0) = 0$ and $f(1) = 1$. The two spaces Lip^α and $\mathcal{M}$ are separable complete metric spaces when equipped with the supremum, L^∞ norm for functions, that we denote by $\|.\|$. Recall that a property is said to

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be typical in a complete metric space E, when it holds on a residual set, i.e. a set with a complement of first Baire category. A set is of first Baire category if it is the union of countably many nowhere dense sets. Most often one can verify that the residual set is dense G_δ, that is, a countable intersection of dense open sets in E. Quite often, typical properties are called generic, and this will be the case in the following. The generic multifractal behavior of functions in $\mathcal{M}$ has been investigated in [3]. In this paper, we investigate the generic behavior of $g \circ f$, where $g \in \text{Lip}^\alpha$ and $f \in \mathcal{M}$. The composition of functions with monotone functions can be interpreted as time subordination, hence our result describes the generic local regularity properties of α-Hölder regular functions after time subordination.

Before exposing our results, let us recall how the local regularity of a function is measured.

Definition 1.1. *Let $Z \in L^\infty([0,1])$. For $h \geq 0$ and $t_0 \in [0,1]$, Z is said to belong to $C^h_{t_0}$ if there are a polynomial P of degree less than $[\alpha]$ and a constant C such that, for t close to t_0,*

$$|Z(t) - P(t - t_0)| \leq C|t - t_0|^\alpha. \tag{2}$$

The pointwise Hölder exponent of Z at t_0 is $h_Z(t_0) = \sup\{\alpha \geq 0 : \ f \in C^\alpha_{t_0}\}$.

Observe that when $h_Z(t_0) < 1$, the pointwise Hölder exponent of Z at t_0 is also given by the formula

$$h_Z(t_0) = \liminf_{t \to t_0} \frac{\log |Z(t) - Z(t_0)|}{\log |t - t_0|}.$$

As proved in many cases (and in generic cases, will be shown below), the Hölder exponent $h_Z(t_0)$ of a given function Z may depend on t_0. In order to describe the local behavior of f, it is then relevant to estimate the singularity spectrum of Z defined by

$$d_Z(h) = \dim E_Z(h), \quad \text{where } E_Z(h) := \{t : h_Z(t) = h\}.$$

Here dim stands for the Hausdorff dimension, and $\dim \emptyset = -\infty$ by convention. The singularity spectrum of Z describes the geometric repartition of the singularities of Z.

The generic properties of functions in $\mathcal{M}$ and Lip^α are summarized in the following theorem, proved in [3] and [9] (Proposition 5). It concerns both *a priori results*, i.e. upper bounds for the singularity spectrum of all functions in $\mathcal{M}$ and Lip^α, and *generic results* in $\mathcal{M}$ and Lip^α.

Theorem 1.2. *Consider the space of monotone continuous functions $\mathcal{M}$.*

(i) *For every $f \in \mathcal{M}$, for every $h \geq 0$, one has $d_f(h) \leq \min(h, 1)$.*

(ii) *There exists a residual set $\mathcal{R}_1$ in $\mathcal{M}$ such that for every $f \in \mathcal{R}_1$,*

$$\text{for every } h \in [0,1], \quad d_f(h) = h$$

and $E_f(h) = \emptyset$ if $h > 1$.

Let $\alpha \in (0,1)$. The results concerning Lip$^\alpha$ functions are the following.

(i) *For every $g \in$ Lip$^\alpha$, for every $h < \alpha$, $d_g(h) = -\infty$ (i.e. $E_g(h) = \emptyset$), and for every $h \geq \alpha$, $d_g(h) \leq 1$.*

(ii) *There exists a residual set $\mathcal{R}_\alpha$ in Lip$^\alpha$ such that for every $g \in \mathcal{R}_\alpha$, $d_g(\alpha) = 1$ and $E_g(h) = \emptyset$ if $h \neq \alpha$.*

Observe that the first item in Theorem 1.2 serves as a complement of the Lebesgue theorem, which states that monotone functions are Lebesgue-almost everywhere differentiable (i.e. they have a.e. exponents greater than 1). Theorem 1.2 deals with the exponents h between 0 and 1.

As a consequence of Theorem 1.2, a generic function Z in Lip$^\alpha$ is *monofractal*, i.e. d_Z is well defined for only one value of h, while a generic monotone function is *multifractal*, i.e. the spectrum d_Z has its support not reduced to a single point.

We also remark that generic functions in $\mathcal{M}$, or in Lip$^\alpha$ have the worst possible local behavior, in the sense that the maximal possible values of the spectrum are reached for generic functions.

Remark 1.3. *In [9], Jaffard proves that, in C^α equipped with its natural norm $\|f\|_\alpha = \|f\|_\infty + \sup_{x \neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}$, generic functions are monofractal with exponent α. A rapid argument implies that the same holds true in Lip$^\alpha$ equipped with $\|f\|_\infty$. Nevertheless one cannot directly infer generic results in C^α equipped with $\|\cdot\|_\alpha$ from generic results in Lip$^\alpha$ equipped with $\|\cdot\|_\infty$.*

Jaffard proved in [8] that similar phenomena occur for Besov spaces $B^s_{p,q}(\mathbb{R}^d)$: As soon as $s - d/p > 0$, for every $f \in B^s_{p,q}(\mathbb{R}^d)$, one has

$$\text{for every } h \geq s - d/p, \qquad d_f(h) \leq \min(d, s - d/p + h/p),$$

and for generic functions $f \in B^s_{p,q}(\mathbb{R}^d)$ there is equality for all exponents $h \in [s - d/p, s]$. Again, generic functions have the worst possible local behavior. As Theorem 1.2 for the Lebesgue Theorem, this result provides

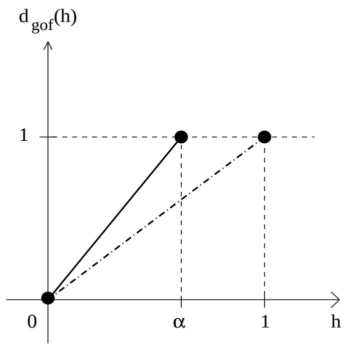

Figure 1: Generic spectrum of compositions of Lip^α functions with monotone functions (continuous line), and for monotone functions (dash-dot line).

us with complementary information on the compact Sobolev embedding $B^s_{p,q}(\mathbb{R}^d) \hookrightarrow C^{s-d/p}(\mathbb{R}^d)$. For instance, the set of points x at which $f \in B^s_{p,q}(\mathbb{R}^d)$ reaches its worst possible local regularity $s - d/p$ has Hausdorff dimension at most 0.

In this paper we investigate the generic multifractal properties of compositions of Lip^α-functions with monotone functions in $\mathcal{M}$. Composing a Lip^α-function g with a continuous monotone function f amounts to performing a time subordination in g, i.e. to consider the function g after a time substitution by f. Processing a time change in a function, or a stochastic process is a widespread issue in analysis and probability theory [1, 2, 6, 7, 10]. From a theoretical standpoint, time subordination and its effect on local behavior has been investigated for instance in [2, 11, 15, 16]. In [7], time subordination is used to perform the local regularity analysis of the famous Pólya function.

Definition 1.4. *Let $\alpha \in (0,1)$. We denote by $\mathcal{E}^\alpha$ the space of couples of functions $\mathcal{E}^\alpha = \mathcal{M} \times Lip^\alpha$. We also set $\mathcal{E}^0 = \mathcal{M} \times C^0$, where C^0 is the space of continuous functions on* $[0,1]$.

The function spaces $\mathcal{E}^\alpha$ and $\mathcal{E}^0$ are complete separable metric spaces when equipped with the L^∞ norm:

$$\text{if } (f,g) \in \mathcal{E}^\alpha \cup \mathcal{E}^0, \quad \|(f,g)\| = \max(\|f\|, \|g\|). \tag{3}$$

The main theorem describes the upper bounds *a priori* for the exponents and for the singularity spectrum of the function $g \circ f$ (where $(f,g) \in \mathcal{E}^\alpha$), as well as the *generic* local behavior of such a composition.

Theorem 1.5. *Let $\alpha \in (0,1)$.*

(i) *Let $(f,g) \in \mathcal{E}^\alpha$. Then for every $h \geq 0$, $d_{g\circ f}(h) \leq \min(h/\alpha, 1)$.*

(ii) *There is a residual set $\mathcal{R}^\alpha$ in $\mathcal{E}^\alpha$ such that for every $(f,g) \in \mathcal{R}^\alpha$, then*

$$d_{g\circ f}(h) = h/\alpha \quad \text{for every } h \in [0,\alpha],$$

and $E_{g\circ f}(h) = \emptyset$ if $h > \alpha$.

(iii) *There is a residual set $\mathcal{R}^0$ in $\mathcal{E}^0$ such that for every $(f,g) \in \mathcal{R}^0$, $d_{g\circ f}(0) = 1$ and $E_{g\circ f}(h) = \emptyset$ if $h > 0$.*

Hence, generically, after a composition by a monotone function, a monofractal function is transformed into a multifractal function. The generic spectrum is the spectrum of a generic monotone function divided by α.

Several questions lead us to the study of local regularity properties of time subordinated functions. Let us detail some of them.

1. A question raised by Yves Meyer to the second author was the following: given a multifractal real function $Z : \mathbb{R} \to \mathbb{R}$, is there a monofractal function $g : \mathbb{R} \to \mathbb{R}$ such that Z can be written $Z = g \circ f$, where f is a monotone function? In other words, given a multifractal function, does there exist an underlying monofractal function g such that Z is only g subordinated in time f? Partial answers to this question are given in [16], and also in the rest of this paper.

2. It is very challenging to understand how the local structure of a function in a given functional space (here Lip^α) is modified after composition. Here we are going to prove that Lip^α functions, which are generically monofractal, are generically multifractal with a given spectrum after a time change. Same questions could be asked for instance for subordinations of functions belonging to Besov spaces.

3. As far as applications are concerned, time subordination has proved to be an efficient method to generate functions or stochastic processes with interesting properties, especially used in applications for creating models for financial data [13, 14], or Physics [11], among other scientific fields.

One of the most encountered subordinated stochastic process, both from a theoretical and a practical standpoint, is the so-called "multifractal random walk", which consists of the composition of a Brownian motion with the integral of random multifractal measures. The sample paths of Brownian motions belonging almost surely to all spaces $C^{1/2-\varepsilon}(\mathbb{R}^+)$, $\varepsilon > 0$, the study of the local regularity properties of multifractal random walks naturally leads, again, to investigate the properties of compositions of C^α and Lip^α functions with monotone functions.

All these considerations gave rise to this article.

The paper is organized as follows. Section 2 contains preliminary results. The proof of Theorem 1.5 is divided in several parts:

• Section 3.1 provides us with a residual set $\mathcal{R}_\alpha^\alpha \subset \mathcal{E}^\alpha$ such that for every couple of functions $(f,g) \in \mathcal{R}_\alpha^\alpha$, the pointwise Hölder exponents $h_{g\circ f}(x)$ of $g \circ f$ are all less or equal than α. It also yields the existence of a set of full Lebesgue measure on which $g\circ f$ has exactly exponent α (Proposition 3.6).

• In Section 3.2, we build a second residual set $\mathcal{R}_\alpha^0 \subset \mathcal{E}^\alpha$ such that for every couple of functions $(f,g) \in \mathcal{R}_\alpha^0$, there is an uncountable set of points with exponent 0 for the composition $g \circ f$.

• Then, in Section 4, we exhibit a third residual set $\mathcal{R}_\alpha \subset \mathcal{E}^\alpha$. For every couple of functions $(f,g) \in \mathcal{R}_\alpha$, we build simultaneously an uncountable number of Cantor sets $X_{f,g}^h$ on which $g\circ f$ has exponent exactly h. This part is very delicate. More precisely, we will be able to construct Cantor sets $X_{f,g}^h$ on which $g\circ f$ has exponent less or equal than h, and by a "geometric measure theory" argument (developed in Corollary 4.8), we extract from each of these Cantor sets a smaller set $Y_{f,g}^h \subset X_{f,g}^h$ such that $\dim Y_{f,g}^h = \dim X_{f,g}^h$ and every $y \in Y_{f,g}^h$ satisfies $h_{g\circ f}(y) = h$.

• Item (ii) of Theorem 1.5 is finally obtained, since all the claimed properties hold on $\mathcal{R}_\alpha^\alpha \bigcap \mathcal{R}_\alpha^0 \bigcap \mathcal{R}_\alpha$, which is obviously a residual set in $\mathcal{E}^\alpha$ as intersection of residual sets.

