Micro Tangent sets of typical continuous functions

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Abstract

In this paper we continue our study of the micro tangent set system of typical (generic) continuous functions in $C[0,1]$. We show that at a typical point of any level set of such functions we have two-sided vertical universality. At one-sided accumulation points of the level sets we have one-sided vertical universality with the exception of Hausdorff dimension zero of α's when the level set $L_{\alpha,f}$ contains one exceptional one-sided accumulation point where we do not have one-sided vertical universality. Based on the Bruckner-Garg characterization of the

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total level set structure of the typical continuous functions the micro tangent sets at the local extrema of $f_\gamma(x) = f(x) + \gamma x, (\gamma \in \mathbb{R})$ are investigated. It is shown that for the typical $f \in C[0,1]$ for any $\gamma \in \mathbb{R}$ we have half-universality at any local extrema of f_γ.

1 Introduction

This paper is a continuation of [3]. We consider $C[0,1]$ endowed with the topology coming form the supremum norm. In case $f \in C[0,1]$ is differentiable at $(x_0; f(x_0))$ its micro tangent set system $f_{MT}(x_0)$ consists of one element, the piece of line $y = f'(x_0)x$ belonging to $\{(x; y) : |x| \le 1, |y| \le 1\}$. The concept of micro tangent sets is interesting for functions with irregular behavior, like "fractal" functions, or the typical (generic) continuous function in the Baire category sense. In [3] we denoted by $UMT(f)$ the set of those points $(x_0; f(x_0))$ where the micro tangent set system of f is universal. By this we mean that if we take any $g \in C[-1,1]$ with $g(0) = 0$ then $graph(g) \cap Q^2$ belongs to $f_{MT}(x_0)$. It was an interesting result of [3], (Theorem 5 of [3]), that for the typical continuous function for almost every $x_0 \in [0,1]$ the point $(x_0; f(x_0))$ belongs to $UMT(f)$.

On the other hand, by Theorem 7 of [3] for the typical continuous function for almost every $y \in \mathbb{R}$ there is no $x_0 \in [0,1]$ such that $(x_0; f(x_0)) \in UMT(f)$ and $f(x_0) = y$, that is, the projection of $UMT(f)$ onto the y-axis is of zero Lebesgue-measure. Furthermore, Theorem 2 of [3] implies that for any continuous function $UMT(f)$ is of σ-finite $\mathcal{H}^1$-measure (one-dimensional Hausdorff measure). By a result of R.D. Mauldin and S.C. Williams [6] the graph of the typical continuous function is of Hausdorff dimension one, but is not of σ-finite-$\mathcal{H}^1$-measure.

Therefore, results of [3] are not giving more information about the micro tangent set system when we consider "most points" on the graph of the typical continuous function, or we consider level sets.

In this paper we want to explore the behaviour of the typical continuous function at some of those points where we do not have universal micro tangent system. We obtain results which apply to all, or almost all level sets and to points on the graph which are of non-σ-finite $\mathcal{H}^1$-measure.

Theorem 2 of [3] shows, in fact, that for any continuous function the graph-like, or central graph-like micro tangent points are of σ-finite $\mathcal{H}^1$-measure. Therefore, if we are interested in larger sets on the graph of the typical continuous function then instead of looking for points $(x_0; f(x_0))$ where the micro tangent set system contains the intersection of the graph of a continuous function g with Q^2 we look for points $(x_0; f(x_0))$ where $f_{MT}(x_0)$ contains sets which consist of vertical line segments, that is, sets which are of the form $F \times [-1, 1]$, where $F \subset [-1, 1]$ is an arbitrary closed set with $0 \in F$. From Theorem 18 it follows for the typical continuous function that if $\alpha \in (min_f, Max_f)$ then at a typical point on the level set, $L_{\alpha,f}$ we have two-sided vertical universality. This means that at these points any set $F \times [-1, 1]$ of the above form belongs to the micro tangent set system of f.

By the well-known Bruckner-Garg result [2] for the typical continuous function a level set $L_{\alpha,f}$ either equals a nowhere dense perfect set $P_{\alpha,f}$, or contains an isolated point corresponding to a local extrema and a nowhere dense perfect set $P_{\alpha,f} \neq \emptyset$. It is natural to explore the behavior of typical continuous functions at some specific points of $L_{\alpha,f}$. In Section 2 of our paper we are interested in the local extrema.

Motivated by the Bruckner-Garg total level set structure characterization [2] we consider local extrema of $f_\gamma(x) = f(x) + \gamma x$ for all $\gamma \in \mathbb{R}$ simultaneously. At a local maximum we cannot except universality, or vertical universality but it is natural to ask whether for a function $g \in C[-1, 1]$, $g \leq 0, g(0) = 0$, the intersection of the graph of g with Q^2 belongs to the micro tangent set system of f. Indeed, in Theorem 1 we show that for typical continuous functions for any $\gamma \in \mathbb{R}$ at any local maximum of f_γ for any above g, $graph(g) \cap Q^2$ belongs to the micro tangent set system of f_γ, that is, we have "half-universality" at this local maxima. Similar results hold about local minima. The main difficulty in the proof of these theorems is that we want to obtain half-universality at all local extrema of f_γ for all $\gamma \in \mathbb{R}$ simultaneously. In case we consider one fixed γ we need to consider only countably many points, while the result concerning all γ's gives information about the micro tangent system at points belonging to a c-dense subset on the graph of f.

Apart from local extrema there are other points on a level set $L_{\alpha,f}$ where we cannot expect universality, half-universality, or two-sided vertical universality. These are the endpoints of intervals contiguous to $P_{\alpha,f}$. If $(d; f(d)) = (d; \alpha)$ is the left endpoint of an interval contiguous to $P_{\alpha,f}$ then it is a right accumulation point of $P_{\alpha,f}$ and one can ask whether we have right vertical universality at $(d; \alpha)$. By this we mean that if we take an arbitrary closed set $F \subset [0, 1]$, with $0 \in F$ then $F \times [-1, 1]$ belongs to the micro tangent set of f at $(d; \alpha)$.

In Theorem 21 we show that for the typical continuous function for most level sets at any point of $P_{\alpha,f}$ which is not a two-sided accumulation point of $P_{\alpha,f}$ we have one-sided vertical universality, while there will be a small (of zero Hausdorff dimension) exceptional set of α's where there is just one exceptional endpoint of an interval continguous to $P_{\alpha,f}$ where we do not have one-sided vertical universality. These exceptional α's exist, since the local extrema of $f_\gamma, \gamma \neq 0$ are endpoints of intervals contiguous to $P_{\alpha,f}$ but we cannot have one-sided vertical universality for f at these points. This also shows that the c-dense subset of the graph of f consisting of $(x_0; f(x_0))$ for which there exists $\gamma \in \mathbb{R} \setminus \{0\}$ such that $(x_0; f_\gamma(x_0))$ is a local extrema of f_γ projects on the y-axis into a very small set and each level set $L_{\alpha,f}$ can contain at most one such point.

Next we recall some notation and a few definitions from [3]. Points in $\mathbb{R}^2$ will be denoted by $(x; y)$, while (x, y) denotes the open interval in $\mathbb{R}$ with endpoints x, y. The closed cube of side length $2\delta > 0$ centered at $(x; y)$ will be denoted by $Q((x; y), \delta)$, that is, $Q((x; y), \delta) = \{(x'; y') : |x' - x| \leq \delta \text{ and } |y' - y| \leq \delta\}$. We have already mentioned $Q^2 = Q((0; 0), 1)$. The Hausdorff distance of two compact sets $A, B \subseteq \mathbb{R}^2$ will be denoted by $dist_{\mathcal{H}}(A, B)$.

Given $f \in C[0, 1], x_0 \in [0, 1], \delta > 0$ we put

$$F(f, x_0, \delta) = \frac{1}{\delta}\Big(\big(graph(f) \cap Q((x_0; f(x_0)), \delta)\big) - (x_0; f(x_0))\Big), \tag{1}$$

that is, to obtain $F(f, x_0, \delta)$ we translate the part of $graph(f)$ belonging to $Q((x_0; f(x_0)), \delta)$ into Q^2 and enlarge it $1/\delta$-times. The set F^* is a *micro tangent set* of f at x_0, that is, $F^* \in f_{MT}(x_0)$ if there exists $\delta_n \searrow 0$ such that $dist_{\mathcal{H}}(F(f, x_0, \delta_n), F^*)$ converges to 0 as $n \to \infty$.

2 Half-universality at local extrema

First we recall some results about the level sets of typical continuous functions. For an arbitrary real function f put $L_{\alpha,f} = \{(x; f(x)) : f(x) = \alpha\}$, that is, $L_{\alpha,f}$ is the horizontal level set at height α.
Denote by $P_{\alpha,f}$ the accumulation points of $L_{\alpha,f}$, by $P^+_{\alpha,f}$ the points in $P_{\alpha,f}$ which are only right, and by $P^-_{\alpha,f}$ the points in $P_{\alpha,f}$ which are only left accumulation points of $P_{\alpha,f}$. Finally, set $P^\pm_{\alpha,f} = P^-_{\alpha,f} \cup P^+_{\alpha,f}$.

For an $f \in C[0,1]$, min_f and Max_f denote the minimum and maximum of f on $[0,1]$. Given a property we say that the typical continuous function has this property if the set of functions in $C[0,1]$ having this property is residual. By the well-known Bruckner-Garg result [2] the typical continuous function f in $C[0,1]$ satisfies the following BG-property:

- there exists a denumerable dense set S_f in (min_f, Max_f) such that if $\alpha \notin S_f \cup \{min_f, Max_f\}$ then $L_{\alpha,f} = P_{\alpha,f}$ which is nowhere dense and perfect,
- $L_{\alpha,f}$ is a single point if $\alpha \in \{min_f, Max_f\}$,
- $L_{\alpha,f}$ is the union of a single point and $P_{\alpha,f} \neq \emptyset$, a nowhere dense and perfect set, if $\alpha \in S_f$.

In the latter case the isolated point on the level set corresponds to a local extremum.

The set of functions satisfying the BG-property will be denoted by BG. We also refer to [1] for the total level set structure of typical continuous functions. Given $f \in C[0,1]$ and $\gamma \in \mathbb{R}$ put $f_\gamma(x) = f(x) + \gamma x$. For the typical continuous function there exists Γ_f, a countable dense subset in $\mathbb{R}$ such that if $\gamma \in \mathbb{R} \setminus \Gamma_f$, f_γ has the BG-property and if $\gamma \in \Gamma_f$ then f_γ meets all conditions of the BG-property, with the exception of one level set which contains two isolated points instead of one.
We denote by BG^* the set of those functions in $C[0,1]$ which satisfy the above property about the total level set structure of the typical continuous functions.
By $C[-1,1]_{0,-}$, or $C[-1,1]_{0,+}$ we denote the set of those functions g in $C[-1,1]$ for which $g(0) = 0$ and $g \leq 0$, or $g \geq 0$, respectively.

We say that f is *half-universal* at a local maximum, $(x_0; f(x_0))$ if

$$graph(g) \cap Q^2 \in f_{MT}(x_0)$$

for every $g \in C[-1,1]_{0,-}$. One can similarly define half-universality at a local minimum.

Theorem 1. *For the typical continuous function $f \in C[0,1]$ for any $\gamma \in \mathbb{R}$ the function f_γ is half-universal at all of its extrema.*

Definition 2. Given $\kappa \in (0, \frac{1}{2})$ we denote by $\mathcal{G}(\kappa)$ those $g \in C[-2, 2]$ for which

- $g(0) = 0$, $g(x) \leq -\kappa|x|$ for all $x \in [-2, 2]$, $g(-2) = g(2) = -1$,
- g is piecewise linear without intervals of constancy,
- g has no local extrema on the boundary of Q^2,
- all of its extrema are of different y coordinate and
- the graph of g is not going through any of the vertices of Q^2.

Clearly, we can choose countable sets $g_\nu, \kappa_\nu, \nu = 1, 2, 3, ...$ such that, $\kappa_\nu > 0$, $g_\nu \in \mathcal{G}(\kappa_\nu)$ and $g_\nu \mid_{[-1,1]}$, $\nu = 1, 2, ...$ is dense in $C[-1, 1]_{0,-}$.

Lemma 3. *Given $\varepsilon > 0$ and $g \in \mathcal{G}(\kappa)$ for a $\kappa \in (0, \frac{1}{2})$ there exists $\eta_{\varepsilon,g} > 0$ such that if $|\phi(x) - g(x)| < \eta_{\varepsilon,g}$ for $x \in [-1, 1]$ and $\phi \in C[-1, 1]$ then*

$$dist_{\mathcal{H}}(graph(\phi) \cap Q^2, graph(g) \cap Q^2) < \varepsilon.$$

Proof. Since $g \in \mathcal{G}(\kappa)$ we can choose $\eta_0 \in (0, 1/2)$ such that if $0 < \eta < \eta_0$ then whenever $(x_0; g(x_0))$ is a local extremum of g on $[-1, 1]$ then from $(x_0; g(x_0)) \in Q^2$ it follows that $(x_0; g(x_0) + \eta) \in Q^2$ and $(x_0; g(x_0) - \eta) \in Q^2$, moreover from $(x_0; g(x_0)) \notin Q^2$ it follows that $(x_0; g(x_0) + \eta) \notin Q^2$ and $(x_0; g(x_0) - \eta) \notin Q^2$.

Put $L_\eta = \{(x; y) : x \in [-3/2, 3/2], |y - g(x)| \leq \eta\}$.
Since $g \in \mathcal{G}(\kappa)$ we can assume that η_0 is chosen so small that for $\eta \leq \eta_0$, L_η does not contain a vertex of Q^2. Then $Q^2 \cap L_\eta$ can be divided into parallelograms and triangles so that two sides of the parallelograms are parts of the graphs of $g(x) + \eta$ and $g(x) - \eta$ while the other two sides are parallel to the y axis, one side of the small triangles is part of $g(x) + \eta$ or $g(x) - \eta$ while the other two sides are parallel to the x, or y axis. See Figure 1. The horizontal side of these triangles is part of a horizontal side of Q^2, the vertical sides of these triangles are connecting the graphs of $g(x) + \eta$ and $g(x) - \eta$ and hence are of length 2η. Since g is piecewise linear and has no interval of constancy there exists $\tau > 0$ such that the absolute value of the slope of any line segment making up the graph of g is at least τ. Hence the horizontal side of the above small triangles is of length at most $2\eta/\tau$.

This implies that the diameter of the small triangles is less than $2\eta\sqrt{1 + \tau^{-2}}$.

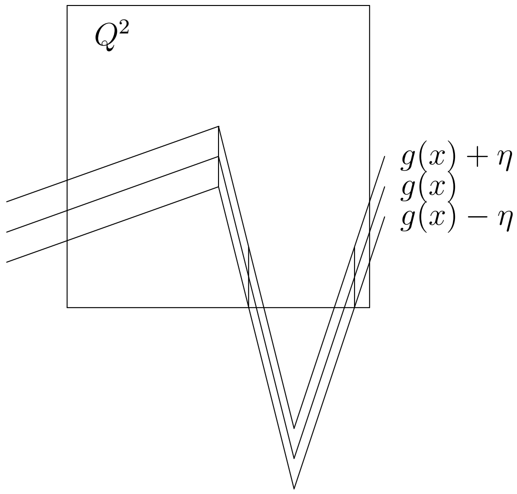

Figure 1: Division of L_η

We choose $\eta_{\varepsilon,g} < \eta_0$ such that $2\eta_{\varepsilon,g}\sqrt{1+\tau^{-2}} < \varepsilon$.

