Micro Tangent Sets of Continuous Functions

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Abstract
Motivated by the concept of tangent measures and by H. Fürstenberg’s definition of microsets of a compact set A we introduce micro tangent sets and central micro tangent sets of continuous functions. It turns out that the typical continuous function has a rich (universal) micro tangent set structure at many points. The Brownian motion on the other hand with probability one does not have graph like, or central graph like micro tangent sets at all. Finally we show that at almost all points Takagi’s function is graph like, and Weierstrass’s nowhere differentiable function is central graph like.

1 Introduction

In Mathematical Reviews 97i:28009, the reviewer (Joan Verdera), wrote the following: “Tangent measures play, with respect to measures, the same role that derivatives play with respect to functions. Given a measure μ (locally finite Borel measure on \( \mathbb{R}^n \)) and a point, one looks at the measure in a small neighborhood of the point, blows it up, normalizes suitably and takes a weak star limit in the space of measures. The result is a tangent measure for μ at the given point.”

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In this paper from measures we return to continuous functions and we see that this concept of blowing up and taking suitable limits, this time in the Hausdorff metric, might be useful to obtain information about tangential “regularity” of irregular functions. To do so we will introduce the concept of micro tangent sets of a continuous function. If the function is differentiable then the only micro tangent set we can obtain at a point is a straight line segment with the slope of the derivative. This shows that we have a generalization of the derivative.

For more irregular functions a much wider class of micro tangent sets may exist for many points. First we consider the typical continuous functions, then the Brownian motion.

It turns out that the micro tangent structure of the typical (in the sense of Baire category) \( f \in \mathcal{C}[0,1] \) for almost every \( x \in [0,1] \) is universal. Though \( UMT(f) \), the set of points \( \{x; f(x)\} \) which are universal, have some pathological properties. Its projection onto the \( y \)-axis is of measure zero and it is of \( \sigma \)-finite one dimensional Hausdorff measure while the graph of the typical continuous function is of non-\( \sigma \)-finite one dimensional Hausdorff measure. We also show that the packing dimension of \( UMT(f) \) is two, which is also the packing dimension of the graph of the typical continuous function. We remark that differentiability properties of typical continuous functions were studied for a long while and still are subject of current research, see for example Zajíček [20], and Zajíček and Preis [18] and the references in these papers.

The Brownian motion turns out to be wilder than the typical continuous function. With probability one the Brownian path \( W(t) \) does not have any universal micro tangent points. If vertical distortion is allowed in taking the weak limits of tangent spaces of random processes see also the recent papers of K. Falconer [4], [5]. We introduce some weaker conditions than universality, called graph like, or central graph like points, but the Brownian motion path is not behaving well with respect to these concepts either.

In Theorem 2 we see that on the graph of any continuous function the set of graph or central graph like points is of \( \sigma \)-finite \( H^1 \)-measure.

Finally we look at two examples of “irregular functions”: Takagi’s function and Weierstrass’s nowhere differentiable function. We see that these nowhere differentiable functions are tamer than the Brownian motion and almost every point is graph like for Takagi’s function and central graph like for Weierstrass’s nowhere differentiable function. Takagi’s function has also a property which we call “micro-self similarity”. This means that at almost every point the graph of the original function is a micro tangent set. The universality of typical continuous functions implies that they are also “micro-self similar”.

We denote by \( Q^{(m)} \) the closed unit cube of \( \mathbb{R}^m \).

Working with a compact subset \( A \subseteq \mathbb{R}^m \) in his talk [6] H. Fürstenberg defined the microsets of \( A \) the following way: \( A' \) is a microset of \( A \) if there exist sequences of scaling constants \( \gamma_n \in \mathbb{R} \) and translation vectors \( t_n \in \mathbb{R}^m \) such that \( \gamma_n A + t_n \cap Q^{(m)} \) converges to \( A' \) in the Hausdorff metric.

In this paper we will deal with special compact sets of \( \mathbb{R}^2 \), namely with graphs of continuous functions defined on \([0,1]\). This definition and the defi-
nition of tangent measures (see Preiss [17], Mattila [15], and Falconer [3] Ch. 9.) motivated our concept of micro tangent (M-tangent) sets of continuous functions.