• Finally, Section 5 explains how to adapt the proof to the case of $\mathcal{E}^0$.

2. Preliminary results

The open ball centered at x and of radius r is denoted by $B(x,r)$. The closure of the set $A \subset \mathbb{R}$ is denoted by $\overline{A}$, moreover $|A|$ and $\lambda(A)$ denote its diameter and Lebesgue measure, respectively. A set A is c-dense in $[0,1]$ if for any $0 \le a < b \le 1$ the cardinality of $A \cap (a,b)$ equals continuum.

Throughout the paper, we will need the oscillating 1-periodic function $H : \mathbb{R} \to \mathbb{R}$ defined by

$$\forall\, x \in \mathbb{R}, \quad H(x) = d(x,\mathbb{Z}) - 1/4, \tag{4}$$

where $d(x,\mathbb{Z})$ stands for the distance between x and the set of all integers $\mathbb{Z}$. The function H has mean 0 on each period, is piecewise affine, and has everywhere left and right derivatives equal to 1 or -1.

2.1. Hausdorff dimensions of sets and dimension of measures

We refer to [4, 5] for the standard definition of Hausdorff measures $\mathcal{H}^s(E)$ and Hausdorff dimensions $\dim_{\mathcal{H}}(E)$ of a set E.

Recall that the lower local dimension of a Borel measure μ at x is defined as (see [5])

$$\underline{\dim}_{\rm loc}\mu(x) = \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \tag{5}$$

and $\dim \mu \stackrel{\rm def}{=} \sup\{s : \underline{\dim}_{\rm loc}\mu(x) \geq s \text{ for } \mu \text{ a.e. } x\}$. By Proposition 10.2 of [5]

$$\dim(\mu) = \inf\{\dim(E) : E \subset [0,1] \text{ Borel and } \mu(E) > 0\}.$$

The following property will be particularly relevant:

$$\begin{array}{c}\text{if } \dim(\mu) \geq h, \text{ then for every Borel set } E \subset [0,1] \\ \text{of dimension strictly less than } h,\ \mu(E) = 0.\end{array} \tag{6}$$

We recall the Mass Distribution Principle, see for example [4], Chapter 4.

Theorem 2.1. *Let μ be a finite mass distribution (measure) on $E \subset \mathbb{R}$. Suppose that for some $s \geq 0$, there exist $c > 0$ and $\delta > 0$ such that $\mu(U) \leq c|U|^s$ for all sets U with $|U| \leq \delta$. Then $\dim E \geq \dim \mu \geq s$.*

2.2. Oscillations of order 1 and local regularity

Oscillations of order 1 arise naturally in our approach, since by a classical result they characterize the pointwise Hölder exponents strictly less than 1 (see Lemma 2.2). For every interval $I \subset [0,1]$, consider the oscillation of order 1 of a continuous function $Z : [0,1] \to \mathbb{R}$ on I defined by

$$\omega_I(Z) = \sup_{t,t'\in I} |Z(t) - Z(t')| = \sup_{t\in I} Z(t) - \inf_{t\in I} Z(t).$$

For every $j \geq 1$, $k \in \{0, ..., 2^j - 1\}$, we will also consider the dyadic intervals $I_{j,k} = [k \cdot 2^{-j}, (k+1)\cdot 2^{-j})$, so that $\bigcup_{k=0,...,2^j-1} I_{j,k} = [0,1)$, the union being disjoint. For every $j \geq 1$ and $k \in \{0, ..., 2^j - 1\}$, for simplicity we write $\omega_{j,k}(Z) = \omega_{I_{j,k}}(Z)$ (which also equals $\omega_{\overline{I_{j,k}}}(Z)$ since Z is continuous).

Recall the characterization of the pointwise Hölder exponents, which is very easy to check.

Lemma 2.2. *Let $Z : [0,1] \to \mathbb{R}$ be a continuous function. Assume that for some $t \in [0,1]$, $h_Z(t) < 1$. Then*

$$h_Z(t) = \liminf_{r\to 0^+} \frac{|\log \omega_{B(t,r)}(Z)|}{|\log r|} = \liminf_{j\to +\infty} \frac{|\log_2 \omega_{B(t,2^{-j})}(Z)|}{j}. \tag{7}$$

Hence, if $\alpha \in (0,1)$, for every $g \in \mathrm{Lip}^\alpha$, for every $t \in [0,1]$, $h_g(t) \geq \alpha$.

For our purpose it is natural to investigate what happens for the pointwise exponents when two functions are composed.

Lemma 2.3. *Let $f : [0,1] \to [0,1]$ and $g : [0,1] \to \mathbb{R}$ be two continuous functions. Assume that for $t \in [0,1]$, $h_f(t) < 1$ and $h_g(f(t)) < 1$. Then*

$$h_{g \circ f}(t) \geq h_g(f(t)) \cdot h_f(t). \tag{8}$$

Proof: This follows from the trivial remark that when $\varepsilon > 0$ and h is small,

$$|g \circ f(t+h) - g \circ f(t)| \leq |f(t+h) - f(t)|^{h_g(f(t))-\varepsilon} \leq |h|^{(h_g(f(t))-\varepsilon)(h_f(t)-\varepsilon)}. \ \square$$

The case where the exponents are greater than one is more delicate, since in formula (2) a polynomial may interfere. Such cases will not arise in our approach hereafter.

2.3. Lip$^{1,\alpha}$ functions, and oscillating perturbations of Lip$^{1,\alpha}$ functions

It is a small technical difficulty that there are C^1 functions which do not satisfy (1) and hence do not belong to Lip^α.

Definition 2.4. *For $\alpha \in (0,1)$ we denote by Lip$^{1,\alpha}$ the set of those functions $g \in C^1([0,1])$ for which*

$$|g(x) - g(y)| < |x-y|^\alpha, \text{ for all } (x,y) \in [0,1]^2, \ x \neq y. \tag{9}$$

We impose strict inequality in (9). When $g \in \mathrm{Lip}^{1,\alpha} \subset C^1([0,1])$, we have $|g(x) - g(y)|/|x-y|^\alpha \to 0$ as $y \to x$. Hence, letting $G_\alpha(x,y) = |g(x) - g(y)|/|x-y|^\alpha$ when $x \neq y$ and $G_\alpha(x,x) \stackrel{\text{def}}{=} 0$, we obtain a continuous function on $[0,1]^2$. Therefore, there exists a constant $\gamma_g < 1$ such that $|G_\alpha(x,y)| \leq \gamma_g$ for all $(x,y) \in [0,1]^2$. This implies

$$|g(x) - g(y)| \leq \gamma_g \, |x-y|^\alpha \text{ for all } (x,y) \in [0,1]^2. \tag{10}$$

Lemma 2.5. *Lip$^{1,\alpha}$ is dense in Lip$^\alpha$.*

The proof is obvious. Indeed, given $\varepsilon > 0$ and $g \in \mathrm{Lip}^\alpha$, one can choose by uniform continuity of g finitely many points $0 = x_0 < x_1, ..., < x_n = 1$ such that the piecewise linear function g_0 passing through the points $(x_k, g(x_k))$ belongs to $B(g, \varepsilon/2) \cap \mathrm{Lip}^\alpha$. Then it is a small exercise to show that by smoothing the sharp corners of g_0 at the points $(x_k, g(x_k))$, one can obtain the required function $g_\varepsilon \in B(g,\varepsilon) \cap \mathrm{Lip}^{1,\alpha}$.

Lemma 2.6. *For any function $g \in Lip^{1,\alpha}$, there exists a constant $W_g > 2$ such that for any choice of $w \geq W_g$, $1 \geq \alpha' \geq \alpha$ and $1 \geq \alpha''$, the function*

$$\phi(x) = g(x) + w^{-\alpha'} H(w^{\alpha''} x) \tag{11}$$

belongs to Lip^{α} (recall that the oscillating function H is defined by (4)).

Proof: The function g being given, we choose γ_g as in (10). By assumption, $g \in C^1([0,1])$, hence $|g'| \leq M_g$ for some constant M_g. Let

$$C_g = \min\left\{ \left(\frac{1 - 2^{\alpha-1}}{M_g} \right)^{\frac{1}{1-\alpha}}, \frac{1}{2} \right\} \quad \text{(so that } M_g C_g^{1-\alpha} + 2^{\alpha-1} \leq 1), \tag{12}$$

and choose W_g so large that

$$W_g^{-1} \leq 2(1-\gamma_g)^{1/\alpha} C_g \quad \text{and} \quad M_g (2W_g)^{\alpha-1} + 2^{\alpha-1} \leq 1. \tag{13}$$

We separate three cases according to the value of $|x-y|$:

- $\mathbf{C_g \leq |x-y| \leq 1}$**:** Using (10), (12) and (13) and the fact that the function $|H|$ is less than $1/4$, we get (recall that $\alpha' \geq \alpha$)

$$\begin{aligned} |\phi(x) - \phi(y)| &\leq |g(x) - g(y)| + |w^{-\alpha'} H(w^{\alpha''} x)| + |w^{-\alpha'} H(w^{\alpha''} y)| \\ &\leq \gamma_g |x-y|^{\alpha} + w^{-\alpha'}/2 \leq \gamma_g |x-y|^{\alpha} + (1-\gamma_g)^{\alpha'/\alpha} (C_g)^{\alpha'} \\ &\leq \gamma_g |x-y|^{\alpha} + (1-\gamma_g)|x-y|^{\alpha'} \leq |x-y|^{\alpha}. \end{aligned} \tag{14}$$

- $\mathbf{\frac{1}{2w} \leq |x-y| \leq C_g}$**:** Since $\alpha' \geq \alpha$, we have by (12)

$$\begin{aligned} |\phi(x) - \phi(y)| &\leq |g(x) - g(y)| + |w^{-\alpha'} H(w^{\alpha''} x)| + |w^{-\alpha'} H(w^{\alpha''} y)| \\ &\leq M_g |x-y| + \frac{1}{2} w^{-\alpha'} \leq M_g |x-y| + \frac{1}{2}(2|x-y|)^{\alpha'} \\ &\leq M_g |x-y| + \frac{1}{2}(2|x-y|)^{\alpha} = (M_g |x-y|^{1-\alpha} + 2^{\alpha-1})|x-y|^{\alpha} \\ &\leq (M_g C_g^{1-\alpha} + 2^{\alpha-1})|x-y|^{\alpha} \leq |x-y|^{\alpha}. \end{aligned} \tag{15}$$

- $\mathbf{|x-y| \leq \frac{1}{2w}}$**:** This implies that $w^{\alpha''-\alpha'}|x-y|^{1-\alpha} \leq w^{1-\alpha}(1/2w)^{1-\alpha} = 2^{\alpha-1}$. Using that H is 1-lipschitz (i.e. $H \in \mathrm{Lip}^1$), the same kind of computation yields:

$$\begin{aligned} |\phi(x) - \phi(y)| &\leq |g(x) - g(y)| + w^{-\alpha'} |H(w^{\alpha''} x) - H(w^{\alpha''} y)| \\ &\leq M_g |x-y| + w^{-\alpha'+\alpha''} |x-y| \\ &\leq (M_g |x-y|^{1-\alpha} + w^{\alpha''-\alpha'} |x-y|^{1-\alpha}) |x-y|^{\alpha} \\ &\leq (M_g (2w)^{\alpha-1} + 2^{\alpha-1})|x-y|^{\alpha} \leq |x-y|^{\alpha}. \end{aligned} \tag{16}$$

The lemma follows from (14), (15), and (16). □

2.4. Properties of the singularity spectrum for monotone functions

We recall standard results on singularity spectra of monotone functions [3].

Proposition 2.7. *Let $f \in \mathcal{M}$. Then for every $h \in [0,1]$, $d_f(h) \leq h$.*

Proposition 2.7 allows us to find easily an upper bound for the singularity spectrum of the composition of functions of $\mathcal{E}^\alpha$.