Assume that $|\phi(x) - g(x)| < \eta_{\varepsilon,g}$ for $x \in [-1,1]$. Then we consider two cases. If $(x;\phi(x)) \in Q^2 \cap L_{\eta_{\varepsilon,g}}$ and $(x;\phi(x))$ belongs to a parallelogram in the above subdivision of $Q^2 \cap L_{\eta_{\varepsilon,g}}$ then $(x; g(x))$ also belongs to this parallelogram and $dist((x;\phi(x)),(x;g(x))) < 2\eta_{\varepsilon,g}$.

If $(x;\phi(x)) \in Q^2 \cap L_{\eta_{\varepsilon,g}}$ and $(x;\phi(x))$ belongs to a triangle in the above subdivision of $Q^2 \cap L_{\eta_{\varepsilon,g}}$ then the intersection point of the graph of $g(x)$ and the horizontal side of the triangle also belongs to this triangle. Furthermore, the graph of $\phi(x)$ also intersects this triangle by our assumption. The diameter of this triangle is less than $2\eta_{\varepsilon,g}\sqrt{1+\tau^{-2}} < \varepsilon$.

Hence, for any point of $graph(\phi) \cap Q^2$ there is a point of $graph(g) \cap Q^2$ of distance less than $2\eta_{\varepsilon,g}\sqrt{1+\tau^{-2}} < \varepsilon$ and for any point of $graph(g) \cap Q^2$ there is a point of $graph(\phi) \cap Q^2$ of distance less than $2\eta_{\varepsilon,g}\sqrt{1+\tau^{-2}} < \varepsilon$.

This implies $dist_{\mathcal{H}}(graph(g) \cap Q^2, graph(\phi) \cap Q^2) < \varepsilon$. □

Given a perturbation vector (x_q, y_q, γ_q) and a function g set $g_{(x_q,y_q,\gamma_q)}(x) = g(x - x_q) + \gamma_q x + y_q$.

Lemma 4. *Given $\eta > 0, g \in \mathcal{G}(\kappa)$ there exists $\delta_\eta > 0$ such that*

$$\text{if } |x_q|, |y_q|, |\gamma_q| < \delta_\eta \text{ then } |g_{(x_q,y_q,\gamma_q)}(x) - g(x)| < \eta \text{ for } x \in [-1,1].$$

Proof. Since g is continuous, it is uniformly continuous on $[-2,2]$. Obviously, if $\delta_\eta < \min(1/2, \eta/6)$ is chosen so that $|g(x-x_q) - g(x)| < \eta/3$ for all $x, x-x_q \in [-2,2]$, with $|x_q| < \delta_\eta$ then $|g_{(x_q,y_q,\gamma_q)}(x) - g(x)| \leq |g(x-x_q) - g(x)| + |\gamma_q| + |y_q| < 3\eta/3$ holds for any $x \in [-1,1]$. □

Lemma 5. *Given any interval $(a,b) \subset [0,1]$ and $f \in C[0,1]$ there exist $\alpha, \beta \in \mathbb{R} \cup \{+\infty, -\infty\}$, $\alpha \leq \beta$, with the property that for all $\gamma \in (\alpha,\beta)$, f_γ takes its maximum on $[a,b]$ at a point $x_{Max,\gamma} \in (a,b)$, $f_\gamma(x_{Max,\gamma}) > \max\{f_\gamma(a), f_\gamma(b)\}$ and for $\gamma \notin [\alpha,\beta]$, f_γ does not take its maximum on $[a,b]$ in (a,b) .*

Proof. Set $\alpha = sup\{\gamma : f_\gamma(x)$ has its maximum on $[a,b]$ at $a\}$, and $\beta = inf\{\gamma : f_\gamma(x)$ has its maximum on $[a,b]$ at $b\}$.

It is left to the reader to verify that the interval (α,β) satisfies the lemma. □

Definition 6. For a fixed (a,b) and a function $f \in C[0,1]$ we denote by (α_f, β_f) the interval given by Lemma 5.

Lemma 7. *Assume $f \in C[0,1], (a,b) \subset [0,1]$ and $\varepsilon > 0$ are given. Then there exists $\eta > 0$ such that if $\tilde{f} \in B(f,\eta)$ then $\alpha_{\tilde{f}} < \alpha_f + \varepsilon$ for $\alpha_f \in \mathbb{R}$ and $\alpha_{\tilde{f}} < -\frac{1}{\varepsilon}$ for $\alpha_f = -\infty$, similarly $\beta_{\tilde{f}} > \beta_f - \varepsilon$ for $\beta_f \in \mathbb{R}$ and $\beta_{\tilde{f}} > \frac{1}{\varepsilon}$ for $\beta_f = +\infty$.*

Proof. Assume $\alpha_f \in \mathbb{R}$. Choose $\gamma > \alpha_f$ for which $\gamma < \alpha_f + \varepsilon/2$. Then f_γ takes its maximum on $[a,b]$ at a point $x_{Max,\gamma} \in (a,b]$. Therefore, $f_\gamma(x_{Max,\gamma}) > f_\gamma(a)$. Choose $\eta > 0$ such that

$$f_\gamma(x_{Max,\gamma}) - \eta > f_\gamma(a) + \eta.$$

Then if $\tilde{f} \in B(f,\eta)$ then $\tilde{f}_\gamma(x_{Max,\gamma}) > \tilde{f}_\gamma(a)$ and hence the proof of Lemma 5 implies that $\alpha_{\tilde{f}} \leq \gamma < \alpha_f + \varepsilon$. The other cases of the proof of this lemma are proved analogously and are left to the reader. □

Lemma 8. *Assume $f \in C[0,1]$, $0 \leq a < a' < b' < b \leq 1, -\infty < \alpha < \beta < +\infty$ and we have a finite subset $\mathcal{M} \subset [a',b']$ such that for all $\gamma \in [\alpha,\beta]$ either f_γ takes its absolute maximum on $[a,b]$, at an $x^* \in \mathcal{M}$, or no absolute maximum of f_γ on $[a,b]$ belongs to $[a',b']$. Given $\delta > 0$ there exists an $\eta > 0$ such that if $\phi \in B(f,\eta)$ and ϕ_γ has an absolute maximum on $[a,b]$ at an $x^{**} \in [a',b']$ for a $\gamma \in [\alpha,\beta]$ then there exists $x^* \in \mathcal{M}$ such that $|x^* - x^{**}| < \delta$.*

Proof. Proceeding towards a contradiction assume that for all $m \in \mathbb{N}$ there exists $\phi_m \in B(f, 1/m)$ such that $(\phi_m)_{\gamma_m}$ has an absolute maximum on $[a, b]$ at an $x_m^{**} \in [a', b']$, for a $\gamma_m \in [\alpha, \beta]$, and $dist(x_m^{**}, \mathcal{M}) \geq \delta$. Turning to a suitable subsequence we can assume $\gamma_m \to \gamma^* \in [\alpha, \beta]$, $x_m^{**} \to x^{**} \in [a', b']$. It is also clear that x^{**} is an absolute maximum of $(\phi)_{\gamma^*}$ on $[a, b]$ and $x^{**} \notin \mathcal{M}$, a contradiction. □

Assume that for a function $f \in BG^*$ the maximum of f_γ on $[a, b]$ is taken at a point $x_{Max,\gamma} \in (a, b)$. If $\gamma \notin \Gamma_f$ then it is a strict maximum of f_γ on $[a, b]$, if $\gamma \in \Gamma_f$ there can be at most one other point $x'_{Max,\gamma} \in (a, b)$ where f takes the same maximum value. In both cases $x_{Max,\gamma}$ is a strict local maximum of f_γ.

We suppose that

$$p' \text{ is even and } p_a, p_b \text{ are odd, } a = \frac{p_a}{p'} < b = \frac{p_b}{p'}, (a, b) \subset [0, 1], \qquad (2)$$

and $g \in \mathcal{G}(\kappa)$ is fixed.

Definition 9. Denote by $\mathcal{P}_{a,b,Max}$ the set of those $f \in C[0, 1]$ which satisfy the following property:
For any $\gamma \in \mathbb{R}$ if f_γ has a maximum on $[a, b]$ at a point $x_{Max,\gamma} \in (a, b)$ then $graph(g) \cap Q^2 \in f_{\gamma,MT}(x_{Max,\gamma})$.

Lemma 10. *The set $\mathcal{P}_{a,b,Max}$ is residual in $C[0, 1]$.*

Proof. First we choose and fix a countable set of $\{f_m\}_{m=1}^\infty \subset C^2[0, 1]$ such that $\{f_m\}_{m=1}^\infty$ is dense in $C[0, 1]$. For each m and $n \in \mathbb{N}$ we choose functions $f_{m,n} \in C[0, 1]$ and $\eta_{m,n} > 0$ such that $B(f_{m,n}, \eta_{m,n}) \subset B(f_m, \frac{1}{m+n})$. We set $\mathcal{G}_n = \cup_m B(f_{m,n}, \eta_{m,n})$ and $\mathcal{G} = \cap_{n=1}^\infty \mathcal{G}_n$. Then the set $\mathcal{G}_n$ is dense open in $C[0, 1]$ and $\mathcal{G}$ is residual.

Our aim is to choose $f_{m,n}$ and $\eta_{m,n}$ so that any $f \in \mathcal{G} \cap BG^*$ has property $\mathcal{P}_{a,b,Max}$. Assume m, n are fixed.

We need to choose $f_{m,n}$ and $\eta_{m,n}$. Since $f_m \in C^2[0, 1]$ its derivative and second derivative are bounded, so there exist $\Delta_{m,1}, \Delta_{m,2} \in \mathbb{R}$ such that $|f'_m| < \Delta_{m,1}, |f''_m| < \Delta_{m,2}$.

If $|f'_m| < \Delta_{m,1} \leq \gamma$ then $(f_m)'_\gamma = f'_m + \gamma > 0$. This implies that f_γ is monotone increasing on $[a, b]$ and hence takes its maximum on $[a, b]$ at b. Therefore, $\beta_{f_m} < \Delta_{m,1}$ and similarly, $\alpha_{f_m} > -\Delta_{m,1}$, that is, $[\alpha_{f_m}, \beta_{f_m}] \subset (-\Delta_{m,1}, \Delta_{m,1})$. By the Mean Value Theorem applied to f'_m we have: $|f'_m(x) -$

$f'_m(y)| < \Delta_{m,2}|x-y|$. If $x_1 < x_2 < x_3$ then there exist $x'_1 \in (x_1, x_2)$, and $x'_2 \in (x_2, x_3)$ such that:

$$\frac{f_m(x_2)-f_m(x_1)}{x_2-x_1} = f'(x'_1), \text{ and } \frac{f_m(x_3)-f_m(x_2)}{x_3-x_2} = f'(x'_2).$$

Therefore,

$$\left|\frac{f_m(x_2)-f_m(x_1)}{x_2-x_1} - \frac{f_m(x_3)-f_m(x_2)}{x_3-x_2}\right| < \Delta_{m,2}|x_3-x_1|. \tag{3}$$

Next using $\varepsilon(m,n,1) = \frac{1}{m+n}$ in Lemma 3 choose

$$\eta^* = \eta_{\varepsilon(m,n,1),g} > 0 \tag{4}$$

such that if $|\phi(x)-g(x)| < \eta^*$ for $x \in [-1,1]$ and $\phi \in C[-1,1]$ then

$$dist_{\mathcal{H}}(graph(\phi) \cap Q^2, graph(g) \cap Q^2) < \frac{1}{m+n}.$$

By using Lemma 4 with $\eta = \eta^*/2$ choose $0 < \delta_{\eta^*} < 1$ such that $\delta_{\eta^*} < \kappa/2$, (recall $g \in \mathcal{G}(\kappa)$) and

$$\text{if } |x_q|, |y_q|, |\gamma_q| < \delta_{\eta^*} \text{ then } |g_{(x_q,y_q,\gamma_q)}(x) - g(x)| < \eta^*/2 \text{ for } x \in [-1,1]\,. \tag{5}$$

Using p' from (2) choose an odd p'' such that $m+n < p''$ and if $p = p'p''$ then

$$\frac{1}{p} = \frac{1}{p'p''} < \frac{\delta_{\eta^*}}{8\Delta_{m,2}}. \tag{6}$$

Choosing p'' sufficiently large we can find a piecewise linear $\widehat{f}_{m,n} \in B(f_m, \frac{1}{2(m+n)})$ such that

- $\widehat{f}_{m,n}$ is linear on the intervals $[\frac{i-1}{p}, \frac{i}{p}]$ and the absolute value of the slope of these linear pieces is larger than $\max\{n, 2\Delta_{m,1}\}$;
- $\widehat{f}_{m,n} \leq f_m$,;
- $\widehat{f}_{m,n}(x) = f_m(x)$ if $x = \frac{i}{p} \in [0,1]\,,\ i$ odd;
- $\widehat{f}_{m,n}$ has its local maxima at $x = \frac{i}{p} \in [0,1]\,,\ i$ odd;

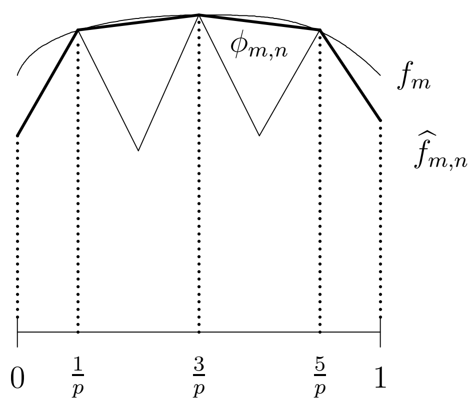

Figure 2: f_m, $\widehat{f}_{m,n}$, and $\phi_{m,n}$

- $\widehat{f}_{m,n}$ has its minimum on the intervals $[\frac{i-1}{p}, \frac{i}{p}] \cap [0,1]$ and $[\frac{i}{p}, \frac{i+1}{p}] \cap [0,1]$ at the points $\frac{i}{p}$ with i even.

See Figure 2. Choose $\widehat{\Delta}$ such that it is an upper bound of the absolute values of the slopes of the linear pieces of $\widehat{f}_{m,n}$. We also remark that $a = \frac{p_a p''}{p}$, $b = \frac{p_b p''}{p}$ with $p_a p''$, $p_b p''$ odd and p even.

From (3) and (6) it follows that if $3 \leq i \leq p-3$ is odd then

$$\left| \frac{\widehat{f}_{m,n}(\frac{i}{p}) - \widehat{f}_{m,n}(\frac{i-2}{p})}{\frac{2}{p}} - \frac{\widehat{f}_{m,n}(\frac{i+2}{p}) - \widehat{f}_{m,n}(\frac{i}{p})}{\frac{2}{p}} \right| < \frac{\delta_{\eta^*}}{2}. \tag{7}$$

Observe that if $\gamma \in [\alpha_{f_m}, \beta_{f_m}] \subset (-\Delta_{m,1}, \Delta_{m,1})$ then $(\widehat{f}_{m,n})_\gamma$ has its absolute maximum on $[a,b]$ at one of the points $\frac{i}{p}$ with i odd.

Denote by $\mathcal{I}$ the set of those odd i for which $\frac{i-2}{p}, \frac{i+2}{p} \in [a,b]$.