2 Notation and Basic Definitions

Points in $\mathbb{R}^2$ will be denoted by $(x; y)$ while the open interval with endpoints $x$ and $y$ will be denoted by $(x, y)$.

Given $A \subset \mathbb{R}^2$ by $|A|$, $\text{int}(A)$, and $cl(A)$ we mean its diameter, interior, and closure, respectively.

The closed cube of side length $2\delta > 0$ centered at $(x; y)$ will be denoted by $Q((x; y), \delta)$, that is, $Q((x; y), \delta) = \{ (x'; y') : |x' - x| \leq \delta \text{ and } |y' - y| \leq \delta \}$. Let $Q^2$ be the closed cube of side length 1 centered at $(0, 0)$, that is, $Q((0; 0), 1)$.

If $F \subset \mathbb{R}^2$ then we denote by $CENT(F)$ the connected component of $F \cap Q^2$ which contains $(0; 0)$, this component is the central component of $F$ in $Q^2$.

The projections of the coordinate plane onto the $x$, or $y$ axis are denoted by $\pi_x$, or $\pi_y$, respectively.

By $\text{dist}_H(A, B)$ we mean the Hausdorff distance of two compact sets $A$ and $B$.

The one-dimensional Hausdorff measure in $\mathbb{R}^2$ will be denoted by $H^1$, the Lebesgue measure on $\mathbb{R}$ will be denoted by $\lambda$.

It is not difficult to see that Vitali’s covering theorem is also valid for coverings by closed squares, that is, the following variant of Theorem 2.8 of [14] holds.

**Theorem 1.** Let $\mu$ be a Radon measure on $\mathbb{R}^2$, $A \subset \mathbb{R}^2$ and $Q$ a family of closed squares such that each point of $A$ is the centre of arbitrarily small squares of $Q$, that is,

$$\inf\{ r : Q((x; y), r) \in Q \} = 0 \text{ for } (x; y) \in A.$$

Then there are disjoint squares $Q_i \in Q$ such that

$$\mu\left(A \setminus \bigcup_i Q_i\right) = 0.$$

By $C[-1, 1]_0$ we mean the set of those functions $g$ in $C[-1, 1]$ for which $g(0) = 0$.

**Definition 1.** The micro tangent (M-tangent) set system of $f \in C[0, 1]$ at the point $x_0 \in (0, 1)$ will be denoted by $f_{MT}(x_0)$ and it is defined as follows.

For $\delta_n > 0$ we put

$$F(f; x_0, \delta_n) = \frac{1}{\delta_n}\left((\text{graph}(f) \cap Q((x_0; f(x_0)), \delta_n)) - (x_0; f(x_0))\right),$$

that is, $F(f; x_0, \delta_n)$ is the $1/\delta_n$-times enlarged part of $\text{graph}(f)$ belonging to $Q((x_0; f(x_0)), \delta_n)$ translated into $Q^2$. 

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The set $F$ is a micro tangent set (M-tangent set) of $f$ at $x_0$, that is, $F \in f_{MT}(x_0)$ if there exists $\delta_0 > 0$ such that $F(f, x_0, \delta) \text{ converges to } F$ in the Hausdorff metric. The set $F$ is a central-micro tangent set (CM-tangent set) of $f$ at $x_0$, that is $F \in f_{CMT}(x_0)$ if there exists $\delta_0 > 0$ such that $CENT(F(f, x_0, \delta_0)) \text{ converges to } F$ in the Hausdorff metric.

It is easy to see that if $f$ is differentiable at $x_0$ then $f_{MT}(x_0) = f_{CMT}(x_0)$ consists of one line segment of slope $f'(x_0)$ passing through the origin.

**Definition 2.** We say that $x_0$ is a graph like, or a central graph like MT-point for $f$ if there exists $g \in C[-1, 1]_0$ such that $\text{graph}(g) \cap Q^2 \in f_{MT}(x_0)$, or $\text{graph}(g) \cap Q^2 \in f_{CMT}(x_0)$, respectively. We denote by $GLMT(f)$, or by $CGLMT(f)$ the set of graph like micro tangent points, or the set of central graph like micro tangent points of $f$, that is, the set of those $(x_0; f(x_0))$ for which $x_0$ is a graph like, or central graph like MT-point of $f$, respectively.