Proposition 2.8. *Let $(f,g) \in \mathcal{E}^\alpha$, for some $\alpha \in (0,1)$. Then*

$$\text{for every } h \in [0,\alpha], \qquad d_{g\circ f}(h) \leq \dim\{t : h_{g\circ f}(t) \leq h\} \leq h/\alpha. \tag{17}$$

Proof: Let $(f,g) \in \mathcal{E}^\alpha$ and $h \in [0,\alpha]$. Since $\{t : h_{g\circ f}(t) = h\} \subset \{t : h_{g\circ f}(t) \leq h\}$, the first inequality is obvious. Next, we remark that (8) yields

$$\{t : h_{g\circ f}(t) \leq h\} \subset \{t : h_g(f(t))h_f(t) \leq h\}.$$

Moreover, as noticed above, if $g \in \mathrm{Lip}^\alpha$, then for every $t \in [0,1]$, $h_g(t) \geq \alpha$. This implies that

$$\{t : h_{g\circ f}(t) \leq h\} \subset \{t : \alpha h_f(t) \leq h\} = \{t : h_f(t) \leq h/\alpha\}.$$

Combining the last inequality with Proposition 2.7 yields the result. □

3. The extremal points of the singularity spectrum in $\mathcal{E}^\alpha$

Let $\alpha \in (0,1)$ be a real number, and consider $\mathcal{E}^\alpha = \mathcal{M}\times\mathrm{Lip}^\alpha$. This set $\mathcal{E}^\alpha$ is a complete separable metric space for the norm (3). Let us denote by $\{(f_n,g_n)\}_{n\geq 1}$ a dense sequence in $\mathcal{E}^\alpha$, consisting of pairs of functions.

As explained at the end of the introduction, Theorem 1.5 will be obtained as the consequence of several results gathered in Sections 3.1, 3.2 and 4.

3.1. Uniform upper bound for $h_f(x)$ and existence of points with exponent α

Proposition 3.1. *There is a residual set $\mathcal{R}_\alpha^\alpha$ in $\mathcal{E}^\alpha$ such that for every $(f,g) \in \mathcal{R}_\alpha^\alpha$, if we set $Z = g \circ f$, then $h_Z(x) \leq \alpha$ for every $x \in [0,1]$, and hence $E_Z(h) = \emptyset$ for $h > \alpha$.*

In order to establish Proposition 3.1, we need to build the dense G_δ set $\mathcal{R}_\alpha^\alpha$ in $\mathcal{E}^\alpha$ on which the claimed property is realized. Several steps are needed to find such a set. We first fix $\varepsilon > 0$ such that

$$\varepsilon < (1-\alpha)/4. \tag{18}$$

At first, we build for all $j \geq 1$ a dense open set $\Omega^{\alpha}_{j,\varepsilon}$ in $\mathcal{E}^{\alpha}$ such that for every couple of functions $(f,g) \in \Omega^{\alpha}_{j,\varepsilon}$, there exists a positive integer $J_{f,g} \geq j$ with the following property: for every $k \in \{0,1,...,2^{J_{f,g}}-1\}$, one has

$$\omega_{J_{f,g},k}(g \circ f) \geq 2^{-J_{f,g}(\alpha+\varepsilon)}. \tag{19}$$

Subsequently, the oscillations of $g \circ f$ at scale $J_{f,g}$ are bounded by below by the same quantity $2^{-J_{f,g}(\alpha+\varepsilon)}$, for all $(f,g) \in \Omega^{\alpha}_{j,\varepsilon}$. The residual set $\mathcal{R}^{\alpha}_{\alpha}$ will then be obtained as countable intersection of well-chosen sets $\Omega^{\alpha}_{j,\varepsilon}$.

Let $j \geq 1$, and let $n \geq 1$. Consider the couple of functions $(f_n, g_n) \in \mathcal{E}^{\alpha}$. We are going to find another couple of functions $(f_{n,j}, g_{n,j}) \in \mathcal{E}^{\alpha}$ and a real number $\delta_{n,j} > 0$ such that, with respect to the supremum norm in $\mathcal{E}^{\alpha}$,

$$B\Big((f_{n,j}, g_{n,j}), \delta_{n,j}\Big) \subset B\Big((f_n, g_n), 2^{-(j+n)}\Big) \tag{20}$$

and for every $(f,g) \in B\Big((f_{n,j}, g_{n,j}), \delta_{n,j}\Big)$, property (19) holds for the couple of functions (f,g) for some well-chosen integer $J_{f,g} \geq j$.

Remark 3.2. *Since $\{(f_n, g_n)\}_{n\geq 1}$ is a dense sequence in $\mathcal{E}^{\alpha}$, and since $\delta_{n,j}$ decreases to 0 when $n \to +\infty$, the union $\bigcup_{n\geq 1} B\Big((f_{n,j}, g_{n,j}), \delta_{n,j}\Big)$ forms a dense open set in $\mathcal{E}^{\alpha}$. $\mathcal{R}^{\alpha}_{\alpha}$ will be a countable intersection of such sets.*

Let $f_{n,j}$ be a continuous function satisfying the following properties:

- $f_{n,j}$ is strictly increasing, $f_{n,j}(0) = 0$ and $f_{n,j}(1) = 1$,
- there is an integer $J_1 \geq j+n+2$ such that $f_{n,j}$ is affine on each dyadic interval $I_{J_1,k}$, $k \in \{0,1,...,2^{J_1}-1\}$,
- $\|f_n - f_{n,j}\| \leq 2^{-j-n-2}$.

The existence of such a function $f_{n,j}$ is just an exercise. See also the left half of Figure 2. We denote by $s_{\min,j}$ and $s_{\max,j}$ the minimal and the maximal slopes of the affine parts of $f_{n,j}$, respectively. Observe that one necessarily has $s_{\min,j} \leq 1 \leq s_{\max,j}$.

Let $\bar{g}_n$ be any function in $\mathrm{Lip}^{1,\alpha}$ satisfying $\|g_n - \bar{g}_n\| \leq 2^{-j-n-2}$ (the existence is guaranteed by Section 2.3). Denote by W_n the constant $W_{\bar{g}_n}$ obtained from Lemma 2.6. We denote by M the maximal value of the derivative of $\bar{g}_n$, that is,

$$M := \max\{|\bar{g}'_n(x)| : x \in [0,1]\}. \tag{21}$$

Let $J_{n,j}$ be an integer satisfying

$$2^{\varepsilon J_{n,j}} > 4\max(M, 2^{J_1}, 1/s_{\min,j}, s_{\max,j}, (W_n)^\varepsilon) \text{ and } \varepsilon J_{n,j} > 4. \tag{22}$$

We define $g_{n,j}$ as follows: For every $x \in [0,1]$,

$$g_{n,j}(x) = \bar{g}_n(x) + 2^{-(\alpha+\varepsilon)J_{n,j}} H(2^{J_{n,j}(1-\varepsilon)}x), \tag{23}$$

where H is the periodic function defined in (4). By (22), we have $2^{J_{n,j}} > W_n$ and by Lemma 2.6, $g_{n,j} \in \mathrm{Lip}^\alpha$. The key property is that the function $g_{n,j}$ is oscillating at frequency $2^{J_{n,j}(1-\varepsilon)}$, and $\overline{g}_n$ plays the role of a slow trend. The construction of $g_{n,j}$ is illustrated on the right half of Figure 2.

Lemma 3.3. *Using $f_{n,j}$, $g_{n,j}$, and $J_{n,j}$ defined above, we have:*

$$\text{for every } k \in \{0,1,...,2^{J_{n,j}}-1\}, \quad \omega_{J_{n,j},k}(g_{n,j} \circ f_{n,j}) \geq 2^{-J_{n,j}(\alpha+4\varepsilon)}.$$

Proof: It is obvious from (18) and (22) that $J_{n,j} \geq J_1$, hence any interval $I_{J_{n,j},k}$ is included in a larger dyadic interval $I_{J_1,k'}$ on which $f_{n,j}$ is affine.

Let $k \in \{0,1,...,2^{J_{n,j}}-1\}$, and let us consider the interval $f_{n,j}(I_{J_{n,j},k})$.

By construction the length of the interval $f_{n,j}(I_{J_{n,j},k})$ is smaller than $s_{\max,j}2^{-J_{n,j}}$, hence by (22) it is smaller than $2^{-J_{n,j}(1-\varepsilon)-2}$, that is, we have

$$\omega_{J_{n,j},k}(f_{n,j}) < 2^{-J_{n,j}(1-\varepsilon)-2}. \tag{24}$$

Consequently, the interval $f_{n,j}(I_{J_{n,j},k})$ contains at most one real number from $2^{-J_{n,j}(1-\varepsilon)-1} \cdot \mathbb{Z} := \{p \cdot 2^{-J_{n,j}(1-\varepsilon)-1} : p \in \mathbb{Z}\}$.

Since $f_{n,j}$ is increasing, there is a dyadic interval $I_k \subset I_{J_{n,j},k}$ of length $2^{-J_{n,j}-2}$ such that $f_{n,j}(I_k)$ does not contain any point of $2^{-J_{n,j}(1-\varepsilon)-1} \cdot \mathbb{Z}$. Recall that $f_{n,j}$ is piecewise affine, and is affine on I_k. The length of the interval $f_{n,j}(I_k)$, that is, $|f_{n,j}(I_k)|$ is greater than $s_{\min,j}2^{-J_{n,j}-2}$. By (22) we obtain

$$|f_{n,j}(I_k)| > 2^{-J_{n,j}(1+\varepsilon)-2}. \tag{25}$$

Finally, recalling the definition of $g_{n,j}$ and its oscillating properties, we remark that the derivative of $g_{n,j}$ has a constant sign on the interval $f_{n,j}(I_k)$ (since it does not contain any dyadic point $k'2^{-J_{n,j}(1-\varepsilon)-1} \in 2^{-J_{n,j}(1-\varepsilon)-1} \cdot \mathbb{Z}$).

From (4), (21), and (23) it follows that the absolute value of this derivative is greater than $2^{J_{n,j}(1-\alpha-2\varepsilon)} - M$. By (18) and (22), we have

$$|g'_{n,j}| \geq 2^{J_{n,j}(1-\alpha-2\varepsilon)} - M \geq 2^{J_{n,j}(1-\alpha-2\varepsilon)} - 2^{J_{n,j}\varepsilon-1} \geq 2^{J_{n,j}(1-\alpha-2\varepsilon)-1}. \tag{26}$$

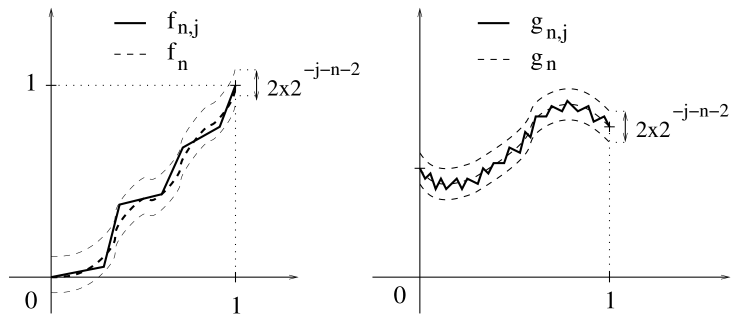

Figure 2: Construction of $f_{n,j}$ and $g_{n,j}$.

Summarizing the above, we find by (22), (25) and (26) that the oscillation of the composition $g_{n,j} \circ f_{n,j}$ on the interval $I_{J_{n,j},k}$ satisfies

$$\begin{aligned} \omega_{I_{J_{n,j},k}}(g_{n,j} \circ f_{n,j}) &\geq \omega_{I_k}(g_{n,j} \circ f_{n,j}) = \omega_{f_{n,j}(I_k)}(g_{n,j}) \\ &\geq 2^{J_{n,j}(1-\alpha-2\varepsilon)-1} 2^{-J_{n,j}(1+\varepsilon)-2} \\ &\geq 2^{-J_{n,j}(\alpha+3\varepsilon)-3} \geq 2^{-J_{n,j}(\alpha+4\varepsilon)}. \qquad \square \end{aligned}$$

Now we demonstrate that the lower bound we obtained in Lemma 3.3 on the oscillations of the composition of functions on dyadic intervals of length $2^{-J_{n,j}}$ is stable with respect to the supremum norm. Arguments analogous to the proof of the next lemma will be needed later several times.

Lemma 3.4. *Suppose $f_{n,j}$, $g_{n,j}$, and $J_{n,j}$ are defined as above. Let*

$$0 < \delta_{n,j} := \min\Big\{2^{-2J_{n,j}}, \min\{\omega_{I_{J_{n,j},k}}(f_{n,j}) : k \in \{0, ..., 2^{J_{n,j}} - 1\}\Big\}/16. \quad (27)$$

Consider any couple of functions $(f, g) \in B\Big((f_{n,j}, g_{n,j}), \delta_{n,j}\Big) \subset \mathcal{E}^\alpha$. Then, for every integer $k \in \{0, ..., 2^{J_{n,j}} - 1\}$, we have

$$\omega_{J_{n,j},k}(g \circ f) \geq 2^{-J_{n,j}(\alpha+5\varepsilon)}.$$

Proof: We have $\|f - f_{n,j}\| < \delta_{n,j}$ and $\|g - g_{n,j}\| < \delta_{n,j}$. Fix $k \in \{0, ..., 2^{J_{n,j}} - 1\}$, and let us consider once again the dyadic interval $I_{J_{n,j},k}$. By (24), the construction ensures that the oscillation of f on $I_{J_{n,j},k}$ satisfies

$$\omega_{I_{J_{n,j},k}}(f) \leq \omega_{I_{J_{n,j},k}}(f_{n,j}) + 2\delta_{n,j} < 2^{-J_{n,j}(1-\varepsilon)-2} + 2\delta_{n,j}. \qquad (28)$$

Again, using that $\delta_{n,j}$ is small enough by (27), such $f(I_{J_{n,j}k})$ contains at most one real number from $2^{-J_{n,j}(1-\varepsilon)-1}\cdot\mathbb{Z}$.