For $i \in \mathcal{I}$ let $[\widehat{\alpha}(i), \widehat{\beta}(i)]$ consist of those γ's for which $(\widehat{f}_{m,n})_\gamma(\frac{i}{p}) \geq (\widehat{f}_{m,n})_\gamma(x)$ holds for all $x \in [a,b]$. If there is no such γ choose arbitrary $\widehat{\beta}(i) < \widehat{\alpha}(i)$, which implies $[\widehat{\alpha}(i), \widehat{\beta}(i)] = \emptyset$.

Similarly, for $i \in \mathcal{I}$ let $[\alpha(i), \beta(i)]$ consist of those γ's for which $(\widehat{f}_{m,n})_\gamma(\frac{i}{p}) \geq (\widehat{f}_{m,n})_\gamma(x)$ holds for all $x \in \{\frac{i-2}{p}, \frac{i}{p}, \frac{i+2}{p}\}$. If there is no such γ choose arbitrary $\beta(i) < \alpha(i)$, which implies $[\alpha(i), \beta(i)] = \emptyset$.

Clearly, $[\widehat{\alpha}(i), \widehat{\beta}(i)] \subset [\alpha(i), \beta(i)]$.

Denote by $\mathcal{I}'$ the set of those $i \in \mathcal{I}$ for which $[\alpha(i), \beta(i)] \neq \emptyset$ and put $\gamma(i) = \frac{\alpha(i)+\beta(i)}{2}$.

For $i \in \mathcal{I}'$ we have

$$\alpha(i) = -\frac{\widehat{f}_{m,n}(\frac{i}{p}) - \widehat{f}_{m,n}(\frac{i-2}{p})}{\frac{2}{p}}, \text{ and } \beta(i) = -\frac{\widehat{f}_{m,n}(\frac{i+2}{p}) - \widehat{f}_{m,n}(\frac{i}{p})}{\frac{2}{p}}. \qquad (8)$$

From (7) it follows that $\beta(i)-\alpha(i) < \frac{\delta_{\eta^*}}{2}$ and it is also clear that $[\widehat{\alpha}(i), \widehat{\beta}(i)] \subset [\alpha(i), \beta(i)]$ implies $\widehat{\beta}(i) - \widehat{\alpha}(i) < \frac{\delta_{\eta^*}}{2}$.

We introduce the following auxiliary function $\phi_{m,n}(0) = \widehat{f}_{m,n}(0), \phi_{m,n}(1) = \widehat{f}_{m,n}(1), \phi_{m,n}(\frac{i}{p}) = \widehat{f}_{m,n}(\frac{i}{p})$ if i is odd, $\frac{i}{p} \in (0,1)$ and $\phi_{m,n}$ is linear on intervals $[\frac{i-2}{p}, \frac{i}{p}] \subset [0,1]$ with i odd, and also on $[0, \frac{1}{p}], [1-\frac{1}{p}, 1]$. Observe that $\phi_{m,n} > \widehat{f}_{m,n}$ if $x \in (\frac{1}{p}, 1-\frac{1}{p}), x \neq \frac{i}{p}, i$ odd.

Now we interrupt the proof of Lemma 10 by a proposition.

Proposition 11. *Assume $\phi^* \leq \phi_{m,n}$ on $[0,1]$, $\phi^*(\frac{i}{p}) = \phi_{m,n}(\frac{i}{p})$ if $\frac{i}{p} \in [0,1]$, i odd, and from $x \in (\frac{1}{p}, 1-\frac{1}{p})$, $x \neq \frac{i}{p}$, i odd it follows that $\phi^*(x) < \phi_{m,n}(x)$, suppose furthermore that $\gamma \in \mathbb{R}$ and $(\phi^*)_\gamma$ has its absolute maximum on $[a,b]$ at a point $x^* \in [a+\frac{1}{p}, b-\frac{1}{p}]$. Then $x^* = \frac{i}{p}$ for an i odd.*

Proof. Assume $x^* \neq \frac{i}{p}$ with i odd. Then $\phi_{m,n}(x^*) + \gamma x^* > \phi^*(x^*) + \gamma x^* \geq \phi^*(\frac{i}{p}) + \gamma \frac{i}{p} = \phi_{m,n}(\frac{i}{p}) + \gamma\frac{i}{p}$ for all $\frac{i}{p} \in [a,b]$, i odd. Since $\phi_{m,n}(x) + \gamma x$ is linear on intervals $[\frac{i-2}{p}, \frac{i}{p}]$ we infer that $\phi_{m,n}(x^*) + \gamma x^* > \phi_{m,n}(x) + \gamma x$ for all $x \in [a+\frac{1}{p}, b-\frac{1}{p}]$. But it is impossible since $x^* \in [a+\frac{1}{p}, b-\frac{1}{p}]$. □

Next we return to the proof of Lemma 10. Since $g(x)$ is bounded choose $M_g \geq max\{1, \widehat{\Delta}, |\alpha_{f_m}|, |\beta_{f_m}|\}$ such that $M_g/2 \geq |g(x)|$ for all $x \in [-2,2]$. Choose $q > 6p$ such that for all $\frac{i}{p} \in [0,1]$ with i odd, the rectangle

$$R_i = \left[\frac{i}{p} - \frac{3}{q}, \frac{i}{p} + \frac{3}{q}\right] \times \left[f_m(\frac{i}{p}) - \frac{3M_g}{q}, f_m(\frac{i}{p}) + \frac{3M_g}{q}\right]$$

is a subset of

$$B^*(f_m, \frac{1}{2(m+n)}) \stackrel{\text{def}}{=} \left\{(x;y) : x \in [0,1], f_m(x) - \frac{1}{2(m+n)} < y < f_m(x) + \frac{1}{2(m+n)}\right\}.$$

From $q > 6p$ it follows that $R_i \subset [0,1] \times \mathbb{R}, R_i \cap R_j = \emptyset$, if $i \neq j$. We also put $U_{m,n} = \{(x;y) : x \in [0,1], y \leq \phi_{m,n}(x)\}$ and for $\frac{i}{p} \in [0,1], i$ odd we put $U_{m,n,i} = U_{m,n} \cap ([\frac{i-2}{p}, \frac{i+2}{p}] \times \mathbb{R})$, $U^*_{m,n,i} = int(U_{m,n}) \cap ([\frac{i-2}{p}, \frac{i+2}{p}] \times \mathbb{R}) \subseteq U_{m,n,i}$. Observe that $\phi_{m,n}$ has two linear pieces on the boundary of $U_{m,n,i}$ and if $i \in \mathcal{I}'$ then $U_{m,n,i}$ and $U^*_{m,n,i}$ are convex. The choice of M_g implies that $\widehat{f}_{m,n}$ intersects the vertical sides of R_i. Recall that $f_m(\frac{i}{p}) = \widehat{f}_{m,n}(\frac{i}{p}) = \phi_{m,n}(\frac{i}{p})$ if i is odd. Assume $i \in \mathcal{I}'$. For $x \in [\frac{i}{p} - \frac{2}{q}, \frac{i}{p} + \frac{2}{q}]$ put

$$f_{m,n}(x) = -\gamma(i)(x - \frac{i}{p}) + \frac{1}{q} g(q(x - \frac{i}{p})) + f_m(\frac{i}{p}). \tag{9}$$

Since $g \in \mathcal{G}(\kappa)$ we have $f_{m,n}(x) \leq -\gamma(i)(x - \frac{i}{p}) - \kappa|x - \frac{i}{p}| + f_m(\frac{i}{p})$ for $x \in [\frac{i}{p} - \frac{2}{q}, \frac{i}{p} + \frac{2}{q}]$ and $f_{m,n}$ is piecewise linear on $[\frac{i}{p} - \frac{2}{q}, \frac{i}{p} + \frac{2}{q}]$.

Since $\gamma(i) \in (\alpha(i), \beta(i))$;

$$\beta(i) - \alpha(i) < \frac{\delta_{\eta^*}}{2} < \frac{\kappa}{4}, \tag{10}$$

we infer that $f_{m,n}(x) < \phi_{m,n}(x)$ if $x \neq \frac{i}{p}$, $x \in [\frac{i}{p} - \frac{2}{q}, \frac{i}{p} + \frac{2}{q}]$. Hence, the graph of $f_{m,n}(x)$ on $[\frac{i}{p} - \frac{2}{q}, \frac{i}{p} + \frac{2}{q}]$ is a subset of $R_i \cap U_{m,n,i}$.

It is clear that the points $(\frac{i}{p} - \frac{3}{q}; \widehat{f}_{m,n}(\frac{i}{p} - \frac{3}{q}))$, $(\frac{i}{p} - \frac{2}{q}; f_{m,n}(\frac{i}{p} - \frac{2}{q}))$, $(\frac{i}{p} + \frac{2}{q}; f_{m,n}(\frac{i}{p} + \frac{2}{q}))$, $(\frac{i}{p} + \frac{3}{q}; \widehat{f}_{m,n}(\frac{i}{p} + \frac{3}{q}))$ all belong to $R_i \cap U^*_{m,n,i}$.

We define $f_{m,n}$ on the intervals $[\frac{i}{p} - \frac{3}{q}, \frac{i}{p} - \frac{2}{q}], [\frac{i}{p} + \frac{2}{q}, \frac{i}{p} + \frac{3}{q}]$ so that it is linear and takes at the endpoints of these intervals the previously defined four values. Since $R_i \cap U^*_{m,n,i}$ is convex the graph of $f_{m,n}$ on $[\frac{i}{p} - \frac{3}{q}, \frac{i}{p} - \frac{2}{q}] \cup [\frac{i}{p} + \frac{2}{q}, \frac{i}{p} + \frac{3}{q}]$ belongs to $R_i \cap U^*_{m,n,i}$. See Figure 3.

If x does not belong to $[\frac{i}{p} - \frac{3}{q}, \frac{i}{p} + \frac{3}{q}]$ for an $i \in \mathcal{I}'$ then set $f_{m,n} = \widehat{f}_{m,n}$.

The choices of R_i and $f_{m,n}$ imply that $f_{m,n} \in B(f_m, \frac{1}{2(m+n)})$ and the choice of $U_{m,n,i}$, $U^*_{m,n,i}$ and $f_{m,n}$ imply that $f_{m,n}(x) \leq \phi_{m,n}(x)$ and $f_{m,n}(x) = \phi_{m,n}(x)$ implies $x = \frac{i}{p}$ with i odd when $x \in [\frac{1}{p}, 1 - \frac{1}{p}]$.

By Proposition 11 if $\gamma \in \mathbb{R}$ and $(f_{m,n}(x))_\gamma$ has an absolute maximum on $[a,b]$ at a point $x^* \in [a + \frac{1}{p}, b - \frac{1}{p}]$ then $x^* = \frac{i}{p}$ with i odd. Also observe that the definition of $\mathcal{I}'$ yields that if $i \in \mathcal{I}$ and $(f_{m,n}(x))_\gamma$ has an absolute maximum at $\frac{i}{p}$ on an interval containing $[\frac{i-2}{p}, \frac{i+2}{p}]$ then $i \in \mathcal{I}'$.

By the above definition $f_{m,n}$ is piecewise linear, choose $\Delta > 1$ such that the absolute value of the slope of any linear piece of $f_{m,n}$ is less than Δ.

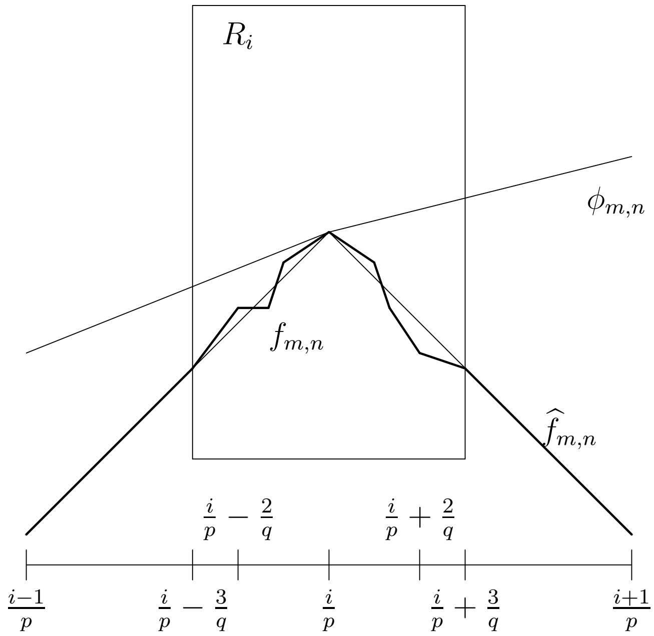

Figure 3: R_i and $f_{m,n}$

Assume that $\Delta^* > \max\{\Delta_{m,1}, n\}$, Lemma 8 is applicable to $f_{m,n}$ with $\alpha = -\Delta^*, \beta = \Delta^*, a' = a + \frac{1}{p}, b' = b - \frac{1}{p}$ with $\delta = \delta_{\eta^*} \frac{1}{4(\Delta+\Delta^*)q}$.

We obtain an $\eta' > 0$ such that if $\phi \in B(f_{m,n}, \eta')$ and ϕ_γ has an absolute maximum on $[a, b]$ for a $\gamma \in [-\Delta^*, \Delta^*]$, at an $x^{**} \in [a + \frac{1}{p}, b - \frac{1}{p}]$ then

$$\left| x^{**} - \frac{i}{p} \right| < \delta_{\eta^*} \frac{1}{4(\Delta + \Delta^*)q} \tag{11}$$

for an $i \in \mathcal{I}'$, this also implies

$$|(f_{m,n})_\gamma(x^{**}) - (f_{m,n})_\gamma(\frac{i}{p})| < \delta_{\eta^*} \frac{1}{4q}. \tag{12}$$

We can also assume that $\eta' < \frac{\delta_{\eta^*}}{8q}$. If $\phi \in B(f_{m,n}, \eta')$ we also have

$$|\phi_\gamma(x^{**}) - \phi_\gamma(\frac{i}{p})| < \delta_{\eta^*} \frac{1}{4q} + 2\eta' < \frac{\delta_{\eta^*}}{2q}.$$

It can also be supposed that $\eta' < \frac{\eta^*}{4q}$ with η^* from (4). Assume now that $\phi \in B(f_{m,n}, \eta')$ and for a $\gamma \in [-\Delta^*, \Delta^*]$, ϕ_γ has its absolute maximum on $[a,b]$ at an $x^{**} \in [a + \frac{1}{p}, b - \frac{1}{p}]$. Then there exists $i \in \mathcal{I}'$ such that $|x^{**} - \frac{i}{p}| < \delta_{\eta^*} \frac{1}{4(\Delta + \Delta^*)q}$. We have $\phi_\gamma(x^{**}) \geq \phi_\gamma(\frac{i-2}{p}) > (f_{m,n})_\gamma(\frac{i-2}{p}) - \eta'$ and similarly $\phi_\gamma(x^{**}) \geq \phi_\gamma(\frac{i+2}{p}) > (f_{m,n})_\gamma(\frac{i+2}{p}) - \eta'$.

We also have $|(f_{m,n})_\gamma(\frac{i}{p}) - \phi_\gamma(x^{**})| \leq |(f_{m,n})_\gamma(\frac{i}{p}) - \phi_\gamma(\frac{i}{p})| + |\phi_\gamma(\frac{i}{p}) - \phi_\gamma(x^{**})| \leq \eta' + \frac{\delta_{\eta^*}}{2q}$.