Clearly, $CGLMT(f) \supset GLMT(f)$.

**Definition 3.** We say that $x_0$ is a universal MT-point for $f$ if $\text{graph}(g) \cap Q^2 \in f_{MT}(x_0)$ for every $g \in C[-1, 1]_0$.

The collection of those points $(x_0; f(x_0))$ for which $x_0$ is a universal MT-point of $f$ will be denoted by $UMT(f)$.

**Definition 4.** We denote by $GLMT_g(f)$, or by $CGLMT_g(f)$ for a fixed $g \in C[-1, 1]_0$ the set of those $(x_0; f(x_0))$ for which $\text{graph}(g) \cap Q^2$ belongs to $f_{MT}(x_0)$, or $CENT(\text{graph}(g) \cap Q^2)$ belongs to $f_{CMT}(x_0)$, respectively.

Clearly, $GLMT_g(f) \subset GLMT(f)$, $GLMT_g(f) \subset CGLMT_g(f)$. Using the fact that we can always find a $g_1 \in C[-1, 1]_0$ for which $\text{graph}(g_1) \cap Q^2 = CENT(\text{graph}(g) \cap Q^2)$ we also have $CGLMT_g(f) \subset CGLMT(f)$.

### 3 Typical continuous functions

We start with a result which is valid for an arbitrary continuous function. Theorem 2 shows that on the graph of $f$ the set of graph like or central graph like points cannot be too large.

**Theorem 2.** For any function $f \in C[0, 1]$ the sets $GLMT(f)$ and $CGLMT(f)$ are of $\sigma$-finite $H^1$-measure.

**Proof.** By $CGLMT(f) \supset GLMT(f)$ it is enough to prove the theorem for $CGLMT(f)$. Given $\delta, \epsilon > 0$ denote by $E_{\delta, \epsilon}$ the set of those $(x_0; f(x_0))$ for which $CENT(F(f, x_0, \delta))$ does not intersect the line segments $L_{1, \epsilon} = \{t; 1 : |t| \leq \epsilon\}$ and $L_{-1, \epsilon} = \{(t; -1) : |t| \leq \epsilon\}$.

Using that $f$ is continuous and that $L_{1, \epsilon} \cup L_{-1, \epsilon}$ is a closed set one can easily see that $E_{\delta, \epsilon}$ is a relatively open subset of $\text{graph}(f)$.

Next, assume that $g \in C[-1, 1]_0$ and $\text{graph}(g) \cap Q^2 \in f_{CMT}(x_0)$. Then there exists $\epsilon > 0$ such that $\text{graph}(g) \cap (L_{1, \epsilon} \cup L_{-1, \epsilon}) = \emptyset$. Since
graph(g) ∩ Q^2 and L_{1,ε} ∪ L_{-1,ε} are both compact by the triangle inequality there exists ε' > 0 such that if dist_{H}(graph(g) ∩ Q^2, CENT(F(f, x_0, δ))) < ε' then CENT(F(f, x_0, δ)) does not intersect L_{1,ε} ∪ L_{-1,ε}. Using the definition of f_CMT(x_0) we can choose δ_n → 0 such that CENT(F(f, x_0, δ_n)) does not intersect L_{1,ε} ∪ L_{-1,ε}.

Hence,

\[ (x_0; f(x_0)) \in \bigcap_{n=1}^{∞} \bigcup_{0<δ<1/n} E_{δ,ε}. \]

Therefore, from \((x_0; f(x_0)) \in CGLMT(f)\) it follows that

\[ (x_0; f(x_0)) \in \bigcap_{m=1}^{∞} \bigcup_{n=1}^{∞} \bigcup_{0<δ<1/n} E_{δ,1/m} = H_0. \]

Next, we verify that \(H = \bigcap_{n=1}^{∞} \bigcup_{0<δ<1/n} E_{δ,1/m}\) is of σ-finite \(H^1\)-measure for every \(m \in \mathbb{N}\). This will imply that \(H_0\), and hence \(CGLMT(f)\) is also of σ-finite \(H^1\)-measure.