Hence the same procedure as above can be applied: We consider one dyadic interval $I_k \subset I_{J_{n,j},k}$ of length $2^{-J_{n,j}-2}$ such that $f(I_k)$ does not contain any real number from $2^{-J_{n,j}(1-\varepsilon)-1}\cdot\mathbb{Z}$.

We now bound by below the oscillation of f on I_k. By (27) and the piecewise linearity of $f_{n,j}$ we have $\omega_{I_k}(f_{n,j}) > 4\delta_{n,j}$. Recalling that $\|f - f_{n,j}\| \le \delta_{n,j}$, we notice that

$$\omega_{I_k}(f) \ge \omega_{I_k}(f_{n,j}) - 2\delta_{n,j},$$

which by (27) is greater than $\omega_{I_k}(f_{n,j})/2$. Therefore, by (25),

$$\omega_{I_{J_{n,j},k}}(f) \ge 2^{-J_{n,j}(1+\varepsilon)-2}/2 = 2^{-J_{n,j}(1+\varepsilon)-3}.$$

We then apply the same argument as in Lemma 3.3 to $(f, g_{n,j})$ on the interval I_k defined in this proof. We find by using (22) and (26) that

$$\begin{aligned}\omega_{I_{J_{n,j},k}}(g_{n,j}\circ f) &\ge \omega_{I_k}(g_{n,j}\circ f) = \omega_{f(I_k)}(g_{n,j}) \ge 2^{J_{n,j}(1-\alpha-2\varepsilon)-1}2^{-J_{n,j}(1+\varepsilon)-3}\\ &\ge 2^{-J_{n,j}(\alpha+3\varepsilon)-4} \ \ge\ 2^{-J_{n,j}(\alpha+4\varepsilon)}.\end{aligned}$$

To find the lower bound for the oscillation of $g\circ f$, we only need to use $||g - g_{n,j}|| < \delta_{n,j}$, (18), (22) and (27) to obtain

$$\begin{aligned}\omega_{I_{J_{n,j},k}}(g\circ f) &\ge \omega_{I_{J_{n,j},k}}(g_{n,j}\circ f) - 2\delta_{n,j}\\ &\ge 2^{-J_{n,j}(\alpha+4\varepsilon)} - 2^{-2J_{n,j}} \ \ge\ 2^{-J_{n,j}(\alpha+5\varepsilon)}.\end{aligned}$$ □

To keep notation sufficiently simple we did not emphasize that the choice of $f_{n,j}$, $g_{n,j}$ and $\delta_{n,j}$ also depended on $\epsilon > 0$ fixed at (18).

Definition 3.5. *We define the two subsets of $\mathcal{E}^\alpha$, $\Omega^\alpha_{j,\varepsilon}$ and $\mathcal{R}^\alpha_\alpha$, by*

$$\Omega^\alpha_{j,\varepsilon} := \bigcup_{n\ge1} B\Big((f_{n,j}, g_{n,j}), \delta_{n,j}\Big) \quad \textit{and} \quad \mathcal{R}^\alpha_\alpha := \bigcap_{j\ge1} \Omega^\alpha_{j,1/j}.$$

Let us list the properties of $\Omega^\alpha_{j,\varepsilon}$:

- $\Omega^\alpha_{j,\varepsilon}$ is open in $\mathcal{E}^\alpha$ since it is the union of open balls,
- $\Omega^\alpha_{j,\varepsilon}$ is dense in $\mathcal{E}^\alpha$, since the set of couples $\{(f_n, g_n)\}_{n\ge1}$ is dense and the distance $\|(f_n, g_n) - (f_{n,j}, g_{n,j})\| \le 2^{-j-n-2}$ tends to zero when n tends to infinity,

- any couple of functions $(f,g) \in \Omega^{\alpha}_{j,\varepsilon}$ fulfills the following property: there is an integer $J_{n,j} \geq j$ such that for every integer $k \in \{0,1,...,2^{J_{n,j}}-1\}$, we have
$$\omega_{J_{n,j},k}(g \circ f) \geq 2^{-J_{n,j}(\alpha+5\varepsilon)}.$$

We now have the tools to prove Proposition 3.1.

Proof: By construction, $\mathcal{R}^{\alpha}_{\alpha}$ is a residual set, in fact it is a dense G_δ set in $\mathcal{E}^{\alpha}$ equipped with the supremum norm.

Let $(f,g) \in \mathcal{R}^{\alpha}_{\alpha}$. Our goal is to prove that $g \circ f$ has pointwise Hölder exponent less than α at every $x \in [0,1]$.

The couple (f,g) belongs to an infinite number of sets $\Omega^{\alpha}_{j,1/j}$, $j \geq 1$. This means that there exist strictly increasing sequences of integers $(j_m)_{m\geq 1}$ and $(J_m)_{m\geq 1}$ for which $J_m \geq j_m$ and the following property holds:

$$\text{for every } k \in \{0,1,...,2^{J_m}-1\}, \quad \omega_{I_{J_m,k}}(g \circ f) \geq 2^{-J_m(\alpha+5/j_m)}.$$

Let $x \in [0,1]$. Obviously, for every $m \geq 1$, the interval $B(x, 2^{-J_m})$ contains a dyadic interval of length 2^{-J_m}, hence

$$\omega_{B(x,2^{-J_m})}(g \circ f) \;\geq\; \omega_{I_{J_m,k}}(g \circ f) \;\geq\; 2^{-J_m(\alpha+5/j_m)}.$$

By (7), we obtain that

$$\begin{aligned} h_{g\circ f}(x) &= \liminf_{r\to 0+} \frac{\log \omega_{B(x,r)}(g \circ f)}{\log r} \leq \liminf_{m\to+\infty} \frac{\log \omega_{B(x,2^{-J_m})}(g \circ f)}{\log 2^{-J_m}} \\ &\leq \lim_{m\to\infty} \alpha + 5/j_m = \alpha. \end{aligned}$$ □

Proposition 3.1 combined with Proposition 2.8 provides us with the existence of points with Hölder exponents exactly equal to α, and even more, with the fact that such points form a set of Lebesgue measure 1:

Proposition 3.6. *Let $(f,g) \in \mathcal{R}^{\alpha}_{\alpha}$. The set of points $\{x \in [0,1] : h_{g\circ f}(x) = \alpha\}$ has Lebesgue measure 1. In particular, $d_{g\circ f}(\alpha) = \dim E_{g\circ f}(\alpha) = 1$.*

Proof: By Proposition 2.8, for every $h < \alpha$, $\dim\{x \in [0,1] : h_{g\circ f}(x) \leq h\} \leq h/\alpha < 1$. Hence the Lebesgue measure of such a set, $\lambda(\{x \in [0,1] : h_{g\circ f}(x) \leq h\})$ is zero. Observe that

$$\{x \in [0,1] : h_{g\circ f}(x) < \alpha\} = \bigcup_{n\geq 1} \{x \in [0,1] : h_{g\circ f}(x) \leq \alpha - 1/n\}.$$

This yields

$$\lambda\big(\{x \in [0,1] : h_{g\circ f}(x) < \alpha\}\big) \le \sum_{n\ge1} \lambda\big(\{x \in [0,1] : h_{g\circ f}(x) \le \alpha - 1/n\}\big) = 0.$$

Hence, recalling that $E_{g\circ f}(\alpha) = \{x \in [0,1] : h_{g\circ f}(x) = \alpha\}$, we have

$$\begin{aligned}\lambda\big(E_{g\circ f}(\alpha)\big) &= \lambda\big(\{x \in [0,1] : h_{g\circ f}(x) \le \alpha\}\big) - \lambda\big(\{x \in [0,1] : h_{g\circ f}(x) < \alpha\}\big)\\ &= \lambda([0,1]) - 0 = 1.\end{aligned}$$

□

3.2. Existence of points with Hölder exponents zero

Proposition 3.7. *There exists a residual set $\mathcal{R}_0^\alpha$ in $\mathcal{E}^\alpha$ such that for every $(f,g) \in \mathcal{R}_0^\alpha$, there is a set of Hausdorff dimension zero, $X_{f,g}^0 \subset [0,1]$ which is c-dense in $[0,1]$ and $h_{g\circ f}(x) = 0$ for any $x \in X_{f,g}^0$ (i.e. $X_{f,g}^0 \subset E_{g\circ f}(0)$).*

We construct a countable intersection of dense open sets in $\mathcal{E}^\alpha$ on which the announced property holds.

Let $0 < \varepsilon < 1 - \alpha$. First, we find a dense open set $\Omega_{j,\varepsilon}^0$ for which the following property holds: for every $(f,g) \in \Omega_{j,\varepsilon}^0$, there are $J_{f,g} \ge j$, some real numbers $(x_p)_{p=1,\dots,2^{J_{f,g}}-1}$ and a real number $0 < r \ll 2^{-J_{f,g}}$ satisfying:

$$\text{for every } p \in \{1,\dots,2^{J_{f,g}}-1\}, \quad \begin{cases} 2^{-J_{f,g}-1} \le |x_p - x_{p+1}| \le 2^{-J_{f,g}+1} \\ \quad r^\varepsilon \le \omega_{B(x_p,r)}(g\circ f). \end{cases} \tag{29}$$

Heuristically $g \circ f$ has poor regularity around the real numbers x_p at the scale r, which is much smaller than $|x_p - x_{p+1}|$.

Take again a dense sequence $\{(f_n, g_n)\}_{n\ge1}$ in $\mathcal{E}^\alpha$. Let $n \ge 1$, and consider the couple (f_n, g_n). First, let $\bar g_n$ be a function in $\mathrm{Lip}^{1,\alpha}$ satisfying

$$\|g_n - \bar g_n\| \le 2^{-j-n-2}.$$

We denote by M the maximal value of the derivative of the mapping $\bar g_n$, that is, $M := \max\{|\bar g_n'(x)| : x \in [0,1]\}$. Denote again by W_n the constant $W_{\bar g_n}$ obtained from Lemma 2.6.

Then, let $\bar f_n : [0,1] \to [0,1]$ be a monotone, discontinuous (piecewise constant) function satisfying:

- $\|f_n - \bar f_n\| \le 2^{-j-n}$, $\bar f_n(0) = 0$ and $\bar f_n(1) = 1$,
- there exists an integer $J_n \ge j+n+2$ such that $\bar f_n$ is constant on every dyadic interval $I_{J_n,k} = [k2^{-J_n}, (k+1)2^{-J_n})$, $k \in \{0,1,\dots,2^{J_n}-1\}$,

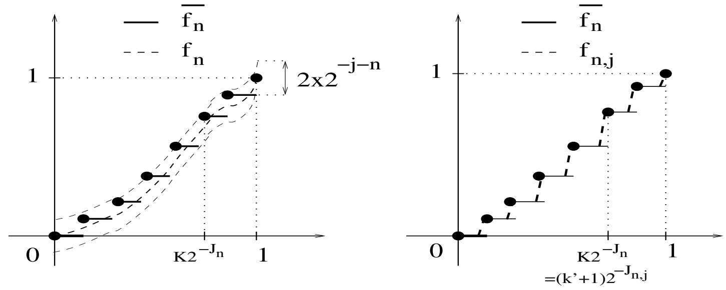

Figure 3: Construction of $\bar{f}_n$ and $f_{n,j}$.

- for every $k \in \{0, 1, ..., 2^{J_n} - 1\}$, $\bar{f}_n(k2^{-J_n}) < \bar{f}_n((k+1)2^{-J_n})$.