Hence

$$-\frac{\delta_{\eta^*}}{2q} - 2\eta' \leq (f_{m,n})_\gamma(\frac{i}{p}) - (f_{m,n})_\gamma(\frac{i-2}{p}) = \tag{13}$$

$$(f_{m,n})_{\gamma(i)}(\frac{i}{p}) - (f_{m,n})_{\gamma(i)}(\frac{i-2}{p}) + (\gamma - \gamma(i))\frac{2}{p}$$

and

$$\frac{\delta_{\eta^*}}{2q} + 2\eta' \geq (f_{m,n})_\gamma(\frac{i+2}{p}) - (f_{m,n})_\gamma(\frac{i}{p}) = \tag{14}$$

$$(f_{m,n})_{\gamma(i)}(\frac{i+2}{p}) - (f_{m,n})_{\gamma(i)}(\frac{i}{p}) + (\gamma - \gamma(i))\frac{2}{p}.$$

By using (8) and recalling that $f_{m,n}(\frac{i}{p}) = \widehat{f}_{m,n}(\frac{i}{p})$, $\frac{\alpha(i)+\beta(i)}{2} = \gamma(i)$ for $i \in \mathcal{I}'$ with

$$\beta(i) = -\frac{f_{m,n}(\frac{i+2}{p}) - f_{m,n}(\frac{i}{p})}{\frac{2}{p}},$$

$$\alpha(i) = -\frac{f_{m,n}(\frac{i}{p}) - f_{m,n}(\frac{i-2}{p})}{\frac{2}{p}},$$

we have by using (10)

$$|(f_{m,n})_{\gamma(i)}(\frac{i}{p}) - (f_{m,n})_{\gamma(i)}(\frac{i-2}{p})| < \frac{2}{p}\frac{\delta_{\eta^*}}{2},$$

$$|(f_{m,n})_{\gamma(i)}(\frac{i+2}{p}) - (f_{m,n})_{\gamma(i)}(\frac{i}{p})| < \frac{2}{p}\frac{\delta_{\eta^*}}{2}.$$

Using $q > 6p$, (13), and (14)

$$|\gamma - \gamma(i)|\frac{2}{p} \leq \frac{2}{p}\frac{\delta_{\eta^*}}{2} + \frac{\delta_{\eta^*}}{2q} + 2\eta' < \frac{2}{p}\delta_{\eta^*}.$$

This implies $|\gamma - \gamma(i)| < \delta_{\eta^*} < 1$.

By (11) and (12) we obtain

$$|(f_{m,n})_{\gamma(i)}(\frac{i}{p}) - (f_{m,n})_{\gamma(i)}(x^{**})| <$$

$$|(f_{m,n})_{\gamma}(\frac{i}{p}) - (f_{m,n})_{\gamma}(x^{**})| + |\gamma - \gamma(i)| \cdot |x^{**} - \frac{i}{p}| <$$

$$\delta_{\eta^*}\frac{1}{4q} + \delta_{\eta^*}\frac{1}{4(\Delta + \Delta^*)q} < \delta_{\eta^*}\frac{1}{2q}.$$

Set $\overline{\phi}(x) = q(\phi_\gamma(\frac{x}{q} + x^{**}) - \phi_\gamma(x^{**}))$.
Then

$$|\overline{\phi}(x) - g(x)| \leq q|\phi_\gamma(\frac{x}{q} + x^{**}) - (f_{m,n})_\gamma(\frac{x}{q} + x^{**})| +$$

$$q|\phi_\gamma(x^{**}) - (f_{m,n})_\gamma(x^{**})| + |q((f_{m,n})_\gamma(\frac{x}{q} + x^{**}) - (f_{m,n})_\gamma(x^{**})) - g(x)| \leq$$

$$A_1 + A_2 + A_3.$$

Since $i \in \mathcal{I}'$ by our definition of $f_{m,n}$ on $[\frac{i}{p} - \frac{2}{q}, \frac{i}{p} + \frac{2}{q}]$ in (9) for $x \in [-2, 2]$ we obtain

$$(f_{m,n})_{\gamma(i)}(\frac{x}{q}+\frac{i}{p}) = -\gamma(i)((\frac{x}{q}+\frac{i}{p})-\frac{i}{p})+\frac{1}{q}g\left(q((\frac{x}{q}+\frac{i}{p})-\frac{i}{p})\right)+f_m(\frac{i}{p})+\gamma(i)(\frac{x}{q}+\frac{i}{p}),$$

and

$$(f_{m,n})_{\gamma(i)}(\frac{i}{p}) = \frac{1}{q}g(0) + f_m(\frac{i}{p}) + \gamma(i)\frac{i}{p} = f_m(\frac{i}{p}) + \gamma(i)\frac{i}{p}.$$

Therefore,

$$g(x) = q((f_{m,n})_{\gamma(i)}(\frac{x}{q} + \frac{i}{p}) - (f_{m,n})_{\gamma(i)}(\frac{i}{p})),$$

for $x \in [-2, 2]$.

Choose $x_q = q(\frac{i}{p} - x^{**}), y_q = q((f_{m,n})_{\gamma(i)}(\frac{i}{p}) - (f_{m,n})_{\gamma(i)}(x^{**})), \gamma_q = \gamma - \gamma(i)$. Then $|x_q|, |y_q|, |\gamma_q| \leq \delta_{\eta^*}$ and

$$g_{(x_q, y_q, \gamma_q)}(x) =$$

$$g(x - q(\frac{i}{p} - x^{**})) + (\gamma - \gamma(i))x + q((f_{m,n})_{\gamma(i)}(\frac{i}{p}) - (f_{m,n})_{\gamma(i)}(x^{**})) =$$

$$q((f_{m,n})_{\gamma(i)}(\frac{x-q(\frac{i}{p}-x^{**})}{q}+\frac{i}{p})-(f_{m,n})_{\gamma(i)}(\frac{i}{p}))+(\gamma-\gamma(i))x+$$

$$q((f_{m,n})_{\gamma(i)}(\frac{i}{p})-(f_{m,n})_{\gamma(i)}(x^{**}))=q(f_{m,n})_{\gamma(i)}(\frac{x}{q}+x^{**})+$$

$$q(\gamma-\gamma(i))\frac{x}{q}-q(f_{m,n})_{\gamma(i)}(x^{**})=q\Bigg((f_{m,n})_{\gamma(i)}(\frac{x}{q}+x^{**})+$$

$$(\gamma-\gamma(i))((\frac{x}{q}+x^{**})-x^{**})-(f_{m,n})_{\gamma(i)}(x^{**})\Bigg)=$$

$$q((f_{m,n})_{\gamma}(\frac{x}{q}+x^{**})-(f_{m,n})_{\gamma}(x^{**})).$$

Therefore, $A_3=|g_{(x_q,y_q,\gamma_q)}(x)-g(x)|$.

Now $|A_1|<q\eta',|A_2|<q\eta'$ and for $x\in[-1,1]$ by the previous computation and (5), $|A_3|<\frac{\eta^*}{2}$. This yields $|\overline{\phi}(x)-g(x)|<2q\eta'+\frac{\eta^*}{2}<\eta^*$ when $x\in[-1,1]$. Set $\eta_{m,n}=\min\{\eta',\frac{1}{2(m+n)}\}$.

Assume $f\in\mathcal{G}\cap BG^*$. Then for each n there exists an m_n such that $f\in B(f_{m_n,n},\eta_{m_n,n})$. Suppose that f_γ has an absolute maximum on $[a,b]$ at $x^{**}\in(a,b)$.

By the choice of $\eta_{m_n,n}$ and (6) there exists $p=p_n$ such that $p_n>m+n>n$. We can assume that n is so large that $x^{**}\in[a+\frac{2}{n},b-\frac{2}{n}]$. We denote by q_n the q chosen for $p=p_n$. By our previous argument used with $\phi=f$

$$|q_n(f_\gamma(\frac{x}{q_n}+x^{**})-f_\gamma(x^{**}))-g(x)|<\eta^* \text{ holds for } x\in[-1,1]\,.$$

By Lemma 3 and the choice of η^* we have

$$dist_{\mathcal{H}}(graph(q_n(f_\gamma(\frac{x}{q_n}+x^{**})-f_\gamma(x^{**})))\cap Q^2,graph(g)\cap Q^2)<\frac{1}{m_n+n}.$$

This implies that $graph(g)\cap Q^2\in f_{\gamma,MT}(x^{**})$.

□

Lemma 12. *Let $g\in\mathcal{G}(\kappa)$ be fixed. For the typical continuous function f, for any $\gamma\in\mathbb{R}$ and any local maximum $x_{Max}\in(0,1)$ of f_γ we have $graph(g)\cap Q^2\in f_{\gamma,MT}(x_{Max})$.*

Proof. Denote by $\mathcal{A}$ the set of those $(a,b),a<b$ in $[0,1]$ for which $a=\frac{p_a}{p'},b=\frac{p_b}{p'}$ with p' even and p_a,p_b odd. By Lemma 10, $\bigcap_{(a,b)\in\mathcal{A}}\mathcal{P}_{a,b,Max}$ is residual in

$C[0,1]$. But if x_{Max} is any local maximum of f_γ then there exists $(a,b)\in\mathcal{A}$ such that $x_{Max}\in(a,b)$ and x_{Max} is the absolute maximum of f_γ on $[a,b]$. Hence, $f\in\mathcal{P}_{a,b,Max}$ implies that $graph(g)\cap Q^2\in f_{\gamma,MT}(x_{Max})$. □

Lemma 13. *For the typical continuous function f, for any $\gamma\in\mathbb{R}$ and any local maximum, $x_{Max}\in(0,1)$, of f_γ we have that for all $g\in C\,[-1,1]_{0,-}$, $graph(g)\cap Q^2\in f_{\gamma,MT}(x_{Max})$.*

Proof. Use the countable set $g_\nu\in\mathcal{G}(\kappa_\nu)$ with $g_\nu|_{[-1,1]}$ dense in $C\,[-1,1]_{0,-}$. Taking countable intersection, by Lemma 10 for any g_ν for the typical continuous function f for any $\gamma\in\mathbb{R}$ and any local maximum, $x_{Max}\in(0,1)$ of f_γ we have $graph(g_\nu)\cap Q^2\in f_{\gamma,MT}(x_{Max})$. The triangle inequality and the density of $g_\nu|_{[-1,1]}$ in $C\,[-1,1]_{0,-}$, then implies the lemma. □

Remark 14. It is clear that Lemma 13 holds for local minima as well.

Proof of Theorem 1. Denote by $\mathcal{F}_+$ the set of those $f\in C[0,1]$ for which for any local maximum of f_γ, $x_{Max}\in(0,1)$ we have that for all $g\in C\,[-1,1]_{0,-}$, $graph(g)\cap Q^2\in f_{\gamma,MT}(x_{Max})$. Similarly denote by $\mathcal{F}_-$ the set of those $f\in C[0,1]$ for which for any local minimum of f_γ, $x_{min}\in(0,1)$ we have that for all $g\in C\,[-1,1]_{0,+}$, $graph(g)\cap Q^2\in f_{\gamma,MT}(x_{min})$. By Lemma 13 and Remark 14 the set $\mathcal{F}_+\cap\mathcal{F}_-$ is residual and hence Theorem 1 holds. □

3 Vertical universality

In this section we will assume that $[c,d]$ denotes the closed interval with endpoints c and d (even when $c>d$).

Assume $f\in C[0,1], r\in[0,1]$. Denote by π_x the projection onto the x axis.

Put $P_f^+=\bigcup_{\alpha\in\mathbb{R}}\pi_x(P_{\alpha,f}^+), P_f^-=\bigcup_{\alpha\in\mathbb{R}}\pi_x(P_{\alpha,f}^-), P_f^\pm=\bigcup_{\alpha\in\mathbb{R}}\pi_x(P_{\alpha,f}^\pm)$ and $P_f=\pi_x(\bigcup_{\alpha\in\mathbb{R}}P_{\alpha,f}\setminus P_{\alpha,f}^\pm)$.

We also put $P_{f,r,\le}^+=\{x\in[r,1]:x\in P_f^+, f(x')\le f(x),\ \forall x'\in(r,x)\}$, and $P_{f,r,\le}^-=\{x\in[0,r]:x\in P_f^-, f(x')\le f(x),\ \forall x'\in(x,r)\}$, one can define similarly the sets $P_{f,r,\ge}^+, P_{f,r,\ge}^-$.

Next observe that

$$P_f^+=\bigcup_{r\in\mathbb{Q}\cap[0,1]}(P_{f,r,\le}^+\cup P_{f,r,\ge}^+)\text{ and }P_f^-=\bigcup_{r\in\mathbb{Q}\cap[0,1]}(P_{f,r,\le}^-\cup P_{f,r,\ge}^-).$$

It is not difficult to see that $f\,|_{P^+_{f,r,\leq}}$ is monotone increasing. Hence, being the graph of a function of bounded variation, the $\mathcal{H}^1$-measure of $f|_{P^+_{\alpha,r,\leq}}$ is finite, [4], p. 146. Using this, one can easily see that $\cup_{\alpha\in\mathbb{R}}P^{\pm}_{\alpha,f}$, that is, the set of those $(x_0; f(x_0))$ which are only one-sided accumulation points of $P_{\alpha,f}$ is of σ-finite-$\mathcal{H}^1$ measure.

Lemma 15. *For the typical continuous function* $\lambda(P^+_{f,r,\leq}) = 0$*.*

Proof. If $\lambda(P^+_{f,r,\leq}) > 0$ then there exists an x which is a Lebesgue density point of $P^+_{f,r,\leq}$ such that $f|_{P^+_{f,r,\leq}}$ is differentiable at x. This would imply that f is approximately differentiable at x. But [1] p.212 tells that the typical continuous function does not have at any point a finite upper aproximate derivative and hence it is not approximately differentiable at any point. $\square$

A similar argument shows that $\lambda(P^+_{f,r,\geq}) = 0$ for the typical continuous function. This implies $\lambda(P^+_f) = 0$ and similarly $\lambda(P^-_f) = 0$.

An alternative way of verifying that $\lambda(P^{\pm}_f) = 0$ is by using Theorem 5 from [3]. If $x_0 \in P^{\pm}_f$ then one can easily see that $(x_0; f(x_0)) \notin UMT(f)$. Indeed, let $g(x) = \sin(2\pi x)$. If $(x_0, f(x_0)) \in UMT(f)$ then we have $graph(g) \cap Q^2 \in f_{MT}(x_0)$ and this implies that $(x_0, f(x_0))$ is a two-sided accumulation point of $L_{f(x_0),g}$. By Theorem 5 of [3] for the typical continuous function the projection of $UMT(f)$ onto the x-axis is of full measure in $[0, 1]$. Since $P^{\pm}_f$ is a subset of the complement of this projection it is of zero measure.

Given $-1 \leq x_1 < x_2 < ... < x_\nu \leq 1$, set $V_{x_1,...,x_\nu} = \{(x_j; y) \in Q^2 : y \in [-1, 1], j \in \{1, ..., \nu\}\}$.