Clearly, \(H_0\) and \(H\) are Borel sets.

Proceeding towards a contradiction assume that \(H^1(H) = ∞\). Then by [2] Theorem 4.10 and Exercise 4.8 for any fixed \(c\) we can choose a Borel subset \(H_c \subset H\) such that \(H^1(H_c) = c\). Assume \(η > 0\) is given. Now, considering for each \((x_0; f(x_0)) \in H_c\) those squares \(Q((x_0; f(x_0)), δ_n, x_0)\) for which \(CENT(F(f, x_0, δ_n, x_0))\) does not intersect \(L_{1,1/m}∪L_{-1,1/m}\) and \(δ_n, x_0 < η\) we obtain a Vitali covering \(Q\) of \(H_c\). Hence by Vitali’s covering theorem used with the Radon measure \(μ(A) = H^1(A ∩ H_c)\), one can choose a system \((x_k; f(x_k)) \in H_c\) with a \(δ_k \in (0, η)\) such that the squares \(Q_k = Q((x_k; f(x_k)), δ_k)\) are disjoint, are of diameter less than \(2\sqrt{2}η\) and \(H^1(H_c \setminus \bigcup_k Q_k) = 0\).

We have \((x; f(x)) \in Q_k = Q((x_k; f(x_k)), δ_k)\) for \(x \in [x_k - (1/m)δ_k, x_k + (1/m)δ_k]\). Hence, for \(k \neq k'\) the intervals \([x_k - (1/m)δ_k, x_k + (1/m)δ_k]\) and \([x_{k'} - (1/m)δ_{k'}, x_{k'} + (1/m)δ_{k'}]\) are disjoint subintervals of \([0, 1]\). This implies \(\sum_k |Q_k| = \sum_k 2\sqrt{2}δ_k \leq √{2}m\) and our cover of \(H_c \setminus \bigcup_k Q_k\) by \(\bigcup_k Q_k\) consists of squares of diameter less than \(2\sqrt{2}η\). On the other hand, \(H^1(H_c \setminus \bigcup_k Q_k) = 0\) implies that \(\sum_k |Q_k| > c/2\) for small values of \(η\) which is impossible when \(c > 2\sqrt{2}m\).

By a result of R. D. Mauldin and S. C. Williams ([16] Theorem 2) the graph of the typical continuous function is of Hausdorff dimension one, but is not of σ-finite \(H^1\)-measure. So, Theorem 2 says that most points in the sense of \(H^1\)-measure on the graph of the typical continuous function are not graph, or central graph like. The next lemma implies that despite this relative smallness of \(GLMT(f)\) and \(CGLMT(f)\) the projection of these sets onto the \(x\)-axis is of full measure and as we will see in Theorem 5 below the projection of \(UMT(f)\) onto the \(x\)-axis is also of full measure.
Lemma 3. For a fixed $g \in C[-1, 1]_0$ the set of those functions $f \in C[0,1]$ for which $\lambda(\pi_g(GLMT_g(f))) = \lambda(\pi_g(CGLMT_g(f))) = 1 = \lambda([0,1])$ is a dense $G_δ$ set in $C[0,1]$.

**Proof.** From $GLMT_g(f) \subset CGLMT_g(f)$ it follows that it is sufficient to show

$\lambda(\pi_g(GLMT_g(f))) = 1$ for any $f \in \mathcal{G}$ for a dense $G_δ$ set $\mathcal{G}$ of $C[0,1]$.

First choose and fix a countable dense subset $\{f_m\}_{m=1}^\infty$ in $C[0,1]$.

Assume $n \in \mathbb{N}$ is fixed. Our goal is to choose functions $f_{m,n}$ and numbers $\hat{\eta}_{m,n} > 0$ such that if $f \in B(f_{m,n}, \eta_{m,n})$ then there exists a set $X_{m,n} \subset [0,1]$ such that

- $\Lambda([0,1] \setminus X_{m,n}) < 2^{-n}$,
- for any $x_0 \in X_{m,n}$ there exists $\delta_{x_0} \in (0, 1/n)$ for which
dist$_H(F(f; x_0, \delta_{x_0}),$ graph$(f) \cap Q^2) < 1/n$, and
- $B(f_{m,n}, \hat{\eta}_{m,n}) \subset B(f_m, 1/mn)$.