The construction of $\bar{f}_n$ is illustrated on the left half of Figure 3, the details are left to the reader. We set

$$\kappa'_{\min} = \min\left\{\bar{f}_n((k+1)2^{-J_n}) - \bar{f}_n(k2^{-J_n}) : k \in \{0, ..., 2^{J_n} - 1\}\right\}.$$

Then, we select $0 < \kappa_{\min} < \kappa'_{\min}/2$ such that

$$(\kappa_{\min})^{-1} > W_n, \ (\kappa_{\min})^{\alpha-1} \geq 2M \quad \text{and} \quad (\kappa_{\min})^{\alpha-1} \geq 2(\kappa_{\min})^{\alpha+\epsilon-1}. \tag{30}$$

Finally, we choose an integer $J_{n,j}$ such that

$$2^{\varepsilon J_{n,j}} \geq 2\max\{2^{(J_n)^2}, 2^{1/(\alpha+\varepsilon)}/\kappa_{\min}\} \quad \text{and} \quad \epsilon^2 J_{n,j} \geq 2. \tag{31}$$

Definition 3.8. *We define the function $f_{n,j}$ as follows:*

- *$f_{n,j}$ is continuous, piecewise affine on each dyadic interval $I_{J_{n,j},k'}$, $k' \in \{0, 1, ..., 2^{J_{n,j}} - 1\}$, $f_{n,j}(0) = 0$ and $f_{n,j}(1) = 1$,*
- *for every $k' \in \{0, 1, ..., 2^{J_{n,j}} - 1\}$ such that $(k'+1)2^{-J_{n,j}}$ does not belong to $2^{-J_n} \cdot \mathbb{Z}$, $f_{n,j}$ is constant on $I_{J_{n,j},k'}$ and equals $\bar{f}_n$,*
- *for every $k' \in \{0, ..., 2^{J_{n,j}} - 1\}$ such that $(k'+1)2^{-J_{n,j}} \in 2^{-J_n} \cdot \mathbb{Z}$, i.e. $(k'+1)2^{-J_{n,j}} = K2^{-J_n}$, $f_{n,j}$ is affine, strictly increasing on $I_{J_{n,j},k'}$ and $f_{n,j}(k'2^{-J_{n,j}}) = \bar{f}_n((K-1)2^{-J_n})$ and $f_{n,j}((k'+1)2^{-J_{n,j}}) = \bar{f}_n(K2^{-J_n})$.*

Heuristically, $f_{n,j}$ is a continuous function very close to $\overline{f}_n$ (which is discontinuous). Hence the oscillations of $f_{n,j}$ are very large around the dyadic numbers $2^{-J_n} \cdot \mathbb{Z}$. The construction is illustrated on Figure 3.

The construction of $g_{n,j}$ is very similar to the one in Proposition 3.1.

We define $g_{n,j}$ as follows: For every $x \in [0,1]$,

$$g_{n,j}(x) = \bar{g}_n(x) + (1/\kappa_{\min})^{-\alpha} H((1/\kappa_{\min})x), \tag{32}$$

where H was defined in (4). By (30) and Lemma 2.6, $g_{n,j} \in \mathrm{Lip}^\alpha$.

Lemma 3.9. *Consider the two functions $f_{n,j}$ and $g_{n,j}$, as well as the integer $J_{n,j}$. For every integer $k \in \{0,1,...,2^{J_{n,j}}-1\}$ such that $(k+1)2^{-J_{n,j}} \in 2^{-J_n} \cdot \mathbb{Z}$, we have*

$$\omega_{I_{J_{n,j},k}}(g_{n,j} \circ f_{n,j}) \geq 2^{-\varepsilon J_{n,j}(\alpha+\varepsilon)}.$$

Proof: Let $k \in \{0,1,...,2^{J_{n,j}}-1\}$ be such that $(k+1)2^{-J_{n,j}} \in 2^{-J_n} \cdot \mathbb{Z}$. By the construction of $f_{n,j}$ and (31)

$$f_{n,j}((k+1)2^{-J_{n,j}}) - f_{n,j}(k2^{-J_{n,j}}) \geq \kappa'_{\min} \geq 2\kappa_{\min} \geq 2^{-J_{n,j}\varepsilon}.$$

The interval $f_{n,j}(I_{J_{n,j},k})$ contains an interval of the form $[p(\kappa_{\min}/2),(p+1)(\kappa_{\min}/2)]$, $p \in \mathbb{Z}$. The derivative of the function $H((1/\kappa_{\min})x)$ has constant sign on such an interval, and by (30) the same is true for $g_{n,j}$. By (30), (31) and (32) we infer

$$|g'_{n,j}| \geq (1/\kappa_{\min})^{1-\alpha} - M \geq (1/\kappa_{\min})^{1-\alpha}/2 \geq (1/\kappa_{\min})^{1-\alpha-\varepsilon}. \tag{33}$$

Consequently, we have

$$\begin{aligned}\omega_{I_{J_{n,j},k}}(g_{n,j} \circ f_{n,j}) &\geq \omega_{[p(\kappa_{\min}/2),(p+1)(\kappa_{\min}/2)]}(g_{n,j}) \\ &\geq (1/\kappa_{\min})^{1-\alpha-\varepsilon}(\kappa_{\min}/2) = (\kappa_{\min})^{\alpha+\varepsilon}/2 \geq 2^{-\varepsilon J_{n,j}(\alpha+\varepsilon)},\end{aligned}$$

the last inequality being a consequence of (31). □

As above, this property on the oscillations of the function $g_{n,j} \circ f_{n,j}$ is stable with respect to the supremum norm in $\mathcal{E}^\alpha$.

Lemma 3.10. *Consider $f_{n,j}$, $g_{n,j}$ and $J_{n,j}$, as in Lemma 3.9. Let*

$$\delta_{n,j} := \min\left(2^{-J_{n,j}\varepsilon(\alpha+\varepsilon)}, \kappa_{\min}\right)/4. \tag{34}$$

Consider any couple $(f,g) \in \mathcal{E}^\alpha$ satisfying $(f,g) \in B\Big((f_{n,j},g_{n,j}),\delta_{n,j}\Big)$. For every $k \in \{0,...,2^{J_{n,j}}-1\}$ such that $(k+1)2^{-J_{n,j}} \in 2^{-J_n} \cdot \mathbb{Z}$, we have

$$\omega_{I_{J_{n,j},k}}(g \circ f) \geq 2^{-\varepsilon J_{n,j}(\alpha+2\varepsilon)}.$$

Proof: Fix $k \in \{0, 1, ..., 2^{J_{n,j}} - 1\}$ such that $(k+1)2^{-J_{n,j}} \in 2^{-J_n} \cdot \mathbb{Z}$.

Arguments similar to those of Lemma 3.4 yield that $f(I_{J_{n,j},k})$ contains an interval I_p of the form $[p(\kappa_{\min}/2), (p+1)(\kappa_{\min}/2)]$ (just as $f_{n,j}$ does).

We then argue as in Lemma 3.9 for the couple of functions $(f, g_{n,j})$ on the interval I_p. We find by (33) and (31) that

$$\begin{aligned} \omega_{I_{J_{n,j},k}}(g_{n,j} \circ f) &\geq \omega_{[p(\kappa_{\min}/2),(p+1)(\kappa_{\min}/2)]}(g_{n,j}) \geq (1/\kappa_{\min})^{1-\alpha-\varepsilon}(\kappa_{\min}/2) \\ &\geq (1/\kappa_{\min})^{-\alpha-\varepsilon}/2 \geq 2^{-\varepsilon J_{n,j}(\alpha+\varepsilon)}. \end{aligned}$$

Finally, we observe that

$$\begin{aligned} \omega_{I_{J_{n,j},k}}(g \circ f) &\geq \omega_{I_{J_{n,j},k}}(g_{n,j} \circ f) - 2\delta_{n,j} \\ &\geq 2^{-J_{n,j}\varepsilon(\alpha+\varepsilon)} - 2^{-J_{n,j}\varepsilon(\alpha+\varepsilon)-1} \geq 2^{-\varepsilon J_{n,j}(\alpha+2\varepsilon)}, \end{aligned}$$

where (31) and (34) have been used in the last two inequalities. This concludes the proof. □

Again we remind the reader that $f_{n,j}$, $g_{n,j}$ and $\delta_{n,j}$ also depend on ε.

Definition 3.11. *We define the sets $\Omega^0_{j,\varepsilon}$ and $\mathcal{R}^0_\alpha$ by*

$$\Omega^0_{j,\varepsilon} := \bigcup_{n\geq 1} B\Big((f_{n,j}, g_{n,j}), \delta_{n,j}\Big) \quad \text{and} \quad \mathcal{R}^0_\alpha := \bigcap_{j\geq 1} \Omega^0_{j,1/j}.$$

As in the previous section, $\Omega^0_{j,\varepsilon}$ is a dense open set in $\mathcal{E}^\alpha$, and any couple of functions $(f, g) \in \Omega^0_{j,\varepsilon}$ has the following property: there are two integers $J_{n,j} > J_n \geq j$ such that for every integer $k \in \{0, 1, ..., 2^{J_{n,j}} - 1\}$ satisfying $(k+1)2^{-J_{n,j}} \in 2^{-J_n} \cdot \mathbb{Z}$, we have

$$\omega_{I_{J_{n,j},k}}(g \circ f) \geq 2^{-\varepsilon J_{n,j}(\alpha+2\varepsilon)} = |I_{J_{n,j},k}|^{\varepsilon(\alpha+2\varepsilon)}.$$

Next we prove Proposition 3.7.

Proof of Proposition 3.7: $\mathcal{R}^0_\alpha$ is obviously a residual set in $\mathcal{E}^\alpha$. Let $(f, g) \in \mathcal{R}^0_\alpha$. We want to find a c-dense set of real numbers $x \in [0, 1]$ such that for every such x, $h_{g\circ f}(x) = 0$.

The couple (f, g) belongs to an infinite number of sets $\Omega^0_{j,1/j}$, $j \geq 1$. This means that there are strictly increasing sequences of integers $(j_m)_{m\geq 1}$, $(J_m)_{m\geq 1}$, and $(\widehat{J_m})_{m\geq 1}$ for which the following property holds: for every $m \geq 1$, $\widehat{J_m} > J_m > j_m > m$, and for every integer $k \in \{0, 1, ..., 2^{\widehat{J_m}} - 1\}$ such that $(k+1)2^{-\widehat{J_m}} \in 2^{-J_m} \cdot \mathbb{Z}$, we have

$$\omega_{I_{\widehat{J_m},k}}(g \circ f) \geq 2^{-\widehat{J_m}(\alpha+2/j_m)/j_m} > 2^{-\widehat{J_m}(\alpha+2/m)/m} = |I_{\widehat{J_m},k}|^{(\alpha+2/m)/m}. \quad (35)$$

We denote by $\widehat{I}_{\widehat{J_m},k}$ those intervals $I_{\widehat{J_m},k}$ for which (35) holds.

Let us consider the limsup set $X^0_{f,g}$ defined as

$$X^0_{f,g} = \{x \in [0,1] : \ x \text{ belongs to an infinite number of intervals } \widehat{I}_{\widehat{J_m},k}\}.$$

Obviously, when $x \in X^0_{f,g}$, there are infinitely many m's for which $B(x, 2^{-\widehat{J_m}})$ contains a dyadic interval of the form $\widehat{I}_{\widehat{J_m},k}$. Thus

$$\omega_{B(x,2^{-\widehat{J_m}})}(g \circ f) \geq 2^{-\widehat{J_m}(\alpha+2/m)/m}.$$

Considering infinitely many m's by (7), we obtain that $h_{g\circ f}(x) = 0$.

We remark that $X^0_{f,g}$ has a structure of a limsup set, and it is easy to see that it is c-dense, since $\widehat{I}_{\widehat{J_m},k}$ contains more than two intervals of the form $\widehat{I}_{\widehat{J_{m'}},k}$ if $m' \gg m$. Using this property repeatedly, any $\widehat{I}_{\widehat{J_{m'}},k}$ contains a Cantor subset of $X^0_{f,g}$. Moreover, $X^0_{f,g} \subset \{x : h_{g\circ f}(x) = 0\}$. By Proposition 2.8, $0 = \dim\{x : h_{g\circ f}(x) = 0\} \geq \dim X^0_{f,g}$. This concludes the proof. $\square$

4. The interior points of the spectrum of generic functions in $\mathcal{E}^\alpha$: the case $0 < h < \alpha$

At this point, we have obtained an upper bound for the spectrum of a generic composition of functions in $\mathcal{E}^\alpha$, i.e. $d_{g\circ f}(h) \leq h/\alpha$ for every $0 \leq h \leq \alpha$. We proved that, generically, all exponents are less than α, Lebesgue almost-all exponents equal α, and there are uncountably many points with exponents 0. We now need to complete the spectrum, i.e. we need to build simultaneously for every $0 < h < \alpha$ a set of dimension h/α which contains points with Hölder exponents exactly equal to h.