Definition 16. Assume $f \in C[0, 1], x_0 \in P^+_f$, that is, letting $\alpha = f(x_0)$ the point $(x_0; f(x_0))$ is a right-sided accumulation point of the level set $L_{\alpha,f}$. We have *right vertical universality* at $x_0 \in P^+_f$ if for any choice of $0 = x_1 < x_2 < ... < x_\nu \leq 1$ we have

$$V_{x_1,...,x_\nu} \in f_{MT}(x_0). \tag{15}$$

One can define similarly the *left vertical universality* at $x_0 \in P^-_f$.
We have *two-sided vertical universality* at $x_0 \in P_f$ if for any $-1 \leq x_1 < x_2 < ... < x_\nu \leq 1$, $0 \in \{x_1, ..., x_\nu\}$ we have (15). If $F \subset [0, 1], 0 \in F$ then $F \times [-1, 1]$ can be approximated in the Hausdorff-metric by sets of the form $V_{x_1,...,x_\nu}$ with $0 = x_1$. Hence if we have right vertical universality at x_0 then

$F \times [-1,1] \in f_{MT}(x_0)$. Similar statement holds for left-, or two-sided vertical universality, with $0 \in F$ and $F \subset [-1,0]$ or $F \subset [-1,1]$, respectively.

Assume $V = V_{x_1,...,x_\nu}$ is fixed with $0 \in \{x_1, ..., x_\nu\}$, $-1 < x_1$, and $x_\nu < 1$.

Lemma 17. *For the typical continuous function f, for all $\alpha \in \mathbb{R}$ if $P_{\alpha,f} \neq \emptyset$ then there exists $P_{\alpha,f}(V) \subset P_{\alpha,f}$ dense $\mathcal{G}_\delta$ in $P_{\alpha,f}$ such that if $x_0 \in \pi_x(P_{\alpha,f}(V))$ then $V \in f_{MT}(x_0)$.*

Theorem 18. *For the typical continuous function, f, for all $\alpha \in \mathbb{R}$ if $P_{\alpha,f} \neq \emptyset$ then there exists $\widehat{P}_{\alpha,f} \subset P_{\alpha,f}$, dense $\mathcal{G}_\delta$ in $P_{\alpha,f}$ such that if $x_0 \in \pi_x(\widehat{P}_{\alpha,f})$ then we have two-sided vertical universality at x_0.*

First we prove Theorem 18 based on Lemma 17.

Proof. Choose the sets $\{x_1(k), ..., x_{\nu(k)}(k)\}$ for $k \in \mathbb{N}$ so that $0 \in \{x_1(k), ..., x_{\nu(k)}(k)\}$ and any finite sequence of rational numbers in $(0,1)$ which contains zero appears among the sets $\{x_1(k), ..., x_{\nu(k)}(k)\}$ for a certain k. Then for the sequence $V(k) = V_{x_1(k),...,x_{\nu(k)}(k)}$ we have $0 \in \{x_1(k), ..., x_{\nu(k)}(k)\}$, $-1 < x_1(k)$, $x_{\nu(k)}(k) < 1$, and if $0 \in \{x_1, ..., x_\nu\}$ for an arbitrary choice of $-1 \leq x_1 < x_2 < ... < x_\nu \leq 1$ then there is a subsequence k_l such that $V(k_l) \to V_{x_1,...,x_\nu}$ in the Hausdorff metric. Suppose $P_{\alpha,f} \neq \emptyset$. Apply Lemma 17 for each k to obtain for all $\alpha \in \mathbb{R}$ sets $P_{\alpha,f}(V(k))$ which are dense G_δ in $P_{\alpha,f}$. If $x_0 \in \pi_x(P_{\alpha,f}(V(k)))$ then $V(k) \in f_{MT}(x_0)$.

Set $\widehat{P}_{\alpha,f} = \bigcap_{k=1}^{\infty} P_{\alpha,f}(V(k))$. By using the triangle inequality in the Hausdorff metric we infer that for an arbitrary $V_{x_1,...,x_\nu}$ with $0 \in \{x_1, ..., x_\nu\}$, $-1 \leq x_1 < x_2 < ... < x_\nu \leq 1$ for any $x_0 \in \pi_x(\widehat{P}_{\alpha,f})$ we have $V_{x_1,...,x_\nu} \in f_{MT}(x_0)$.

□

Now we turn to the proof of Lemma 17.

Proof of Lemma 17. Choose f_m dense in $C[0,1]$ consisting of piecewise linear functions which all have their extrema at different heights and are nonconstant on any interval. Assume $m, n \in \mathbb{N}$ are fixed and $[a,b] \subset [0,1]$ is a maximal interval on which f_m is linear. Choose a subdivision $a = \beta_0 < \beta_1 < ... < \beta_\kappa = b$. We will assume that if $[a', b']$ is a different maximal interval of linearity for f_m with subdivision $\beta'_0 < ... < \beta'_{\kappa'}$ then $f_m(\beta_i) \neq f_m(\beta'_{i'})$ for $i \in \{0, ..., \kappa\}$, $i' \in \{0, ..., \kappa'\}, \beta_i \neq \beta'_{i'}$. Let $\delta_{min} = \min_{\beta_i \neq \beta'_{i'}} |f_m(\beta(i)) - f_m(\beta'_{i'})|$

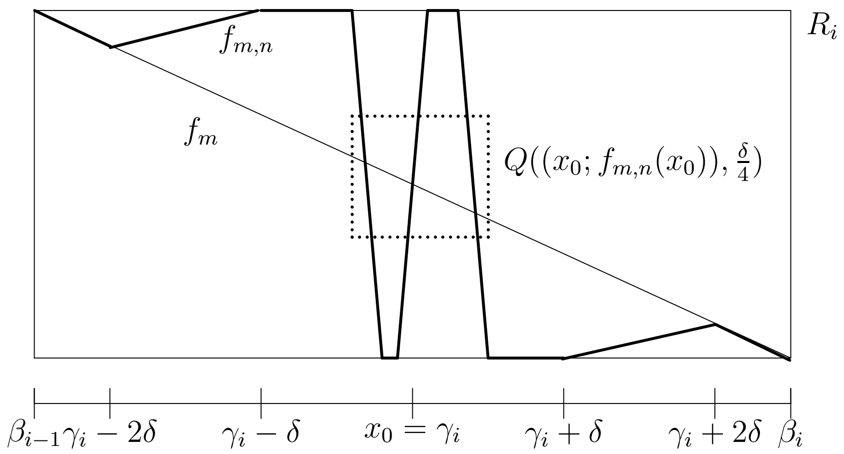

Figure 4: Definition of $f_{m,n}$

where the minimum is taken for all subdvisions of all maximal intervals of linearity of f_m. We also suppose that $\beta_i - \beta_{i-1} < \frac{1}{m+n}$ for $i = 1, ..., \kappa$, moreover if $R_i = [\beta_{i-1}, \beta_i] \times [f_m(\beta_{i-1}), f_m(\beta_i)]$ then

$$R_i \subset B^*(f_m, \frac{1}{m+n}) =$$

$$\{(x; y) : x \in [0, 1], f_m(x) - \frac{1}{m+n} < y < f_m(x) + \frac{1}{m+n}\}$$

for all i.

Next choose $\delta \in (0, \frac{1}{m+n})$ such that taking any two intervals $[a, b]$ and $[a', b']$ with corresponding subdivisions as above for $\delta < \delta_{\min}/4$ we have

$$[f_m(\beta_i) - 2\delta, f_m(\beta_i) + 2\delta] \cap [f_m(\beta'_{i'}) - 2\delta, f_m(\beta'_{i'}) + 2\delta] = \emptyset \qquad (16)$$

for $i \in \{0, ..., \kappa\}, i' \in \{0, ..., \kappa'\}, \beta_i \neq \beta'_{i'}$. In addition, we also suppose that $\delta < \frac{\beta_i - \beta_{i-1}}{4}, i = 1, ..., \kappa$ for all subdivisions of intervals of type $[a, b]$. Assume $\gamma_i = \frac{\beta_i + \beta_{i-1}}{2}$, denotes the midpoint of $[\beta_{i-1}, \beta_i]$.

Recall that we have a fixed $V = V_{x_1,...,x_\nu}$. Choose $\delta' > 0$ so that $x_j - \delta' > x_{j-1} + \delta'$ holds for $j = 2, ..., \nu$ and $\delta' < \frac{1}{2(m+n)}$. Set $x^-_{j,i} = \gamma_i + \frac{\delta}{4}(x_j - \delta')$, $x^+_{j,i} = \gamma_i + \frac{\delta}{4}(x_j + \delta'), j = 1, ..., \nu$. We can also assume that $2\delta' < \min\{x_1 - (-1), 1 - x_\nu\}$.

Assume j is odd. Set $f_{m,n}(x^-_{j,i}) = f_m(\beta_{i-1}), f_{m,n}(x^+_{j,i}) = f_m(\beta_i)$. If j is even set $f_{m,n}(x^-_{j,i}) = f_m(\beta_i), f_{m,n}(x^+_{j,i}) = f_m(\beta_{i-1})$. Set furthermore $f_{m,n}(\gamma_i - \delta) = f_{m,n}(x^-_{1,i}), f_{m,n}(\gamma_i + \delta) = f_{m,n}(x^+_{\nu,i})$. If $x \in [\beta_{i-1}, \beta_i] \setminus (\gamma_i - 2\delta, \gamma_i + 2\delta)$ set $f_{m,n}(x) = f_m(x)$.

Finally, extend the definition of $f_{m,n}$ to points of $[\beta_{i-1}, \beta_i]$ such that it is linear on the subintervals where we have not defined it yet. See Figure 4 (on this figure there is a slight distortion, since $\delta/4$ would be too small and hence the dotted square and the central part of $f_{m,n}$ is slightly enlarged). Observe that for any $x \in [\beta_{i-1},\beta_i]$ we have $(x; f_{m,n}(x)) \in R_i$.

Since $0 \in \{x_1, ..., x_\nu\}$ there exists j_0 such that $x_{j_0} = 0$. Thus $x^-_{j_0,i} = \gamma_i - \frac{\delta}{4}\delta'$, $x^+_{j_0,i} = \gamma_i + \frac{\delta}{4}\delta'$. Suppose $x_0 \in (\gamma_i - \frac{\delta}{4}\delta', \gamma_i + \frac{\delta}{4}\delta')$ and

$$f_{m,n}(x_0) \in [f_{m,n}(\beta_{i-1}) \pm \delta, f_{m,n}(\beta_i) \mp \delta] \tag{17}$$

(if $f_{m,n}(\beta_{i-1}) < f_{m,n}(\beta_i)$ then $\pm$ means $+$, $\mp$ means $-$, while for $f_{m,n}(\beta_{i-1}) > f_{m,n}(\beta_i)$ $\pm$ means $-$, $\mp$ means $+$).

It is not difficult to compute that $dist_{\mathcal{H}}(F(f_{m,n}, x_0, \frac{\delta}{4}), V) < 2\delta' < \frac{1}{m+n}$.

Clearly, if $\eta_i \in (0, \delta)$ is sufficiently small then

$$dist_{\mathcal{H}}(F(f, x_0, \frac{\delta}{4}), V) < \frac{1}{m+n} \tag{18}$$

for any $f \in B(f_{m,n}, \eta_i)$ and $x_0 \in (\gamma_i - \frac{\delta}{4}\delta', \gamma_i + \frac{\delta}{4}\delta')$ satisfying (17).

Do the above steps for all subdivisions of all maximal intervals of linearity of f_m until $f_{m,n}$ is defined on $[0, 1]$ and choose $\eta_{m,n} > 0$ such that it is less than any of the η_i's needed for any subdivision to have (18).

If $P_{\alpha,f} \neq \emptyset$ then by (16) there are at most two consecutive subdivision intervals $[\beta_{i-1}, \beta_i]$ for which (17) does not hold when $\alpha = f_{m,n}(x_0)$. In the next property the exceptional interval $E_{\alpha,f,m,n}$ corresponds to such intervals.

Now by (18) the following property holds: If $f \in B(f_{m,n}, \eta_{m,n})$ for each α there exists $E_{\alpha,f,m,n}$, a possibly empty, exceptional interval of length less or equal than $\frac{2}{m+n}$ such that if $t_0 \in \pi_x(L_{\alpha,f}) \setminus E_{\alpha,f,m,n}$ then there exists an open interval $I(t_0) \subset (t_0 - \frac{1}{m+n}, t_0 + \frac{1}{m+n})$ such that for all $x_0 \in I(t_0)$ there exists $\delta_{x_0} < \frac{1}{m+n}$ for which

$$dist_{\mathcal{H}}(F(f, x_0, \frac{\delta_{x_0}}{4}), V) < \frac{1}{m+n}. \tag{19}$$

Now set $\mathcal{G}_n = \bigcup_m B(f_{m,n}, \eta_{m,n})$ and $\mathcal{G} = \bigcap_n \mathcal{G}_n$.

If $f \in \mathcal{G}$ then there exists m_n such that $f \in B(f_{m_n,n}, \eta_{m_n,n})$. Assume $P_{\alpha,f} \neq \emptyset$. Denote by $H_{f,m_n,n,\alpha}$ the union of all intervals $I(t_0)$ for $t_0 \in \pi_x(L_{\alpha,f})$. Then $H_{f,m_n,n,\alpha}$ is open and if $t_0 \in \pi_x(L_{\alpha,f}) \setminus E_{\alpha,f,m_n,n}$ then $(t_0 - \frac{1}{m_n+n}, t_0 + \frac{1}{m_n+n}) \cap H_{f,m_n,n,\alpha} \neq \emptyset$.

Given $n' \in \mathbb{N}$ set $H'_{n'} = \bigcup_{n \geq n'} H_{f,m_n,n,\alpha}$ and $H' = \bigcap_{n'} H'_{n'}$. Then $H'_{n'}$ is dense and open in $P_{\alpha,f}$ and H' is dense $\mathcal{G}_\delta$. Furthermore, (19) implies that for all $x_0 \in H'$ we have $V \in f_{MT}(x_0)$. □

Denote by $VMT(f)$ the set of those $(x_0; f(x_0))$ for which we have two-sided vertical universality.

Theorem 19. *For the typical continuous function $VMT(f)$ is of non-σ-finite $\mathcal{H}^1$-measure.*

Proof. By Theorem 18 for any typical continuous function f there exists an interval I of positive length such that for any $\alpha \in I$ the intersection of the horizontal line $\ell_\alpha = \{(x;\alpha) : x \in \mathbb{R}\}$ and $VMT(f)$ contains a set of cardinality continuum. Proceeding towards a contradiction suppose that $VMT(f) \subset \cup_{n=1}^{\infty} V_n$ where $\mathcal{H}^1(V_n) < \infty$. Denote by V'_n the set of those $\alpha \in I$ for which $\ell_\alpha \cap V_n$ is infinite, and hence $\mathcal{H}^0(\ell_\alpha \cap V_n) = \infty$. By Theorem 10.10 of [7], $\mathcal{H}^0(\ell_\alpha \cap V_n) < \infty$ for almost all α. Therefore V'_n is of zero Lebesgue measure. Hence $\cup_{n=1}^{\infty} V'_n$ is also of zero measure and does not equal I. This would imply the existence of an $\alpha \in I$ such that $\ell_\alpha \cap VMT(f)$ is countable, a contradiction. □

The next theorem is about a c-dense set in the level set where we do not have two-sided universality.