Then we set $G_n = \bigcup_m B(f_{m,n}, \hat{\eta}_{m,n})$ and $\mathcal{G} = \bigcap_{n=1}^\infty G_n$.

Now, $G_n$ is a dense open set in $C[0,1]$ and if $f \in \mathcal{G}$ then there exist sequences $\{f_{m,n}\}_{m=1}^\infty$, $\{\eta_{m,n}\}_{n=1}^\infty$ such that $f \in B(f_{m,n}, \eta_{m,n})$. Since $\Lambda([0,1] \setminus X_{m,n}) < 2^{-n}$ by the Borel-Cantelli lemma almost every $x_0 \in [0,1]$ belongs to finitely many sets of $\Lambda_{m,n}$. Hence, for almost every $x_0 \in [0,1]$ there exists an $N_{x_0}$ such that $x_0 \in X_{m,n}$ for $n \geq N_{x_0}$. This implies graph$(f) \cap Q^2 \in f_{M,T}(x_0)$, that is, $(x_0; f(x_0)) \in GLMT_g(f)$.

Therefore, to complete the proof of this lemma it is sufficient to show how to find $f_{m,n}$ and $\hat{\eta}_{m,n}$ for a fixed $m, n \in \mathbb{N}$.

Set $f_1 = f_{m,n}$, $\eta_1 = 1/mn$, $X_1^* = \emptyset$, and $\gamma_1 = 1$. Put

$g_1(x) = \begin{cases} g(x) & \text{if } -1 \leq x \leq 1; \\ g(-1) & \text{if } x \leq -1; \\ g(1) & \text{if } x \geq 1. \end{cases}$

Since $g \in C[-1, 1]_0$ we can choose and fix $M$ such that $|g|, |g_1| < M$.

Without limiting generality we can also suppose that $M \geq 1$.

By a small perturbation of $g_1$ choose a function $g_2 \in C(\mathbb{R})$ such that

- $g_2(0) = 0$,
- $|g_2| < M$,
- $g_2$ is continuously differentiable,
- there is no interval on which $g_2$ is constant,

\[ \text{dist}_H(\text{graph}(g_2) \cap Q^2, \text{graph}(g) \cap Q^2) = \frac{1}{2n}, \]

\[ \text{dist}_H(F(g_2, 0, 1), \text{graph}(g) \cap Q^2) < \frac{1}{2n}, \]

[6]
\begin{itemize}
  \item $g_2$ has no extrema on the boundary of $Q^2$, and
  \item $g_2$ does not go through any vertex of $Q^2$.
\end{itemize}

Using the above properties and uniform continuity of $g_2$ on bounded intervals choose $\gamma \in (0, 1/2)$ and $\eta^* > 0$ such that for all $g^* \in B(g_2, \eta^*)$

$$\text{dist}_H(F(g^*, x, 1), F(g_2, 0, 1)) < \frac{1}{2n} \text{ if } |x| \leq \gamma.$$  

(We need the properties of $g_2$ because intervals of constancy, local extrema etc. on the boundary of $Q^2$ might cause “jumps” in the Hausdorff metric when we move from $g_2$ to the nearby function $g^*$ or move from 0 to $x$.) This by (2) implies for all $g^* \in B(g_2, \eta^*)$

$$\text{dist}_H(F(g^*, x, 1), \text{graph}(g) \cap Q^2) < \frac{1}{n} \text{ if } |x| \leq \gamma.$$  

(3)

Suppose $f_j^*, \eta_j, X_j$ are given for a $j \geq 1$.

We will assume that $X^*_j$ when $j \geq 2$ is the union of finitely many disjoint closed intervals and each maximal subinterval of $[0, 1] \setminus X_j$ is of length at least $\tau_j$.