4.1. Construction of a suitable residual set $\mathcal{R}_\alpha$

Let $0 < \varepsilon < 1/2$ and $j \geq 1$. Let $n \geq 1$, and consider the couple $(f_n, g_n) \in \mathcal{E}^\alpha$. The procedure is similar to the ones of Sections 3.1 and 3.2, but the way we obtain simultaneously different Hölder regularities is more delicate than in the previous sections.

Consider $\bar{g}_n$ a function in $\mathrm{Lip}^{1,\alpha}$ satisfying $\|g_n - \bar{g}_n\| \leq 2^{-j-n-2}$. Denote by $M := \max\{|\bar{g}_n'(x)| : x \in [0,1]\}$ the maximal value of the derivative of $\bar{g}_n$ and by W_n the constant $W_{\bar{g}_n}$ which can be obtained from Lemma 2.6.

Then, let $f_n^1 : [0,1] \to [0,1]$ be a C^1 strictly increasing function satisfying:

- $\|f_n - f_n^1\| \leq 2^{-j-n-2}$, $f_n^1(0) = 0$ and $f_n^1(1) = 1$,

- if we set $\kappa_{\max} = \max\{(f_n^1)'(x) : x \in [0,1]\}\}$ and $\kappa_{\min} = \min\{(f_n^1)'(x) : x \in [0,1]\}\}$, then $\kappa_{\min} > 0$.

We then choose a sufficiently large integer $J_{n,j} \geq j$ and a second function $f_n^2 : [0,1] \to [0,1]$ satisfying:

- $\|f_n^2 - f_n^1\| \leq 2^{-j-n-2}$, $f_n^2(0) = 0$ and $f_n^2(1) = 1$,
- f_n^2 is piecewise constant on every dyadic interval $I_{J_{n,j},k}$,
- for every $k \in \{0, 1, ..., 2^{J_{n,j}} - 1\}$, we have

$$(\kappa_{\min}/4)2^{-J_{n,j}} \leq f_n^2((k+1)2^{-J_{n,j}}) - f_n^2(k2^{-J_{n,j}}) \leq (4\kappa_{\max})2^{-J_{n,j}}. \quad (36)$$

This second function f_n^2 is still close to f_n with respect to the supremum norm. We assume without loss of generality that $J_{n,j}$ is chosen so large that

$$2^{J_{n,j}(1-\varepsilon)} > W_n,\ 2^{\varepsilon J_{n,j}} \geq 4\kappa_{\max} \text{ and } 2^{-\varepsilon J_{n,j}} < \kappa_{\min}/4 < 1 \leq \frac{\kappa_{\max}}{\kappa_{\min}} \leq 2^{\varepsilon J_{n,j}}. \quad (37)$$

We set

$$\begin{aligned} s_{\max,j} &= \max\{f_n^2((k+1)2^{-J_{n,j}}) - f_n^2(k2^{-J_{n,j}}) : k \in \{0,1,...,2^{J_{n,j}} - 1\}\} \\ s_{\min,j} &= \min\left\{f_n^2((k+1)2^{-J_{n,j}}) - f_n^2(k2^{-J_{n,j}}) : k \in \{0,1,...,2^{J_{n,j}} - 1\}\right\}. \end{aligned}$$

Obviously, the larger $J_{n,j}$, the smaller $s_{\max,j}$ and $s_{\min,j}$. By (36),

$$(\kappa_{\min}/4)2^{-J_{n,j}} \leq s_{\min,j} \leq s_{\max,j} \leq (4\kappa_{\max})2^{-J_{n,j}} \quad (38)$$

and hence by (37)

$$2^{-J_{n,j}(1+\varepsilon)} \leq s_{\min,j} \leq s_{\max,j} \leq 2^{-J_{n,j}(1-\varepsilon)} < W_n^{-1}. \quad (39)$$

By (38) we can also assume that $J_{n,j}$ is chosen so large that

$$(2/s_{\min,j})^{1-\alpha} - M \geq (2/s_{\min,j})^{1-\alpha-1/j} \text{ and } (s_{\min,j})^{1/j} \leq 1/16. \quad (40)$$

Next, we transform f_n^2 into a continuous piecewise affine function $f_{n,j}$ as in Definition 3.8. We choose an integer $\widehat{J_{n,j}} \gg J_{n,j}$ such that

$$2^{\widehat{J_{n,j}}} > 2^{\varepsilon \widehat{J_{n,j}}} \geq \max\{2^{j(J_{n,j})^2}, 2/s_{\min,j}, (2/s_{\max,j})^j\}. \quad (41)$$

Definition 4.1. *We define $f_{n,j}$ as follows:*

- $f_{n,j}$ *is continuous, piecewise affine on each dyadic interval* $I_{\widehat{J_{n,j}},k'}$, $k' \in \{0,1,...,2^{\widehat{J_{n,j}}}-1\}$, $f_{n,j}(0)=0$ *and* $f_{n,j}(1)=1$,

- *for every* $k' \in \{0,1,...,2^{\widehat{J_{n,j}}}-1\}$ *such that* $(k'+1)2^{-\widehat{J_{n,j}}} \notin 2^{-J_{n,j}}\cdot\mathbb{Z}$, $f_{n,j}(x) = f_n^2(x) = f_n^2(k'2^{-\widehat{J_{n,j}}})$ *for* $x \in I_{\widehat{J_{n,j}},k'}$ *(hence it is a constant)*

- *for every* $k' \in \{0,1,...,2^{\widehat{J_{n,j}}}-1\}$ *such that* $(k'+1)2^{-\widehat{J_{n,j}}} \in 2^{-J_{n,j}}\cdot\mathbb{Z}$, *that is,* $(k'+1)2^{-\widehat{J_{n,j}}} = K2^{-J_{n,j}}$, *the function* $f_{n,j}$ *is affine strictly increasing on* $I_{\widehat{J_{n,j}},k'}$ *and satisfies* $f_{n,j}(k'2^{-\widehat{J_{n,j}}}) = f_n^2((K-1)2^{-J_{n,j}})$ *and* $f_{n,j}((k'+1)2^{-\widehat{J_{n,j}}}) = f_n^2(K2^{-J_{n,j}})$.

As in the previous sections, $f_{n,j}$ is continuous and close to f_n with respect to the supremum norm, but it has a very particular behavior on the dyadic intervals of a certain generation.

Finally, we define $g_{n,j}$ as follows: For every $x \in [0,1]$,

$$g_{n,j}(x) = \bar{g}_n(x) + (2/s_{\min,j})^{-\alpha} H((2/s_{\min,j})x), \tag{42}$$

where H is defined in (4). By (39) we have $2/s_{\min,j} > 2W_n > W_n$ and hence by Lemma 2.6, $g_{n,j}$ belongs to Lip^α. The construction of the oscillating function $g_{n,j}$ is the same as in Sections 3.1 and 3.2, and is illustrated Figure 2.

Remark 4.2. *A key feature is that the functions* $(f_{n,j}, g_{n,j})$ *are the same for all* $h \in (0,\alpha)$. *This will be crucial later, since we aim to construct simultaneously an uncountable number of Cantor sets* $X^h_{f,g}$, *for all* $h \in (0,\alpha)$.

Definition 4.3. *Consider the functions* $f_{n,j}$ *and* $g_{n,j}$, *as well as the integer* $J_{n,j} \geq j$ *defined above. Let*

$$\delta_{n,j} := \min\left(2^{-\widehat{J_{n,j}}}, s_{\min,j}, (s_{\max,j})^j/2\right)/4, \tag{43}$$

where $\widehat{J_{n,j}}$ *was the integer defined by* (41). *We define* $\Omega_{j,\varepsilon}$ *and* $\mathcal{R}_\alpha$ *by*

$$\Omega_{j,\varepsilon} := \bigcup_{n\geq 1} B\Big((f_{n,j}, g_{n,j}), \delta_{n,j}\Big) \quad \text{and} \quad \mathcal{R}_\alpha := \bigcap_{j\geq 1} \Omega_{j,1/j}.$$

The set $\Omega_{j,\varepsilon}$ is a dense open set in $\mathcal{E}^\alpha$, and $\mathcal{R}_\alpha$ is a residual set.

Remark 4.4. *Observe that, as announced,* $\mathcal{R}_\alpha$ *does not depend on* h. *It is noticeable that all functions in* $\mathcal{R}_\alpha$ have the same multifractal behavior.

4.2. Oscillation properties of the composition $g \circ f$ when $(f, g) \in \Omega_{j,1/j}$

The procedure is slightly different from the one developed in the previous section. Here we exhibit several dyadic intervals $I_{p,j}(h)$ on which the oscillations of $g_{n,j} \circ f_{n,j}$ have various behavior, *for the same couple* $(f_{n,j}, g_{n,j})$.

Let $h \in (0, \alpha)$. Suppose that j is so large that we have

$$j > \alpha/h \quad \text{and} \quad (1 - 1/j)\alpha/h > 1. \tag{44}$$

We construct intervals $I_{p,j}(h)$ on which the oscillations of $g_{n,j} \circ f_{n,j}$ (as well as those of the composition of $(f, g) \in \mathcal{R}_\alpha$) will be controlled.

Let $p \in \{1, ..., 2^{J_{n,j}} - 1\}$. We set

$$I_{p,j}(h) = \Big[p2^{-J_{n,j}} - (s_{\max,j})^{\alpha/h}, p2^{-J_{n,j}} - (s_{\max,j})^{\alpha/h}/2\Big]. \tag{45}$$

The length of the interval $I_{p,j}(h)$ is by (38) less than $\frac{1}{2}(4\kappa_{\max}2^{-J_{n,j}})^{\alpha/h}$, which by (37) and $\varepsilon = 1/j$ is less than $\frac{1}{2}(2^{-J_{n,j}(1-1/j)})^{\alpha/h}$. Hence, by (44), $|I_{p,j}(h)|$ is smaller than $2^{-J_{n,j}}/2$, and these intervals are mutually disjoint.

By (39) used with $\varepsilon = 1/j$ we also have

$$2^{-J_{n,j}(1+1/j)(\alpha/h)-1} \leq |I_{p,j}(h)| = \frac{1}{2}(s_{\max,j})^{\alpha/h} \leq 2^{-J_{n,j}(1-1/j)(\alpha/h)-1}. \tag{46}$$

Lemma 4.5. *Suppose that $f_{n,j}$, $g_{n,j}$ and $J_{n,j}$ are defined as above. For every integer $p \in \{1, ..., 2^{J_{n,j}} - 1\}$, for every $x \in I_{p,j}(h)$,*

$$\omega_{B(x,2|I_{p,j}(h)|)}(g_{n,j} \circ f_{n,j}) \geq |I_{p,j}(h)|^{h(1+10/(\alpha j))}$$

when j is sufficiently large.

Proof: Let $p \in \{1, ..., 2^{J_{n,j}} - 1\}$, and consider $x \in I_{p,j}(h)$. Suppose that $p2^{-J_{n,j}} = (k'+1)2^{-\widehat{J_{n,j}}}$. By (41), $|I_{\widehat{J_{n,j}},k'}| = 2^{-\widehat{J_{n,j}}}$ is smaller than $\frac{1}{2}(s_{\max,j})^j$, which is less than $\frac{1}{2}(s_{\max,j})^{\alpha/h}$ when $j > \alpha/h$. Then, we have $I_{\widehat{J_{n,j}},k'} \subset [p2^{-J_{n,j}} - \frac{1}{2} \cdot (s_{\max,j})^{\alpha/h}, p2^{-J_{n,j}}]$. By (45), $p2^{-J_{n,j}}$ belongs to the closure of $B(x, 2|I_{p,j}(h)|)$, and hence by continuity of $f_{n,j}$,

$$\begin{aligned} \omega_{B(x,2|I_{p,j}(h)|)}(f_{n,j}) &\geq f_{n,j}(p2^{-J_{n,j}}) - f_{n,j}(x) \\ &\geq f_{n,j}((k'+1)2^{-\widehat{J_{n,j}}}) - f_{n,j}(k'2^{-\widehat{J_{n,j}}}) \geq s_{\min,j}. \end{aligned}$$

The interval $[f_{n,j}(x), f_{n,j}(p2^{-J_{n,j}})]$ contains a smaller interval of the form $[K(s_{\min,j}/4), (K+1)(s_{\min,j}/4)]$, for some $K \in \mathbb{N}$. On such an interval, the derivative of $g_{n,j}$ has constant sign and it satisfies by (40) and (42)

$$|g'_{n,j}| \geq (2/s_{\min,j})^{1-\alpha} - M \geq (2/s_{\min,j})^{1-\alpha-1/j}.$$

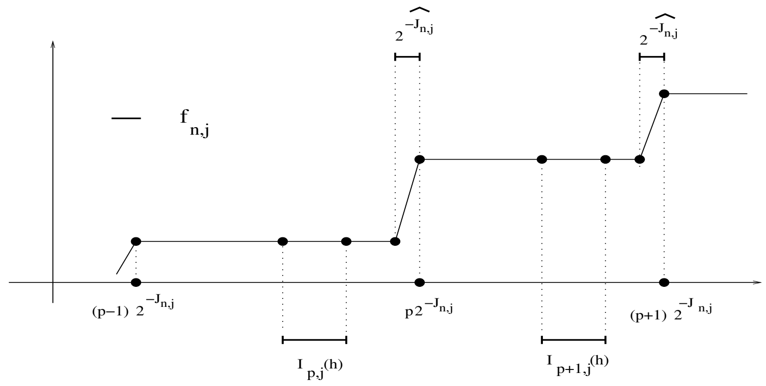

Figure 4: Illustration of $f_{n,j}$ and the relative positions of the intervals $I_p(h)$.