Theorem 20. *For the typical continuous function f, for all $\alpha \in \mathbb{R}$, if $P_{\alpha,f} \neq \emptyset$ then there exists $\widehat{P}'_{\alpha,f} \subset P_{\alpha,f}$, c-dense in $P_{\alpha,f}$ such that if $x_0 \in \pi_x(\widehat{P}'_{\alpha,f})$ then we do not have two-sided vertical universality at x_0.*

Proof. Choose a dense subset $\{f_m\}_{m\in\mathbb{N}}$ in $C[0,1]$ consisting of piecewise linear functions such that all their extrema are at different height and they do not have intervals of constancy. Suppose $a, b \in [0,1] \cap \mathbb{Q}$, $a < b$ are fixed. Assume that $n \in \mathbb{N}$ is given. Put

$$L'_{\alpha,f_m,\eta} = \{x \in [a,b] : |\alpha - f_m(x)| \leq \eta\}.$$

Observe that $L'_{\alpha,f_m,\eta}$ consists of finitely many closed (possibly degenerate) intervals. Denote by $\Delta_{\alpha,m,\eta}$ the maximum of the lengths of these intervals.

The smaller η, the smaller $\Delta_{\alpha,m,\eta}$ and it goes to zero as $\eta \to 0+$. There are two open half lines and finitely many open intervals contiguous to $L'_{\alpha,f_m,\eta}$. Denote the (possibly infinite) length of the shortest and the second shortest of these by $\Delta_{\alpha,m,\eta,1}$ and $\Delta_{\alpha,m,\eta,2}$, respectively. The smaller η, the larger $\Delta_{\alpha,m,\eta,1}$ and $\Delta_{\alpha,m,\eta,2}$. If α is close to an extremum then $\Delta_{\alpha,m,\eta,1}$ might be short compared to $\Delta_{\alpha,m,\eta}$. Given $n \in \mathbb{N}$, using that all the extrema of f_m are of different height, and there are only finitely many of them, we can choose $\eta_{m,n} < \frac{1}{n}$ so small that for all $\alpha \in \mathbb{R}$ from $\pi_x(L_{\alpha,f_m}) \cap [a,b] \neq \emptyset$ it follows that

$$\Delta_{\alpha,m,\eta_{m,n},2} > 100\Delta_{\alpha,m,\eta_{m,n}} \text{ and } \Delta_{\alpha,m,\eta_{m,n}} < \frac{1}{n}. \tag{20}$$

If $\Delta_{\alpha,m,\eta_{m,n},1} \geq 8\Delta_{\alpha,m,\eta_{m,n}}$ then put $L_{\alpha,f_m,\eta_{m,n}} = L'_{\alpha,f_m,\eta_{m,n}}$.

If $\Delta_{\alpha,m,\eta_{m,n},1} < 8\Delta_{\alpha,m,\eta_{m,n}}$ then $L'_{\alpha,f_m,\eta_{m,n}}$ has two different components I'_1 and I'_2 which are separated by a distance of $\Delta_{\alpha,m,\eta_{m,n},1}$ and any other two components of $L'_{\alpha,f_m,\eta_{m,n}}$ are separated by a distance of at least $\Delta_{\alpha,m,\eta_{m,n},2} > 100\Delta_{\alpha,m,\eta_{m,n}}$, by (20). Denote by I' the shortest interval containing $I'_1 \cup I'_2$. Then $\lambda(I') < 10\Delta_{\alpha,m,\eta_{m,n}}$, where λ denotes the Lebesgue measure. Put $L_{\alpha,f_m,\eta_{m,n}} = L'_{\alpha,f_m,\eta_{m,n}} \cup I'$.

Both ways $L_{\alpha,f_m,\eta_{m,n}}$ consists of intervals of length at most $10\Delta_{\alpha,m,\eta_{m,n}}$, moreover we have

Property †: if $I = [x_I, y_I]$ is a component of $L_{\alpha,f_m,\eta_{m,n}}$ then the intervals (or half lines) (x'_I, x_I) and (y_I, y'_I) which are contiguous to $L_{\alpha,f_m,\eta_{m,n}}$ and have one endpoint in I are of length at least $8\lambda(I)$.

Now, set $\mathcal{G}_n = \bigcup_m B(f_m, \eta_{m,n}/2)$ and $\mathcal{G} = \bigcap_n \mathcal{G}_n$. Assume $f \in \mathcal{G} \cap BG$. Choose m_n such that $f \in B(f_{m_n}, \eta_{m_n,n}/2)$. Assume $\pi_x(P_{\alpha,f}) \cap (a,b) \neq \emptyset$ for an $\alpha \in \mathbb{R}$. Then $\pi_x(P_{\alpha,f}) \cap [a,b] \subset L_{\alpha,f_{m_n},\eta_{m_n,n}}$.

Choose a subsequence $n(k) \to \infty$ as $k \to \infty$ such that letting $m(k) = m_{n(k)}, \eta(k) = \eta_{m_{n(k)},n(k)}$ we have

$$\eta(k+1) < \eta(k)/4, \text{ and } \Delta_{\alpha,m(k+1),\eta(k+1)} < \Delta_{\alpha,m(k),\eta(k)}/2. \tag{21}$$

If $x' \in L'_{\alpha,f_{m(k+1)},\eta(k+1)}$ then $|\alpha - f_{m(k+1)}(x')| \leq \eta(k+1)$. We also have $|f_{m(k+1)}(x') - f_{m(k)}(x')| \leq |f_{m(k+1)}(x') - f(x')| + |f(x') - f_{m(k)}(x')| < \eta(k+1)/2 + \eta(k)/2$, this implies $|\alpha - f_{m(k)}(x')| < |\alpha - f_{m(k+1)}(x')| + |f_{m(k)}(x') - f_{m(k+1)}(x')| < 2\eta(k+1) + \eta(k)/2 < \eta(k)$. This implies $L'_{\alpha,f_{m(k+1)},\eta(k+1)} \subset L'_{\alpha,f_{m(k)},\eta(k)}$.

From (21) and the definition of $L_{\alpha,f_{m(k+1)},\eta(k+1)}$ it follows that $L_{\alpha,f_{m(k+1)},\eta(k+1)}$ is a subset of $L_{\alpha,f_{m(k)},\eta(k)}$. It is also clear that $\pi_x(P_{\alpha,f}) \cap [a,b] \subset L_{\alpha,f_{m(k)},\eta(k)}$ for all k.

Denote by $L^P_{\alpha,f_{m(k)},\eta(k)}$ the union of those components of $L_{\alpha,f_{m(k)},\eta(k)}$ which contain points of $\pi_x(P_{\alpha,f})\cap(a,b)$. Then $L^P_{\alpha,f_{m(k)},\eta(k)}$ consists of the union of finitely many disjoint closed intervals. Assume that I is such an interval. Now $I\cap L^P_{\alpha,f_{m(k+1)},\eta(k+1)}$ consists again of finitely many intervals and $\pi_x(P_{\alpha,f})\cap(a,b)\subset L^P_{\alpha,f_{m(k+1)},\eta(k+1)}$ implies that there is at least one such interval.

Denote by $\ell^*(I)$ the (possibly infinite) length of the interval contiguous to $L^P_{\alpha,f_{m(k)},\eta_{m(k)}}$ ending at the left endpoint of I. Since $L^P_{\alpha,f_{m(k)},\eta_{m(k)}}$ consists of some components of $L_{\alpha,f_{m(k)},\eta(k)}$ we have $\ell^*(I)\geq 8\lambda(I)$ by Property †.

CASE A:

If I contains only one component of $L^P_{\alpha,f_{m(k+1)},\eta(k+1)}$ then define the set $\widehat{L}_{\alpha,f_{m(k+1)},\eta(k+1)}$ so that $\widehat{L}_{\alpha,f_{m(k+1)},\eta(k+1)}\cap I$ equals this component of $L^P_{\alpha,f_{m(k+1)},\eta(k+1)}\cap I$.

CASE B:

If $L^P_{\alpha,f_{m(k+1)},\eta(k+1)}\cap I$ consists of more than one component of $L^P_{\alpha,f_{m(k+1)}\eta(k+1)}$ then define the set $\widehat{L}_{\alpha,f_{m(k+1)},\eta(k+1)}$ so that $\widehat{L}_{\alpha,f_{m(k+1)},\eta(k+1)}\cap I$ consists of the two most left components of $L^P_{\alpha,f_{m(k+1),\eta(k+1)}}\cap I$.

Set $\widehat{P}''_{\alpha,f}=\bigcap_{k=1}^{\infty}\widehat{L}_{\alpha,f_{m(k+1)},\eta(k+1)}$ and $\widehat{P}''_{\alpha,f,K}=\bigcap_{k=1}^{K}\widehat{L}_{\alpha,f_{m(k+1)},\eta(k+1)}$. Then $\widehat{P}''_{\alpha,f,K}$ consists of some components of $L^P_{\alpha,f_{m(K+1)},\eta(K+1)}$.

Observe that $\eta(k)\to\infty$ as $k\to\infty$ and by (21), $\Delta_{\alpha,m(k+1),\eta(k+1)}\to 0$ as well. This and $f\in BG$ imply that if $x\in\widehat{P}''_{\alpha,f}$ for the component $I(k,x)$ of $L^P_{\alpha,f_{m(k)},\eta(k)}$ containing x we need to apply *CASE B* infinitely often.

Clearly $\widehat{P}''_{\alpha,f,1}=\widehat{L}_{\alpha,f_{m(2)},\eta(2)}$. If I is a component of $\widehat{P}''_{\alpha,f,K-1}$ then it is a component of $L^P_{\alpha,f_{m(K)},\eta(K)}$ and $I\cap P_{\alpha,f}\neq\emptyset$. Thus I contains at least one component of $L^P_{\alpha,f_{m(K+1)},\eta(K+1)}$ and hence of $\widehat{P}''_{\alpha,f,K}$. Repeating this argument in subcomponents of this component we can see that $I\cap\widehat{P}''_{\alpha,f}\neq\emptyset$.

Since for any $x\in\widehat{P}''_{\alpha,f}$ we need to apply for $I(k,x)$ infinitely often *CASE B*, we also obtain that I contains more than one point of $\widehat{P}''_{\alpha,f}$.

Hence $\widehat{P}''_{\alpha,f}$ is nonempty and perfect. From $\eta(k)\to 0$ it also follows that $\widehat{P}''_{\alpha,f}\subseteq P_{\alpha,f}\cap[a,b]$.

Since $f\in BG$, $\widehat{P}''_{\alpha,f}$ is nowhere dense.

Assume $x_0\in\pi_x(\widehat{P}''_{\alpha,f})\cap(a,b)$, $x_1=-1$, $x_2=-0.5$, $x_3=0$. We claim that $V=V_{x_1,x_2,x_3}\notin f_{MT}(x_0)$.

Proceeding towards a contradiction suppose that $V \in f_{MT}(x_0)$. Then there exists arbitrarily small $\delta > 0$ such that we can choose

$$x'_1 \in (x_0 - 1.01\delta, x_0 - 0.99\delta),\ x'_2 \in (x_0 - 0.51\delta, x_0 - 0.49\delta),\ x'_3 = x_0$$

with $x'_1, x'_2, x'_3 \in \pi_x(P_{\alpha,f}) \cap (a, b)$.

We can also assume that δ is so small that x'_1, x'_2, x'_3 belong to the same component of $\widehat{L}_{\alpha,f_{m(1)},\eta(1)}$. Since $\eta(k) \to 0, f \in BG$ there is k_1 such that x'_1, x'_2, x'_3 do not belong to the same component of $\widehat{L}_{\alpha,f_{m(k_1+1)},\eta(k_1+1)}$, but they belong to the same component, I of $\widehat{L}_{\alpha,f_{m(k_1)},\eta(k_1)}$. Components of $\widehat{L}_{\alpha,f_{m(k_1+1)},\eta(k_1+1)}$ are components of $L^P_{\alpha,f_{m(k_1+1)},\eta(k_1+1)}$. From $x'_1, x'_2, x'_3 \in P_{\alpha,f}$ it follows that $x'_1, x'_2, x'_3 \in I \cap L^P_{\alpha,f_{m(k_1+1)},\eta(k_1+1)}$.

Denote by $I(1)$ and $I(3)$ the components of $I \cap L^P_{\alpha,f_{m(k_1+1)},\eta(k_1+1)}$ containing x'_1 and x'_3 respectively. Thus $I(1) \neq I(3)$. This is possible only if at step k_1 we apply *CASE B* and $I(1), I(3)$ are the two most left components of $I \cap L^P_{\alpha,f_{m(k_1+1)},\eta(k_1+1)}$. Hence x'_2 either belongs to $I(1)$, or to $I(3)$. This is impossible since this would imply that $\lambda(I(1)) > 0.48\delta$, or $\lambda(I(3)) > 0.48\delta$ and the gap between $I(1)$ and $I(3)$ less than 0.52δ, contradicting Property †.

Now for each $a, b \in [0, 1] \cap \mathbb{Q}$, $a < b$ we can find a residual set $\mathcal{F}_{a,b} \subset C[0, 1]$ such that for each $f \in \mathcal{F}_{a,b}$, and $\alpha \in \mathbb{R}$ if $\pi_x(P_{\alpha,f}) \cap (a, b) \neq \emptyset$ then there is a non-empty perfect set $\widehat{P}''_{\alpha,f,a,b}$ such that if $x_0 \in \pi_x(\widehat{P}''_{\alpha,f,a,b}) \setminus \{a\}$ then $V \notin f_{MT}(x_0)$.

Denote by $\mathcal{F}$ the countable intersection of all the above $\mathcal{F}_{a,b}$'s. Then $\mathcal{F}$ is still residual. If $f \in \mathcal{F}$ and for an $\alpha \in \mathbb{R}$, $P_{\alpha,f} \neq \emptyset$ then denote by $\widehat{P}'_{\alpha,f}$ the union of the sets $\widehat{P}''_{\alpha,f,a,b} \setminus \{(a; f(a))\}$ for all $a, b \in [0, 1] \cap \mathbb{Q}$, $a < b$. This will provide the c-dense subset of $P_{\alpha,f}$ required by the Lemma. □

A function $\phi : [0, +\infty) \to [0, +\infty]$ is called a Hausdorff function if it is monotone increasing, continuous from the right and $\phi > 0$ for $x > 0$. (See [8].) By results of Kirchheim [5] for the typical continuous function $L_{\alpha,f}$ is of zero Hausdorff dimension. In fact, if ϕ is an arbitrary Hausdorff function satisfying $\lim_{x \to 0+} \phi(x) = 0$ then $\mathcal{H}^\phi(L_{\alpha,f}) = 0$.

Theorem 21. *Assume ϕ is a Hausdorff function ϕ with $\lim_{x \to 0+} \phi(x) = 0$. For the typical continuous function, $f \in C[0, 1]$ there exists E_f with $\mathcal{H}^\phi(E_f) = 0$ such that*

(i) if $\alpha \notin E_f$ then f has right, or left vertical universality at $(x_0;\alpha)$ for all $(x_0;\alpha) \in P^+_{\alpha,f}$, or $P^-_{\alpha,f}$, respectively;
(ii) if $\alpha \in E_f$ then there is one exceptional point $(x_0^*;\alpha) \in P^+_{\alpha,f} \cup P^-_{\alpha,f}$ where f does not have right, or left vertical universality and for other $(x_0;\alpha) \in P^+_{\alpha,f} \cup P^-_{\alpha,f}$ (i) holds.

Remark 22. All extrema of f_γ with $\gamma \neq 0$ belong to E_f showing that the above exceptional points are c-dense on the graph of f.