Next, choose a large natural number $\kappa_j$ so that it is divisible by four,

$$\frac{1}{\kappa_j} < \min(\tau_j/8, \eta_j/6M), \tag{4}$$

and the oscillation of $f_j^*$ on an interval of length $4/\kappa_j$ is less than $\eta_j/3$. Assume $k$ is an integer and $(4k + 2)/\kappa_j \in [0, 1]$. For $x \in [(4k + 1)/\kappa_j, (4k + 3)/\kappa_j]$ set

$$f^*_{j+1}(x) = f^*_j \left( \frac{4k + 2}{\kappa_j} \right) + \frac{1}{\kappa_j(1 + \gamma)} g_2 \left( \kappa_j(1 + \gamma) \left( x - \frac{4k + 2}{\kappa_j} \right) \right). \tag{5}$$

Then our assumptions on $M$ and $\kappa_j$ imply that

$$|f^*_{j+1}(x) - f^*_j(x)| < \eta_j \text{ for } x \in \left[ \frac{4k + 1}{\kappa_j}, \frac{4k + 3}{\kappa_j} \right]. \tag{6}$$

Next, extend the definition of $f^*_{j+1}$ onto intervals of the form $[4k/\kappa_j, (4k + 1)/\kappa_j)$ and $((4k + 3)/\kappa_j, 4k/\kappa_j]$ so that (6) holds on these intervals as well and $f^*_{j+1}$ is continuous on $[0, 1]$.

Using (3) and (5) choose $\eta_{j+1}$ such that for any $f \in B(f^*_{j+1}, \eta_{j+1})$ we have

$$\text{dist}_H \left(F \left( f, x, \frac{1}{\kappa_j(1 + \gamma)} \right), \text{graph}(g) \cap Q^2 \right) < \frac{1}{n} \text{ if } x \in \left[ \frac{4k + 2}{\kappa_j} - \frac{\gamma}{\kappa_j(1 + \gamma)}, \frac{4k + 2}{\kappa_j} + \frac{\gamma}{\kappa_j(1 + \gamma)} \right]$$

and $B(f^*_{j+1}, \eta_{j+1}) \subset B(f^*_j, \eta_j)$.
Set
\[ X_{j+1}^* = X_j^* \cup \bigcup_{k \in \mathbb{Z}} \left[ \frac{4k + 2}{k_j} - \frac{\gamma}{\kappa_j(1 + \gamma)} \cdot \frac{4k + 2}{k_j} + \frac{\gamma}{\kappa_j(1 + \gamma)} \right] \cap [0, 1]. \]

From (4) and the definition of \( \tau_j \) it follows that any interval contiguous to \( X_j \) contains at least one complete interval of the form \([4k/\kappa_j, 4(k+1)/\kappa_j]\). Thus there exists a constant \( \gamma^* \) depending only on \( \gamma \) and not depending on \( j \) such that
\[ \mathcal{H}^1([0, 1] \setminus X_{j+1}^*) < (1 - \gamma^*)\mathcal{H}^1([0, 1] \setminus X_j^*). \]
Hence, repeating the above procedure, there exists \( j \) such that \( \mathcal{H}^1([0, 1] \setminus X_j^*) < 2^{-n} \). Then we stop and set \( \tilde{f}_{m,n} = f_j^*, \tilde{\eta}_{m,n} = \eta_j \) and \( X_{m,n} = X_j^* \).

**Lemma 4.** If \( g \in C([-1, 1]_0^0 \) and \( f \in C[0, 1] \) then \( GLMT_g(f) \) is a \( G_\delta \) set in the relative topology of \( \text{graph}(f) \).