Consequently, by using (40) again we infer

$$\omega_{B(x,2|I_{p,j}(h)|)}(g_{n,j}\circ f_{n,j}) \geq (2/s_{\min,j})^{1-\alpha-1/j}(s_{\min,j}/4) \geq (s_{\min,j})^{\alpha+2/j}. \quad (47)$$

Applying (39) twice with $\varepsilon = 1/j$ and remembering (45), we see that

$$s_{\min,j} \geq (s_{\max,j})^{(1+1/j)/(1-1/j)} \geq (s_{\max,j})^{1+3/j} \geq |I_{p,j}(h)|^{\frac{h}{\alpha}(1+3/j)}$$

when j is sufficiently large. This and (47) yield

$$\omega_{B(x,2|I_{p,j}(h)|)}(g_{n,j}\circ f_{n,j}) \geq |I_{p,j}(h)|^{(\alpha+2/j)\frac{h}{\alpha}(1+3/j)} \geq |I_{p,j}(h)|^{h(1+10/(\alpha j))}. \quad \square$$

As above, this property of the oscillations of $g_{n,j}\circ f_{n,j}$ is stable with respect to the supremum norm in $\mathcal{E}^\alpha$.

Lemma 4.6. *For sufficiently large j, consider any couple of functions $(f,g) \in B\Big((f_{n,j}, g_{n,j}), \delta_{n,j}\Big)$, where $\delta_{n,j}$ is defined in* (43). *For every integer $p \in \{1,...,2^{J_{n,j}}-1\}$, and for every $x \in I_{p,j}(h)$, we have*

$$\omega_{B(x,2|I_{p,j}(h)|)}(g\circ f) \geq |I_{p,j}(h)|^{h(1+12/(\alpha j))+1/j}. \quad (48)$$

Proof: The proof is similar to the ones of Lemmas 3.4 and 3.10. $\square$

4.3. Statement of the properties of the Cantor sets $X^h_{f,g}$ and the measure μ^h

For a given exponent $h \in (0,\alpha)$ we are going to construct a Cantor set of dimension h/α containing points with Hölder exponents less than h for the composition $g\circ f$, where $(f,g) \in \mathcal{R}_\alpha$. In Section 4.7, we explain why this is achieved simultaneously for all h's.

Proposition 4.7. *Let $(f,g) \in \mathcal{R}_\alpha$ and $h \in (0,\alpha)$.*

There is a set $X^h_{f,g}$ of Hausdorff dimension h/α such that for every $x \in X^h_{f,g}$, $h_{g\circ f}(x) \leq h$.

More precisely, there is a probability measure μ^h supported on $X^h_{f,g}$ such that $\dim(\mu^h) \geq h/\alpha$ and $\mu^h(X^h_{f,g}) = 1$.

Proposition 4.7 is proved in Sections 4.4, 4.5 and 4.6.

Assume for a while that Proposition 4.7 is proved. We deduce the existence of points with exponents exactly equal to h (almost as in Proposition 3.7). Observe that in the next statement, the existence of the measure μ^h is crucial: the sole knowledge of the Hausdorff dimension of $X^h_{f,g}$ would not be sufficient to guarantee the existence of points of exact Hölder exponent h.

Corollary 4.8. *Let $(f,g) \in \mathcal{R}_\alpha$. Let $h \in (0,\alpha)$.*

There is a set $Y^h_{f,g}$ of Hausdorff dimension h/α such that for every $x \in Y^h_{f,g}$, $h_{g\circ f}(x) = h$, i.e. $Y^h_{f,g} \subset E_{g\circ f}(h)$. This implies that $d_{g\circ f}(h) = h/\alpha$.

Proof: By Proposition 4.7, there is a measure μ^h and a set $X^h_{f,g}$ such that $\mu^h(X^h_{f,g}) = 1$, $\dim(\mu^h) \geq h/\alpha$, and every point of $X^h_{f,g}$ has an exponent less than h for the composition $g \circ f$. Since $\dim(\mu^h) \geq h/\alpha$, the property (6) recalled in Section 2 implies that any set E of Hausdorff dimension strictly less than h/α satisfies $\mu^h(E) = 0$. Let us consider the set

$$Y^h_{f,g} = X^h_{f,g} \setminus \bigcup_{p\geq 1} \{x \in [0,1] : h_{g\circ f}(x) \leq h - 1/p\}.$$

By construction, $Y^h_{f,g}$ contains only points x at which the pointwise Hölder exponent of $g \circ f$ is exactly h.

We know by Proposition 2.8 that for every $p \geq 1$, the set $\{x \in [0,1] : h_{g\circ f}(x) \leq h - 1/p\}$ has dimension less than $(h-1/p)/\alpha$. Hence $\mu^h(\{x \in [0,1] : h_{g\circ f}(x) \leq h - 1/p\}) = 0$, and

$$\mu^h\Big(\bigcup_{p\geq 1} \{x \in [0,1] : h_{g\circ f}(x) \leq h - 1/p\} \Big) = 0,$$

and thus

$$\mu^h\Big(Y^h_{f,g}\Big) = \mu^h\Big(X^h_{f,g} \setminus \bigcup_{p\geq 1} \{x \in [0,1] : h_{g\circ f}(x) \leq h - 1/p\}\Big) = \mu^h\Big(X^h_{f,g}\Big) = 1.$$

Since $\dim(\mu^h) \geq h/\alpha$ and $\mu^h\Big(Y^h_{f,g}\Big) = 1$, $\dim Y^h_{f,g} \geq h/\alpha$. We know that $\dim Y^h_{f,g} \leq h/\alpha$ by Proposition 2.8, we conclude that $\dim Y^h_{f,g} = h/\alpha$. $\square$

4.4. Construction of $X^h_{f,g}$ and μ^h

Let $(f,g) \in \mathcal{R}_\alpha$ and $h \in (0,\alpha)$. The couple (f,g) belongs to an infinite number of sets $\Omega_{j,1/j}$, $j \geq 1$. We apply Lemma 4.6 for a suitable subsequence of integers depending on (f,g). In the sequel, the notation from Lemma 4.6 will be adjusted/simplified: J_m will play the role of the integer $J_{n,j}$, k will be used instead of p, and I^m_k will play the role of $I_{p,j}(h)$. We can also suppose that when we choose our suitable subsequence then at the m'th step we use sets $\Omega_{j,1/j}$ with $j \geq m$ and from (48) we will be able to deduce (52). Therefore, we select a strictly increasing sequence $(J_m)_{m\geq 1}$ for which the following properties hold:

- for every $m \geq 1$, $J_{m+1} \geq e^{J_m}$, $J_1 > 1$, which also implies $J_m/m > 1$,
- for every $m \geq 1$, there exist intervals I^m_k, for $k \in \{1, 2, ..., 2^{J_m} - 1\}$, which all have the same length, and which satisfy:
  - by (45) for every $k \in \{1, 2, ..., 2^{J_m} - 2\}$,

$$\text{there is exactly one interval } I^m_k \text{ in each interval } I_{J_m,k-1}, \tag{49}$$

$$\text{and } 2^{-J_m(1+1/m)} \leq \text{dist}(I^m_k, I^m_{k+1}) \leq 2 \cdot 2^{-J_m} = 2^{-J_m+1},$$

    where the left inequality holds for large m's
  - by (46) and $\frac{J_m}{m} \cdot \frac{\alpha}{h} > 1$, for every $k \in \{1, 2, ..., 2^{J_m} - 1\}$, we get

$$2^{-J_m\frac{\alpha}{h}(1+2/m)} \leq 2^{-J_m\frac{\alpha}{h}(1+1/m)-1} \leq |I^m_k| \leq 2^{-J_m\frac{\alpha}{h}(1-2/m)}, \tag{50}$$

    Since $\alpha/h > 1$ for sufficiently large m's, I^m_k and I^m_{k+1} do not intersect, and are far from each other (relatively to their size). We can also suppose that J_m is increasing so fast that

$$2^{-J_{m+1}} < |I^m_k|/100. \tag{51}$$

  - by (48) for every $k \in \{1, 2, ..., 2^{J_m} - 1\}$, for every $x \in I^m_k$,

$$\omega_{B(x,2|I^m_k|)}(g \circ f) \geq |I^m_k|^{h(1+12/(\alpha m))+1/m}. \tag{52}$$

These intervals I^m_k (more precisely, some of them) will constitute the basic intervals of a Cantor set $X^h_{f,g}$, on which we will build a measure μ^h such that the Mass Distribution Principle (Theorem 2.1) is applicable. We shall keep in mind that the intervals I^m_k depend on h, but since h is fixed in this section, for sake of simplicity we omit to denote explicitely this dependence.

The Cantor set $X^h_{f,g}$ and the measure μ^h are built recursively as follows:

- The first generation of intervals of $X^h_{f,g}$ consists of the intervals I^1_k. Then, a measure μ_1 is defined as follows: for every interval I^1_k, $k \in \{1, 2, ..., 2^{J_1} - 1\}$, we set

$$\mu_1(I^1_k) = \frac{1}{2^{J_1} - 2}.$$

  The probability measure μ_1 gives the same weight to each I^1_k (which all have the same length). We call $\mathcal{F}_1$ the set of these intervals, and we set $\Delta_1 = \#\mathcal{F}_1$.

  The measure μ_1 can be extended to a Borel probability measure on the algebra generated by $\mathcal{F}_1$, i.e. on $\sigma(L : L \in \mathcal{F}_1)$.

- Assume that we have constructed the first $n \geq 1$ generations of intervals $\mathcal{F}_1, \mathcal{F}_2, \dots , \mathcal{F}_n$ and the measure μ_n on the algebra $\sigma\big(L : L \in \mathcal{F}_n\big)$. Then we choose the intervals of generation $n+1$ as those intervals of the form I^{n+1}_k which are entirely included in one (and only one) interval I^n_k of generation n. By abuse of notation, we still denote them by I^{n+1}_k, and we call $\mathcal{F}_{n+1}$ the set consisting of them. We also set for every $I' \in \mathcal{F}_n$,

$$\Delta^{I'}_{n+1} = \#\{I \in \mathcal{F}_{n+1} : I \subset I'\}.$$

  Then we define the measure μ_{n+1}: For every interval $I \in \mathcal{F}_{n+1}$, we set

$$\mu_{n+1}(I) = \mu_n(I') \frac{1}{\Delta^{I'}_{n+1}},$$

  where I' is the unique interval of generation n such that $I \subset I'$. Observe that we can use (49), (50) and (51) to obtain

$$\mu_{n+1}(I) \leq \mu_n(I') \frac{1}{(|I'|/2^{-J_{n+1}}) - 2} < \mu_n(I') \, |I'|^{-1} \, 2^{-J_{n+1}+1}. \quad (53)$$

  The probability measure μ_{n+1} can be extended to a Borel probability measure on the algebra $\sigma(L : L \in \mathcal{F}_{n+1})$ generated by $\mathcal{F}_{n+1}$.

By construction, for every $n \geq 1$, for every k, properties (50) and (52) are satisfied by the intervals I^n_k. We set

$$X^h_{f,g} = \bigcap_{n \geq 1} \; \bigcup_{I \in \mathcal{F}_n} \; I.$$

By the Kolmogorov extension theorem, $(\mu_n)_{n \geq 1}$ converges weakly to a Borel probability measure μ^h supported on $X^h_{f,g}$ and for $I \in \mathcal{F}_n$, $\mu^h(I) = \mu_n(I)$.

4.5. Hausdorff dimension of $X^h_{f,g}$ and μ^h

We start by proving that the measure μ^h has an almost monofractal behavior on the intervals belonging to $\bigcup_n \mathcal{F}_n$.