Definition 23. For $\delta > 0, f \in C[0,1], (x_0;y_0) \in \mathbb{R}^2$ put

$$F(f,(x_0;y_0),\delta) = \frac{1}{\delta}\Big(graph(f) \cap Q((x_0;y_0),\delta) - (x_0;y_0)\Big).$$

Observe that $F(f,x_0,\delta) = F(f,(x_0;f(x_0)),\delta)$.

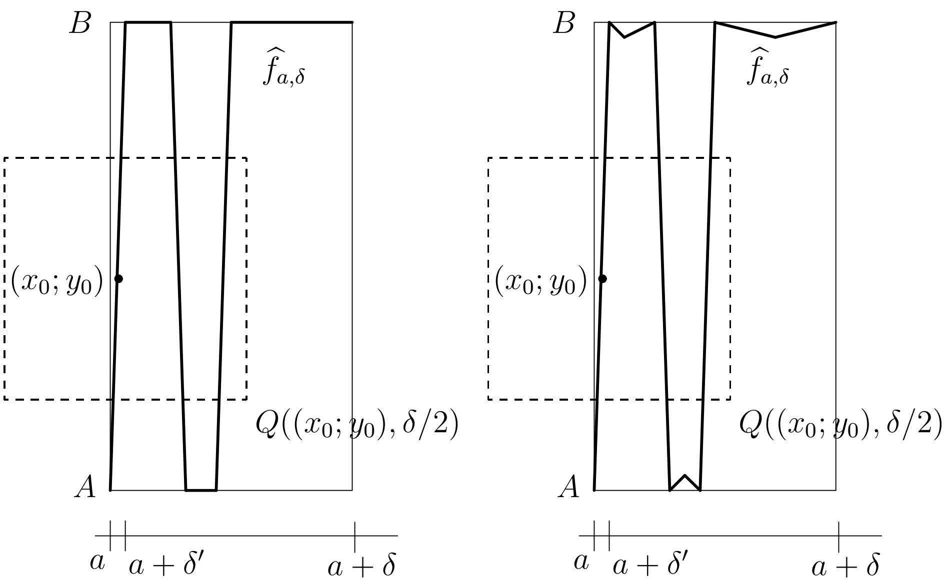

Figure 5: The definition of $\widehat{f}_{a,\delta}$ and its modification.

Lemma 24. *Assume $\nu \in \mathbb{N}$ is odd, $\epsilon_0 \in (0,1), A \neq B, V = V_{x_1,\ldots,x_\nu}$ are given with $x_1 = 0$ and $x_\nu < 1$. Furthermore, $\delta > 0$ and $[a,a+\delta] \subset [0,1]$. Then there* *exists* $\delta' > 0$ *and a piecewise linear continuous function* $\widehat{f}_{a,\delta} : [a, a+\delta] \to [A, B]$ *such that* $\widehat{f}_{a,\delta}(a) = A$, $\widehat{f}_{a,\delta}(a+\delta') = \widehat{f}_{a,\delta}(a+\delta) = B$, $\widehat{f}_{a,\delta}$ *is linear on* $[a, a+\delta']$ *and if* $x_0 \in [a, a+\delta']$, $y_0 \in [A, B]$, $|y_0 - A| \geq \delta$, $|y_0 - B| \geq \delta$ *then for any continuous function* $f : [a, a+\delta] \to [A, B]$ *satisfying* $f(x) = \widehat{f}_{a,\delta}(x)$ *when* $x \in \{a\} \cup [a+\delta', a+\delta]$ *we have*

$$dist_{\mathcal{H}}(F(f, (x_0; y_0), \frac{\delta}{2}), V) < \epsilon_0. \tag{22}$$

Proof. Without limiting generality assume $A < B$. Choose $\delta' > 0$ so that $\frac{4\delta'}{\delta} < \epsilon_0$ and $\frac{\delta}{2} x_\nu + \delta' < \frac{\delta}{2}$. Set

$$\widehat{f}_{a,\delta}(a + \frac{\delta}{2} x_j) = \begin{cases} A & \text{if} \quad j \text{ is odd;} \\ B & \text{if} \quad j \text{ is even} \end{cases}$$

$$\widehat{f}_{a,\delta}(a + \frac{\delta}{2} x_j + \delta') = \begin{cases} B & \text{if} \quad j \text{ odd;} \\ A & \text{if} \quad j \text{ even.} \end{cases}$$

Since ν is odd we have $\widehat{f}_{a,\delta}(a + \frac{\delta}{2} x_\nu + \delta') = B$. Extend the definition of $\widehat{f}_{a,\delta}$ onto $[a, a + \frac{\delta}{2} x_\nu + \delta']$ so that it is piecewise linear. For $x \in [a + \frac{\delta}{2} x_\nu + \delta', a+\delta]$ set $\widehat{f}_{a,\delta}(x) = B$. It is left to the reader to check that $\widehat{f}_{a,\delta}$ satisfies the assumptions of the lemma, see the left side of Figure 5. □

Remark 25. Observe that by a small alteration (see the right side of Figure 5) on the intervals of constancy of $\widehat{f}_{a,\delta}$ one can obtain functions which are piecewise linear and non-constant on any subinterval of $[a, a+\delta]$ and still satisfy the conclusion of Lemma 24. Indeed, if $|y_0 - A| \geq \delta$ and $|y_0 - B| \geq \delta$ then $Q((x_0; y_0), \delta/2)$ contains no points $(x_1; y_1)$ with $|y_1 - A| < \delta/2$ or $|y_1 - B| < \delta/2$. Hence, if we alter the definition of $\widehat{f}_{a,\delta}$ for some of those x's for which

$$|\widehat{f}_{a,\delta}(x) - A| < \delta/2, \text{ or } |\widehat{f}_{a,\delta}(x) - B| < \delta/2 \tag{23}$$

and after the alteration at these points we still have (23) then we can preserve the validity of (22).

One can choose $\widehat{\eta}_{a,\delta}$ sufficiently small such that if $f \in B(\widehat{f}_{a,\delta}, \widehat{\eta}_{a,\delta})$ then from $\alpha \in [A, B]$, $|\alpha - A| \geq \delta$, $|\alpha - B| \geq \delta$ it follows that if x_0 denotes the left endpoint of $\pi_x(L_{\alpha,f} \cap [a, a+\delta])$ then

$$dist_{\mathcal{H}}(F(f, (x_0; \alpha), \frac{\delta}{2}), V) < \epsilon_0. \tag{24}$$

Next choose and fix a sequence $V(n) = V_{x_1(n),...,x_{\nu(n)}(n)}$, such that $x_1(n) = 0$, $x_{\nu(n)}(n) < 1$, $\nu(n)$ is odd and for any $V = V_{x_1,...,x_\nu}$ with $x_1 = 0$ there exists a subsequence n_k such that $V(n_k) \to V$ in the Hausdorff metric.

Lemma 26. *Assume $f_0 \in C[0,1]$ is piecewise linear and non-constant on any interval, its local extrema are at different heights, $\epsilon_0 > 0$, and ϕ is a fixed Hausdorff function. Then there exists $\widehat{f_0} \in B(f_0, \epsilon_0)$, piecewise linear, continuous and non-constant on any subinterval of $[0,1]$, $\widehat{\eta}_0 > 0$, an exceptional set E_0 consisting of finitely many intervals $\mathcal{J}_\beta$ such that $\sum_\beta \phi(\lambda(\mathcal{J}_\beta)) < \epsilon_0$. Moreover, if $f \in B(\widehat{f_0}, \widehat{\eta}_0), \alpha \notin E_0, P_{\alpha,f} \neq \emptyset$ and (c,d), $c < d$ is an interval contiguous to $\pi_x(L_{\alpha,f})$ with $d - c > \epsilon_0$ then from $d \in (0,1)$ it follows that for any $j = 1, ..., n$ there exists $\delta_{j,d} \in (0, \epsilon_0)$ such that*

$$dist_{\mathcal{H}}(F(f,(d;\alpha),\delta_{j,d}), V(j)) < \epsilon_0,$$

and from $c \in (0,1)$ it follows that there exists $\delta_{j,c} \in (0, \epsilon_0)$ such that

$$dist_{\mathcal{H}}(F(f,(c;\alpha),\delta_{j,c}), -V(j)) < \epsilon_0.$$

If $\alpha \in E_0, P_{\alpha,f} \neq \emptyset$ then we have the above property with the possible exception of at most two intervals $(c_1,d_1),(c_2,d_2)$ defined as above. In case we have these two exceptional intervals $c_1 < d_1 < c_2 < d_2$ then $c_2 - d_1 < 2\epsilon_0$ and the endpoints c_1 and d_2 are not exceptional.

Proof. Since f_0 is piecewise linear denote by Γ_0, and γ_0 the maximum and minimum of the absolute values of the slopes of f_0, respectively. By our assumptions $0 < \gamma_0 \leq \Gamma_0 < \infty$. Denote by $\mathcal{E}_{f_0}$ the extrema of f_0 and by D_{extr} the minimum of the distances between different points in $\mathcal{E}_{f_0}$. Choose d_a such that

$$\frac{\Gamma_0}{\gamma_0} d_a < \frac{\min\{D_{extr}, \epsilon_0\}}{8}. \tag{25}$$

First divide $[0,1]$ into subintervals determined by $0 = a_0 < ... < a_t = 1$ so that

$$a_i - a_{i-1} < d_a < \frac{\gamma_0}{\Gamma_0} \frac{\min\{D_{extr}, \epsilon_0\}}{8} \leq \frac{\min\{D_{extr}, \epsilon_0\}}{8}, \tag{26}$$

$f_0(a_{i'}) \neq f_0(a_i)$ if $i \neq i'$, f_0 is linear on all intervals $[a_{i-1}, a_i]$, $i = 1, ..., t$, and each closed rectangle

$$R_i = [a_{i-1}, a_i] \times [f_0(a_{i-1}), f_0(a_i)] \subset \tag{27}$$
$$B^*(f_0, \epsilon_0) = \{(x;y) : x \in [0,1], f_0(x) - \epsilon_0 < y < f_0(x) + \epsilon_0\}.$$

Next choose $\delta_1 > 0$ such that,

$$[f_0(a_{i'}) - 4\delta_1, f_0(a_{i'}) + 4\delta_1] \cap [f_0(a_i) - 4\delta_1, f_0(a_i) + 4\delta_1] = \emptyset \text{ for } i \neq i' \quad (28)$$

and $\phi(8\delta_1)(t+1) < \frac{\epsilon_0}{2}$.

Set $E_0 = \cup_{i=0}^{t}[f_0(a_i) - 4\delta_1, f_0(a_i) + 4\delta_1]$. We can also suppose that

$$a_i - \delta_1 > a_{i-1} + \delta_1 \text{ holds for } i = 1, ..., t. \quad (29)$$

We are going to apply a similar procedure on all intervals of the form $[a_i, a_i + \delta_1], (i = 0, ..., t-1)$ and $[a_i - \delta_1, a_i], (i = 1, ..., t)$.

So assume i is fixed and without limiting generality consider $[a_i, a_i + \delta_1]$. First set $V = V(1)$, $A = f_0(a_i)$, $B = f_0(a_{i+1})$ and use Lemma 24 to find $\delta_1' > 0$ and the piecewise linear function $\widehat{f_1} : [a_i, a_i + \delta_1] \to [A, B]$ such that $\widehat{f_1}(a_i) = f_0(a_i) = A, \widehat{f_1}(a_i + \delta_1) = f_0(a_{i+1}) = B$, $\widehat{f_1}$ is linear on $[a_i, a_i + \delta_1'], \widehat{f_1}(a_i) = A, \widehat{f_1}(a_i + \delta_1') = \widehat{f_1}(a_i + \delta_1) = B$ and if $x_0 \in [a_i, a_i + \delta_1']$, $y_0 \in [A, B], |y_0 - A| \geq \delta_1, |y_0 - B| \geq \delta_1$, then for any continuous function $f : [a_i, a_i+\delta_1] \to [A, B]$ satisfying $f(x) = \widehat{f_1}(x)$ when $x \in \{a_i\} \cup [a_i+\delta_1', a_i+\delta_1]$ we have with $\delta_0' = \delta_1$

$$dist_{\mathcal{H}}(F(f, (x_0; y_0), \frac{\delta_0'}{2}), V(1)) < \epsilon_0. \quad (30)$$

Assume that $2 \leq j \leq n$ and the piecewise linear function $\widehat{f}_{j-1}$ is defined on $[a_i, a_i + \delta_1]$, $0 < \delta_{j-1}' < \delta_{j-2}' < ... < \delta_1' < \delta_0' = \delta_1, \widehat{f}_{j-1}(a_i + \delta_{j'}') = B, j' = 0, ..., j-1, \widehat{f}_{j-1}(x) \in [A, B]$, for all $x \in [a_i, a_i+\delta_1]$, $\widehat{f}_{j-1}$ is linear on $[a_i, a_i+\delta_{j-1}']$ and if $x_0 \in [a_i, a_i + \delta_{j'-1}'], y_0 \in [A, B]$, $|y_0 - A| \geq \delta_1 \geq \delta_{j'-1}', |y_0 - B| \geq \delta_1 \geq \delta_{j'-1}'$ then for any continuous function $f : [a_i, a_i + \delta_1] \to [A, B]$ satisfying $f(x) = \widehat{f}_{j-1}(x)$ when $x \in \{a_i\} \cup [a_i + \delta_{j-1}', a_i + \delta_1]$ we have

$$dist_{\mathcal{H}}(F(f, (x_0; y_0), \frac{\delta_{j'-1}'}{2}), V(j')) < \epsilon_0$$

for $j' = 1, ..., j-1$.

Next apply Lemma 24 with $V = V(j), A = f_0(a_i), B = f_0(a_{i+1})$ on the interval $[a_i, a_i + \delta_{j-1}']$ to find $\delta_j' > 0$ and the piecewise linear function $\widehat{f_j} : [a_i, a_i+\delta_{j-1}'] \to [A, B]$ such that $\widehat{f_j}(a_i) = A, \widehat{f_j}(a_i+\delta_j') = \widehat{f_j}(a_i+\delta_{j-1}') = B$ and if $x_0 \in [a_i, a_i+\delta_j'], y_0 \in [A, B]$, $|y_0-A| \geq \delta_1 \geq \delta_{j-1}', |y_0-B| \geq \delta_1 \geq \delta_{j-1}'$ then for any continuous function $f : [a_i, a_i + \delta'_{j-1}] \to [A, B]$ satisfying $f(x) = \widehat{f}_j(x)$ when $x \in \{a_i\} \cup [a_i + \delta'_j, a_i + \delta'_{j-1}]$ we have

$$dist_{\mathcal{H}}(F(f, (x_0; y_0), \frac{\delta'_{j-1}}{2}), V(j)) < \epsilon_0.$$

Extend the definition of $\widehat{f}_j$ onto $[a_i + \delta'_{j-1}, a_i + \delta_1]$ so that $\widehat{f}_j = \widehat{f}_{j-1}$. We have $\widehat{f}_j(a_i + \delta'_{j'}) = B$ for $j' = 0, ..., j$, $\widehat{f}_j(x) \in [A, B]$ for all $x \in [a_i, a_i + \delta_1]$, $\widehat{f}_j$ is linear on $[a_i, a_i + \delta'_j]$ and if $x_0 \in [a_i, a_i + \delta'_j]$, $y_0 \in [A, B]$, $|y_0 - A| \geq \delta_1 \geq \delta'_{j'}$, $|y_0 - B| \geq \delta_1 \geq \delta'_{j'}$ then for any continuous function $f : [a_i, a_i + \delta_1] \to [A, B]$ satisfying $f(x) = \widehat{f}_j(x)$ when $x \in \{a_i\} \cup [a_i + \delta'_j, a_i + \delta_1]$ we have

$$dist_{\mathcal{H}}(F(f, (x_0; y_0), \frac{\delta'_{j'-1}}{2}), V(j')) < \epsilon_0$$

for $j' = 1, ..., j$.