**Proof.** For a given \( \epsilon > 0 \) set
\[ E_{q,\epsilon} = \{(x_0; f(x_0)): \text{dist}_\mathcal{H}(F(f, x_0, q), \text{graph}(g) \cap Q^2) < \epsilon\} \cdot \]
Denote by \( E_{q,\epsilon} \) the interior of \( E_{q,\epsilon} \) in the relative topology of the graph of \( f \).
Then, clearly
\[ GLMT_g(f) \supset \bigcap_{n \in \mathbb{N}} \bigcap_{m \in \mathbb{N} | q < 1/m} \bigcup E_{q,1/n}. \]
On the other hand, if \( (x_0; f(x_0)) \in GLMT_g(f) \) and \( n, m \in \mathbb{N} \) are given using the definition of \( GLMT_g(f) \) choose \( \delta < 1/m \) such that
\[ \text{dist}_\mathcal{H}(F(f, x_0, \delta), \text{graph}(g) \cap Q^2) < \frac{1}{n}. \]
Recall that \( f \) has only countably many intervals of constancy and countably many strict local extreme values. By choosing a slightly larger \( q \in (\delta, 1/m) \), for which we still have
\[ \text{dist}_\mathcal{H}(F(f, x_0, q), \text{graph}(g) \cap Q^2) < \frac{1}{n} \] \hspace{1cm} (7)
we can assume that \( f \) is not constant and has no extreme values on the boundary of \( Q((x_0; f(x_0)), q) \). Then by the continuity of \( f \) at \( x_0 \) and by (7) one can choose a \( \delta' \in (0, q - \delta) \) such that for any \( x' \in (x_0 - \delta', x_0 + \delta') \) we have \( \text{dist}_\mathcal{H}(F(f, x', q), \text{graph}(g) \cap Q^2) < 1/n \) and hence \((x_0; f(x_0))\) belongs to \( E_{q,1/n} \).

Therefore,
\[ GLMT_g(f) = \bigcap_{n \in \mathbb{N}} \bigcap_{m \in \mathbb{N} | q < 1/m} \bigcup E_{q,1/n}. \]

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Theorem 5. There is a dense \(G_δ\) set \(G\) of \(C[0, 1]\) such that \(\lambda(\pi_x(UMT(f))) = 1\) for all \(f \in G\). Furthermore, \(UMT(f)\) is a dense \(G_δ\) subset in the relative topology of \(\text{graph}(f)\). Hence, for the typical continuous function in \(C[0, 1]\) almost every \(x \in [0, 1]\) is a universal \(MT\)-point and a typical point on the graph of \(f\) is in \(UMT(f)\).

Proof. Choose a countable dense system \(\{g_n\}_{n=1}^{\infty}\) in \(C[-1, 1]_0\). By Lemma 3 for each \(g_n\) there exists a dense \(G_δ\) set \(G^n\) in \(C[0, 1]\) such that \(\lambda(\pi_x(GLMT_{g_n}(f))) = 1\) for any \(f \in G^n\). Set \(G = \bigcap_{n=1}^{\infty} G^n\).

Assume \(f \in G\) and \(g \in C[-1, 1]_0\) are given. We need to show that \(\text{graph}(g) \cap Q^2 \in f_{MT}(x_0)\) for almost every \(x_0 \in [0, 1]\). Let \(X = \bigcap_{n=1}^{\infty} \pi_x(GLMT_{g_n}(f))\). Then \(\lambda(X) = 1\).

Suppose \(x_0 \in X\) and \(\epsilon > 0\) are given. Choose \(g_n\) such that
\[
\text{dist}_H(\text{graph}(g_n) \cap Q^2, \text{graph}(g) \cap Q^2) < \frac{\epsilon}{2} \tag{8}
\]
Since \(x_0 \in X\) and \(f \in \bigcap_{n=1}^{\infty} G^n \subset G^n\) we can choose \(\delta \in (0, \epsilon)\) such that
\[
\text{dist}_H(F(f, x_0, \delta), \text{graph}(g_n) \cap Q^2) < \frac{\epsilon}{2}. \tag{9}
\]
Hence, for any \(\epsilon > 0\) there exists \(\delta \in (0, \epsilon)\) such that
\[
\text{dist}_H(F(f, x_0, \delta), \text{graph}(g) \cap Q^2) < \epsilon.
\]
Therefore, \(\text{graph}(g) \cap Q^2 \in f_{MT}(x_0)\).

In fact, above we have shown that if \((x_0, f(x_0)) \in \bigcap_{n=1}^{\infty} GLMT_{g_n}(f)\) then \((x_0, f(x_0)) \in UMT(f)\). The other inclusion being obvious, we have
\[
\bigcap_{n=1}^{\infty} GLMT_{g_n}(f) = UMT(f).
\]
Since \(f\) is continuous by \(\lambda(X) = 1\) the set \(\bigcap_{n=1}^{\infty} GLMT_{g_n}(f)\) is dense on the graph of \(f\). From Lemma 4 it follows that \(\bigcap_{n=1}^{\infty} GLMT_{g_n}(f)\) is a dense \(G_δ\) subset of \(\text{graph}(f)\). Hence, \(UMT(f)\) is also a dense \(G_δ\) subset of \(\text{graph}(f)\).