Lemma 4.9. *When n is sufficiently large, for every interval $I \in \mathcal{F}_n$,*

$$|I|^{\frac{h}{\alpha}+\frac{1}{|\log |I||}} \leq \mu^h(I) \leq |I|^{\frac{h}{\alpha}-\frac{1}{|\log |I||}}. \tag{54}$$

Proof: Due to the inequalities (49) and (50), for every interval $I \in \mathcal{F}_n$, the number of intervals of generation $n+1$ included in I is bounded by

$$1/4 \cdot 2^{J_{n+1}(1-1/(n+1))} 2^{-J_n(\frac{\alpha}{h})(1+2/n)} \leq \Delta^I_{n+1} \leq 2 \cdot 2^{J_{n+1}+1} 2^{-J_n(\frac{\alpha}{h})(1-2/n)}.$$

Using that $J_{n+1} \geq e^{J_n}$ and $2^{-(\alpha/h)} < 1$, it can be simplified into

$$2^{J_{n+1}(1-2/(n+1))} \leq \Delta^I_{n+1} \leq 2^{J_{n+1}}, \tag{55}$$

when n is large enough, that is $n \geq n_0$. Suppose that $I \in \mathcal{F}_n$, and for $p \leq n$ denote by I_p the unique interval in $\mathcal{F}_p$ containing I. We obtain

$$\mu^h(I_{n_0}) \cdot \Big(\prod_{p=n_0+1}^{n} \Delta_p^{I_{p-1}} \Big)^{-1} = \mu^h(I). \tag{56}$$

The key property is that in equation (55), the bounds are uniform in $I \in \mathcal{F}_n$. Hence, by (55) and (56),

$$\mu^h(I_{n_0}) \cdot \Big(\prod_{p=n_0+1}^{n} 2^{J_p} \Big)^{-1} \leq \mu^h(I) \leq \mu^h(I_{n_0}) \cdot \Big(\prod_{p=n_0+1}^{n} 2^{J_p(1-2/p)} \Big)^{-1}.$$

Recalling that $J_p \geq e^{J_{p-1}}$ for every p, we see that there exists $n_1 \geq n_0$ such that for $n \geq n_1$ we have

$$2^{-J_n(1+1/n)} \leq \mu^h(I) \leq 2^{-J_n(1-3/n)}.$$

This means that, the measure μ^h is almost uniformly distributed on the intervals of the same generation. Since these intervals $I \in \mathcal{F}_n$ have same lengths, we use (50) to get

$$|I|^{\frac{h}{\alpha}(1+1/n)/(1-2/n)} \leq \mu^h(I) \leq |I|^{\frac{h}{\alpha}(1-3/n)/(1+2/n)}. \tag{57}$$

We can choose $n_2 \geq n_1$ such that for $n \geq n_2 - 1$ we have

$$|I|^{\frac{h}{\alpha}(1+4/n)} \leq \mu^h(I) \leq |I|^{\frac{h}{\alpha}(1-6/n)}. \tag{58}$$

Finally, we remark that by (50) and $J_p \geq e^{J_{p-1}}$, $n = o(|\log |I||)$ when $I \in \mathcal{F}_n$ is arbitrary, hence (58) yields (54). $\square$

We will need later that from (58) and from (53) used with $n-1$ instead of n for the limit measure μ^h we obtain that for $n \geq n_2$, if $I' \in \mathcal{F}_{n-1}$, $I \in \mathcal{F}_n$ and $I \subset I'$, then

$$\mu^h(I) \leq |I'|^{\frac{h}{\alpha}(1-\frac{6}{n-1})} \cdot 2^{-J_n+1}|I'|^{-1} = 2^{-J_n+1}\,|I'|^{\frac{h}{\alpha}(1-\frac{6}{n-1})-1}. \tag{59}$$

In order to apply the Mass Distribution Principle to μ^h and $X^h_{f,g}$, we need Lemma 4.10, which allows us to extend (54) and Lemma 4.9 to all Borel subsets of $[0,1]$.

Lemma 4.10. *There is a continuous increasing mapping $\psi : \mathbb{R}^+ \to \mathbb{R}^+$, satisfying $\psi(0) = 0$, and there is $\delta_{MDP} > 0$, such that for any Borel set $B \subset [0,1]$ with $|B| < \delta_{MDP}$ we have*

$$\mu^h(B) \leq |B|^{\frac{h}{\alpha}-\psi(|B|)}. \tag{60}$$

Using $B = B(x,r)$ in (60) and recalling (5) we obtain that $\underline{\dim}_{\mathrm{loc}}\mu(x) \geq h/\alpha$ for μ^h a.e. x and hence $\dim(\mu^h) \geq h/\alpha$.

Proof: Fix $\varepsilon_1 = 2^{-1}$, a Borel set $B \subset [0,1]$ with $|B| < 2^{-J_{n_2}} = \delta_0$. Let $n > n_2$ be the unique integer such that

$$2^{-J_n} \leq |B| < 2^{-J_{n-1}}. \tag{61}$$

- $\mathbf{|B| \geq 2^{-J_{n-1}(1+2/(n-1))(\alpha/h)}}$: By (61), B intersects at most two intervals $I \in \mathcal{F}_{n-1}$. If there is no such interval then $\mu^h(B) = 0$. Otherwise denoting by I one of these intervals, using (50) and (57), we find that

$$\begin{aligned}\mu^h(B) &\leq 2\cdot\mu^h(I) \leq 2\cdot|I|^{\frac{h}{\alpha}(1-3/(n-1))/(1+2/(n-1))}\\ &\leq 2\cdot\left(2^{-J_{n-1}\frac{\alpha}{h}(1-2/(n-1))}\right)^{\frac{h}{\alpha}(1-3/(n-1))/(1+2/(n-1))}\\ &\leq 2\cdot 2^{-J_{n-1}(1-3/(n-1))} \leq 2\cdot|B|^{-J_{n-1}\frac{h}{\alpha}\frac{1-3/(n-1)}{1+2/(n-1)}}.\end{aligned}$$

When n is sufficiently large, the last term in the above inequality is less than $|B|^{\frac{h}{\alpha}-\varepsilon_1}$. Recall that n is related to the diameter of B (the smaller $|B|$ is, the larger n becomes). We can thus choose $\delta_1 \in (0,\delta_0)$ so that

$$\text{when } |B| \leq \delta_1, \quad \mu^h(B) \leq |B|^{\frac{h}{\alpha}-\varepsilon_1}.$$

- $\mathbf{|B| < 2^{-J_{n-1}(1+2/(n-1))(\alpha/h)}}$**:** for large n's, B intersects at most one interval $I' \in \mathcal{F}_{n-1}$. If there is no such interval then $\mu^h(B) = 0$. Hence we need to deal with the case when such an interval I' exists. By (50), $|B| < |I'|$ for large n's. By (49), B can intersect at most $\frac{|B|}{2^{-J_n}} + 2 < 4 \cdot |B| \cdot 2^{J_n}$ intervals $I \in \mathcal{F}_n$. Recalling that $h/\alpha < 1$, using (59) and (61) we can bound by above the μ^h-measure of intervals $I \in \mathcal{F}_n$ in the following way:

$$\begin{aligned} \mu^h(B) &\leq 4|B|2^{J_n} \cdot \mu^h(I) \leq |B|2^{J_n+2} \cdot 2^{-J_n+1} |I'|^{\frac{h}{\alpha}(1-\frac{6}{n-1})-1} \\ &= 8|B| \cdot |I'|^{\frac{h}{\alpha}(1-\frac{6}{n-1})-1} \leq |B| \cdot |B|^{\frac{h}{\alpha}(1-\frac{6}{n-1})-1} \leq |B|^{\frac{h}{\alpha}-\varepsilon_1}, \end{aligned}$$

the last inequality being true for large n, i.e. for Borel sets B of diameter small enough (by the same argument as above).

Fix now $\varepsilon_2 = 2^{-2}$. By the same method as above, we find $0 < \delta_2 < \delta_1$ such that if $|B| \leq \delta_2$,

$$\mu^h(B) \leq |B|^{\frac{h}{\alpha}-\varepsilon_2}.$$

We iterate the procedure: $\forall\ p > 1$, there is $0 < \delta_p < \delta_{p-1}$ such that

$$\text{if } |B| \leq \delta_p, \quad \mu^h(B) \leq |B|^{\frac{h}{\alpha}-\varepsilon_p}, \qquad \text{where } \varepsilon_p = 2^{-p}.$$

In order to conclude, we consider a map ψ built as an increasing continuous interpolation function which goes through the points $(\delta_{p+1}, \varepsilon_p)_{p\geq 1}$ and (δ_1, ϵ_1). The shift in the indices in the sequence is introduced so that $\varepsilon_p \leq \psi(x) \leq \varepsilon_{p-1}$ holds for $x \in [\delta_{p+1}, \delta_p]$. Hence (60) holds true for every Borel set B satisfying $|B| \leq \delta_{MDP} := \delta_1$. □

Obviously, $X^h_{f,g}$ is the support of μ^h. From Lemma 4.10, we see that for all $\varepsilon > 0$ we have

$$\mu^h(B) \leq |B|^{\frac{h}{\alpha}-\varepsilon},$$

for B of diameter small enough. The Mass Distribution Principle allows us to deduce that $\dim X^h_{f,g} \geq \dim \mu^h \geq \frac{h}{\alpha} - \varepsilon$. Letting ε tend to zero leads to $\dim X^h_{f,g} \geq \dim \mu^h \geq \frac{h}{\alpha}$, which finishes the proof of Proposition 4.7. □

4.6. Regularity properties of $g \circ f$ when $(f, g) \in \mathcal{R}_\alpha$

To conclude, we must prove the composition $g \circ f$ has pointwise Hölder exponent less than h at every $x \in X^h_{f,g}$. This is relatively easy.

Lemma 4.11. *For every $x \in X^h_{f,g}$, $h_{g\circ f}(x) \leq h$.*

Proof: If $x \in X^h_{f,g}$, then x belongs to an infinite number of intervals I whose lengths tend to zero satisfying (50) and (52). Therefore,

$$\omega_{B(x,2|I|)}(g \circ f) \geq |I|^{h(1+12/(\alpha m))+1/m}.$$

Since $h(1 + \frac{12}{\alpha m}) + \frac{1}{m} \to h$ when $m \to \infty$, Lemma 2.2 yields $h_{g\circ f}(x) \leq h$. □

4.7. Simultaneous construction of all Cantor sets $Y^h_{f,g}$ when $(f,g)\in\mathcal{R}_\alpha$

Here we just observe that the residual set $\mathcal{R}_\alpha$ constructed above does not depend on the exponent h. Hence, when $(f,g)\in\mathcal{R}_\alpha$ is fixed, the result of Corollary 4.8 is true for every $h\in(0,\alpha)$. This simple remark is the consequence of the careful choice of the functions $f_{n,j}$ and $g_{n,j}$.

5. The case of continuous functions: Last part of Theorem 1.5

Here, all we need to prove is that there is a residual set $\mathcal{R}^0_0\in\mathcal{E}^0$, such that for every couple $(f,g)\in\mathcal{R}^0_0$, for every $x\in[0,1]$, $h_{g\circ f}=0$. In other words, for every couple $(f,g)\in\mathcal{R}^0_0$, $E_{g\circ f}(0)=[0,1]$.

To this end, we only need to establish the analog of Proposition 3.1 with $\alpha=0$. We explain how to adapt Proposition 3.1 to the context of $\mathcal{E}^0$.

Denote $\alpha_j=1/(j+1)$, for every $j\geq 1$.

Fix $\varepsilon>0$ and $j\geq 1$. Applying the method developed in Section 3, we construct the functions $f_{n,j}$ and $g_{n,j}$ and then the set $\Omega^{\alpha_j}_{j,\varepsilon}$ exactly as $\Omega^{\alpha}_{j,\varepsilon}$ was built. The only difference is that exponents α_j are used instead of α.

Consider then the set

$$\mathcal{R}^0_0:=\bigcap_{j\geq 1}\Omega^{\alpha_j}_{j,1/j}.$$

Obviously $\mathcal{R}^0_0$ is a G_δ-set, and since α_j tends to zero when j tends to infinity, the same arguments as in Section 3 yield that for every couple $(f,g)\in\mathcal{R}^0_0$,

$$\text{for every } x\in[0,1],\quad h_{g\circ f}(x)\leq \lim_{j\to+\infty}\alpha_j=0.$$

Since $g\circ f$ is continuous, we conclude that for all $x\in[0,1]$, $h_{g\circ f}(x)=0$.

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