We repeat the above steps for all $j \leq n$.

Next we define $\widehat{f}_n$ on $[0, 1]$. We define it as above on all intervals $[a_i, a_i + \delta_1], (i = 0, ..., t-1)$ and by a "mirror" procedure, using $-V(j), (j = 1, ..., n)$ on $[a_i - \delta_1, a_i], (i = 1, ..., t)$. Finally, we extend $\widehat{f}_n$ to intervals of the form $[a_{i-1} + \delta_1, a_i - \delta_1], (i = 1, ..., t)$ so that it is linear on these intervals. This implies that if $x \in [a_{i-1}, a_i]$ then $\widehat{f}_n(x) \in [\widehat{f}_n(a_{i-1}), \widehat{f}_n(a_i)] = [f_0(a_{i-1}), f_0(a_i)]$, that is $(x; \widehat{f}_n(x)) \in R_i$. Hence, by (27), $\widehat{f}_n \in B(f_0, \epsilon_0)$. We put $\widehat{f}_0 = \widehat{f}_n$. For $x \in [a_{i-1}, a_i]$ we have

$$\begin{aligned} |\widehat{f}_0(x) - f_0(x)| = &|\widehat{f}_n(x) - f_0(x)| \leq |f_0(a_i) - f_0(a_{i-1})| \leq \\ &\Gamma_0 |a_i - a_{i-1}| < \Gamma_0 d_a. \end{aligned} \tag{31}$$

Analogously to Remark 25 we can adjust our function so that $\widehat{f}_0$ is not constant on any subinterval of $[0, 1]$, we still have (31), and can choose

$$\widehat{\eta}_0 > 0, \text{ satisfying } \widehat{\eta}_0 < \frac{\min\{\delta_1, \gamma_0 D_{extr}, \gamma_0 \epsilon_0\}}{16} \tag{32}$$

such that for all $i = 1, ..., t$ if $f \in B(\widehat{f}_0, \widehat{\eta}_0)$ then from $\alpha \in [\widehat{f}_0(a_{i-1}), \widehat{f}_0(a_i)]$, $|\alpha - \widehat{f}_0(a_{i-1})| \geq \delta_1$, $|\alpha - \widehat{f}_0(a_i)| \geq \delta_1$ it follows that if $x_{0,\ell}$ denotes the left endpoint of $\pi_x(L_{\alpha,f}) \cap [a_{i-1}, a_i]$ then for all $j = 1, ..., n$ there exists $0 < \delta_{x_0,j} < \epsilon_0$ such that

$$dist_{\mathcal{H}}(F(f|_{[a_{i-1}, a_i]}, x_{0,\ell}, \delta_{x_0,j}), V(j)) < \epsilon_0.$$

Similarly if $x_{0,r}$ denotes the right endpoint of $\pi_x(L_{\alpha,f}) \cap [a_{i-1}, a_i]$ then for all $j = 1, ..., n$ there exists $0 < \delta_{x_0,j} < \epsilon_0$ such that

$$dist_{\mathcal{H}}(F(f\mid_{[a_{i-1},a_i]}, x_{0,r}, \delta_{x_0,j}), -V(j)) < \epsilon_0.$$

By (31) for all $x \in [0,1]$

$$|f(x) - f_0(x)| < \Gamma_0 d_a + \widehat{\eta}_0. \tag{33}$$

Now assume that $\alpha \notin E_0, P_{\alpha,f} \neq \emptyset$, and $(c,d), c < d$ is an interval contigous to $\pi_x(L_{\alpha,f})$ with $d - c > \epsilon_0$. Without limiting generality we work with the endpoint d, a similar "mirror" argument works for c as well.

Assume $d \in (0,1)$. There exists an i such that $d \in [a_{i-1}, a_i], d \neq a_{i-1}$. First we show that $a_{i-1} \notin \mathcal{E}_{f_0}$. Indeed, proceeding towards a contradiction, without limiting generality, assume that $(a_{i-1}; f_0(a_{i-1}))$ is a local maximum of f_0. Then f_0 is increasing and linear on $[a_{i-1} - D_{extr}, a_{i-1}]$ with slope greater or equal than γ_0 and on $[a_{i-1}, a_{i-1} + D_{extr}]$, f_0 is decreasing and linear with slope greater than $-\Gamma_0$. This, (25), (26) and (33) imply that

$$\begin{aligned} &|f(d) - f_0(a_{i-1})| < |f_0(d) - f_0(a_{i-1})| + \Gamma_0 d_a + \widehat{\eta}_0 \leq \\ &\Gamma_0 |a_i - a_{i-1}| + \Gamma_0 d_a + \widehat{\eta}_0 < 2\Gamma_0 d_a + \widehat{\eta}_0 < \gamma_0 \frac{\min\{D_{extr}, \epsilon_0\}}{4} + \widehat{\eta}_0. \end{aligned} \tag{34}$$

From $\alpha = f(d) \notin E_0$, it follows that $|f(d) - f_0(a_{i-1})| > 4\delta_1$. Hence, using $f_0(a_{i-1}) = \widehat{f}_0(a_{i-1})$, $|\widehat{f}_0(a_{i-1}) - f(a_{i-1})| < \widehat{\eta}_0$ we have

$$f(a_{i-1}) - f(d) > 4\delta_1 - \widehat{\eta}_0 > 3\delta_1 > 0.$$

Set $c' = a_{i-1} - \frac{\min\{D_{extr}, \epsilon_0\}}{2}$. Then $d - c > \epsilon_0$ and (26) imply $c \leq c' < d$ and f_0 is increasing and linear on $[c', a_{i-1}]$ with slope greater than or equal to γ_0. Using this, (25), (32), (33) and (34) we infer

$$\begin{aligned} &f(c') < f_0(c') + \Gamma_0 d_a + \widehat{\eta}_0 \leq f_0(a_{i-1}) - \gamma_0 \frac{\min\{D_{extr}, \epsilon_0\}}{2} + \Gamma_0 d_a + \widehat{\eta}_0 < \\ &f(d) + 2\widehat{\eta}_0 - \frac{1}{8}\gamma_0 \min\{D_{extr}, \epsilon_0\} < f(d) = \alpha. \end{aligned}$$

This would imply $f(c') < \alpha = f(d) < f(a_{i-1})$ and hence the segment connecting $(c; \alpha)$ and $(d; \alpha)$, would intersect the graph of f somewhere on the segment $[c', a_{i-1}] \times \alpha$, contradicting our assumption that (c,d) is contiguous to $\pi_x(L_{\alpha,f})$. Hence we can assume that $a_{i-1} \notin \mathcal{E}_{f_0}$.

From $f(d) = \alpha \notin E_0$ it follows that $|\alpha - \widehat{f}_0(a_{i-1})| = |\alpha - f_0(a_{i-1})| \geq 4\delta_1$ and $|\alpha - \widehat{f}_0(a_i)| = |\alpha - f_0(a_i)| \geq 4\delta_1$. We also have $|\widehat{f}_0(d) - \alpha| < \widehat{\eta}_0 < \delta_1, \widehat{f}_0(d) \in [\widehat{f}_0(a_{i-1}), \widehat{f}_0(a_i)]$, which implies $\alpha \in [\widehat{f}_0(a_{i-1}), \widehat{f}_0(a_i)]$. Therefore, $d = x_{0,\ell}$, equals the left endpoint of $\pi_x(L_{\alpha,f}) \cap [a_{i-1}, a_i]$ and hence for all $j = 1, ..., n$ there exists $\delta_{j,d} < \epsilon_0$ such that

$$dist_{\mathcal{H}}(F(f\mid_{[a_{i-1},a_i]}, d, \delta_{j,d}), V(j)) < \epsilon_0.$$

By (25), (26), (29) and $\delta_{j,d} \leq \delta_1$ we have $d - \delta_{j,d} \in [a_{i-2}, a_i]$ and we need to see that $Q_d = Q((d; f(d)), \delta_{j,d}) = Q((d; \alpha), \delta_{j,d})$ does not contain points of the form $(x; f(x))$ with $x \in [a_{i-2}, a_{i-1}]$.

Since $a_{i-1} \notin \mathcal{E}_{f_0}$, f_0 is linear on $[a_{i-2}, a_i]$. Thus R_{i-1} and R_i intersect each other only in their vertex at $(a_{i-1}; f_0(a_{i-1}))$. From this, $\delta_{j,d} \leq \delta_1$ and $|\alpha - \widehat{f}_0(a_{i-1})| \geq 4\delta_1$, it is not difficult to see that $(x; f(x)) \notin Q_d$ when $x \in [a_{i-2}, a_{i-1}]$. Hence

$$dist_{\mathcal{H}}(F(f, d, \delta_{j,d}), V(j)) < \epsilon_0$$

for $j = 1, ..., n$.

Now if $\alpha \in E_0$ and $P_{\alpha,f} \neq \emptyset$ then there is at most one i' such that $\alpha \in [f_0(a_{i'}) - 2\delta_1, f_0(a_{i'}) + 2\delta_1]$. If (c, d) is an interval continguous to $E_{\alpha,f}$ and the endpoint d (or c) does not belong to $[a_{i'}, a_{i'+1}]$, (or to $[a_{i'-1}, a_{i'}]$, respectively) then we can argue as before. Since $d - c > \epsilon_0$ there can be at most two intervals with endpoints in $[a_{i'-1}, a_{i'+1}]$ and the distance of these endpoints is less than $2\epsilon_0$. □

Next we turn to the proof of Theorem 21.

Proof. Choose a dense set f_m in $C[0, 1]$ consisting of piecewise linear functions such that the extrema of these functions are at different height. Set $\epsilon_{m,n} = 2^{-(m+n)}$ and using Lemma 26 choose functions $\widehat{f}_{m,n} \in B(f_m, 2^{-(m+n)})$, and $\widehat{\eta}_{m,n} < 2^{-(m+n)}$ with exceptional sets $E_{m,n}$ consisting of finitely many intervals $\mathcal{J}_{\beta,m,n}$ such that $\sum_\beta \phi(\lambda(\mathcal{J}_{\beta,m,n})) < 2^{-(m+n)}$. Moreover, if $f \in B(\widehat{f}_{m,n}, \widehat{\eta}_{m,n}), \alpha \notin E_{m,n}, P_{\alpha,f} \neq \emptyset$ and $(c, d), c < d$ is an interval contiguous to $\pi_x(L_{\alpha,f})$ with $d - c > 2^{-(m+n)}$ from $d \in (0, 1)$ it follows that for any $j = 1, ..., n$ there exists $\delta_{j,m,n,d} \in (0, 2^{-(m+n)})$ such that

$$dist_{\mathcal{H}}(F(f, d, \delta_{j,m,n,d}), V(j)) < 2^{-(m+n)} \tag{35}$$

and from $c \in (0,1)$ it follows that there exists $\delta_{j,m,n,c} \in (0, 2^{-(m+n)})$ such that

$$dist_{\mathcal{H}}(F(f,c,\delta_{j,m,n,c}), -V(j)) < 2^{-(m+n)}. \tag{36}$$

If $\alpha \in E_{m,n}, P_{\alpha,f} \neq \emptyset$ then we have the above property with the possible exception of at most two intervals $(c_{1,m,n}, d_{1,m,n}), (c_{2,m,n}, d_{2,m,n})$ defined as in Lemma 26. In case we have these two exceptional intervals $c_{1,m,n} < d_{1,m,n} < c_{2,m,n} < d_{2,m,n}$ then $c_{2,m,n} - d_{1,m,n} < 2 \cdot 2^{-(m+n)}$ and the two other endpoints $c_{1,m,n}$ and $d_{2,m,n}$ are not exceptional.

Put $\mathcal{G}_n = \bigcup_m B(\widehat{f}_{m,n}, \widehat{\eta}_{m,n})$ and $\mathcal{G} = \bigcap_n \mathcal{G}_n$.

Assume $f \in \mathcal{G}$ and choose m_n such that $f \in B(\widehat{f}_{m_n,n}, \widehat{\eta}_{m_n,n})$.

Set $E_f = \bigcap_{N=1}^{\infty} \bigcup_{n \geq N} E_{m_n,n}$. Obeserve that $E_{m_n,n}$ consists of finitely many intervals $\mathcal{J}_{\beta,m_n,n}$ with $\sum_{\beta} \phi(\lambda(\mathcal{J}_{\beta,m_n,n})) < 2^{-(m_n+n)}$. Therefore, $\mathcal{H}^{\phi}(E_f) = 0$. If $\alpha \notin E_f, P_{\alpha,f} \neq \emptyset$ and (c,d) is an interval contiguous to $\pi_x(L_{\alpha,f})$ then we can apply (35) and (36) for infinitely many n and hence for any n' we have $V(n') \in f_{MT}(d)$ when $d \in (0,1)$ and $-V(n') \in f_{MT}(c)$ when $c \in (0,1)$. This implies half universality at c and d.

Assume $\alpha \in E_f, P_{\alpha,f} \neq \emptyset$ and e_0 is an endpoint of an interval (c_0, d_0) contiguous to $\pi_x(L_{\alpha,f})$ and we cannot apply (35), or (36) at e_0 infinitely often. For each n there can be at most one other interval $(c'_{0,n}, d'_{0,n})$ contiguous to $\pi_x(L_{\alpha,f})$ such that we cannot apply (35), or (36) at one endpoint, say $e'_{0,n}$ of this interval and $|e'_{0,n} - e_0| < 2 \cdot 2^{-(m_n+n)}$. Since $2^{-(m_n+n)} \to 0$ as $n \to \infty$ this $e'_{0,n}$ cannot be the same endpoint for infinitely many n's. Thus if $e'_0 \neq e_0$ is an endpoint of an interval contigous to $\pi_x(L_{\alpha,f})$ then we can apply (35) or (36) infinitely often at e'_0 showing that we have half-universality at e'_0. $\square$

References

[1] A.M. Bruckner, *Differentiation of real functions, Second edition, CRM Monograph series 5.*, American Mathematical Society, Providence, RI, 1994.

[2] A. M. Bruckner, K. M. Garg, The level structure of a residual set of continuous functions. *Trans. Amer. Math. Soc.* **232** (1977), 307–321.

[3] Z. Buczolich, Micro tangent sets of continuous functions, Vol. 128, No. 2, pp. 147-167, 2003, *Mathematica Bohemica*, Budapest 2003.

[4] K. Falconer, *Fractal Geometry*, John Wiley & Sons, (1990).

[5] B. Kirchheim, Hausdorff measure and level sets of typical continuous mappings in Euclidean spaces. *Trans. Amer. Math. Soc.* **347** (1995), no. 5, 1763–1777.

[6] R. D. Mauldin and S. C. Williams, On the Hausdorff dimension of some graphs, *Trans. Amer. Math. Soc.*, **298** (1986) no. 2, 793-803.

[7] P. Mattila, *Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability.* Cambridge Studies in Advanced Mathematics, **44.** Cambridge University Press, Cambridge, 1995.

[8] C. A. Rogers, *Hausdorff Measures*, Cambridge University Press 1970.