Assume that \(g_0\) denotes the identically zero function in \([-1, 1]\). Then \(GLMT_{g_0}(f) = GLMT_{g_0}(f)\).

Lemma 6. We have \(\lambda(\pi_y(GLMT_{g_0}(f))) = 0\) for any \(f \in C[0, 1]\).

Proof. By Lemma 4 \(GLMT_{g_0}(f)\) is a Borel set. We use the notation introduced in the proof of Theorem 2. For \(g_0\) we choose \(\epsilon = 1\) and observe that for \(\epsilon' \in (0, 1]\) from \(\text{dist}_H(F(f, x_0, \delta), \text{graph}(g_0) \cap Q^2) < \epsilon'\) it follows that \(F(f, x_0, \delta)\) does not intersect \(L_{-1, 1}^{-1} \cup L_{-1, 1}^{-1}\), moreover \(|f(x) - f(x_0)| < \epsilon'\) holds for \(x \in [x_0 - \delta, x_0 + \delta]\).

It follows that \(GLMT_{g_0}(f) \subset H = \bigcap_{n=1}^{\infty} \cup_{\delta < 1/n} E_{\delta, 1}^{-1}\) where \(E_{\delta, 1}\) was defined at the beginning of the proof of Theorem 2. Hence, by the argument of this proof \(H\) is of finite \(\mathcal{H}^1\)-measure which implies that \(GLMT_{g_0}(f)\) is also of finite \(\mathcal{H}^1\)-measure.
Next, for any fixed $\epsilon' > 0$ considering for each $(x_0; f(x_0)) \in GLMT_{g_0}(f)$ those cubes $Q((x_0; f(x_0)), \delta)$ for which
\[
|f(x) - f(x_0)| < \epsilon'\delta \text{ for } x \in [x_0 - \delta, x_0 + \delta]
\] (10)
we obtain a Vitali cover of $GLMT_{g_0}(f)$. By the Vitali covering theorem one can choose $(x_k, f(x_k)) \in GLMT_{g_0}(f)$ and $\delta_k > 0$ such that the cubes $Q_k = Q((x_k; f(x_k)), \delta_k)$ are disjoint and $H^1(GLMT_{g_0}(f) \setminus \bigcup_k Q_k) = 0$. Then
\[
\lambda(\pi_y(GLMT_{g_0}(f) \setminus \bigcup_k Q_k)) = 0
\] (11)
holds as well. Now the disjointness of $Q_k$ and $Q_{k'}$ for $k \neq k'$ implies that $[x_k - \delta_k, x_k + \delta_k]$ and $[x_{k'} - \delta_{k'}, x_{k'} + \delta_{k'}]$ are also disjoint. Hence $\sum_k 2\delta_k < 1$ and by (10) we obtain $\lambda(\pi_y(\text{graph}(f) \cap \bigcup_k Q_k)) < \epsilon'\sum_k 2\delta_k < \epsilon'$. Using this and (11) we obtain that $\lambda(\pi_y(GLMT_{g_0}(f))) < \epsilon'$ holds for any $\epsilon' > 0$ and this concludes the proof.

The next theorem shows that though $UMT(f)$ has large $x$-projection, it has small $y$-projection.

**Theorem 7.** There is a dense $G_\delta$ set $G$ of $\mathbb{C}[0,1]$ such that $\lambda(\pi_y(UMT(f))) = 0$ for all $f \in G$. Hence any preimage of almost every $y$ in the range of the typical continuous function is not a $UMT$-point.

**Proof.** Since $UMT(f) \subset GLMT_{g_0}(f)$ the Theorem follows from Lemma 6. □

For the definition of packing dimension we will use the notation introduced in [2] 3.4 p.47, or [3] 2.1 pp. 22-23 and we recall that
\[
P^s_\delta(F) = \sup\left\{ \sum_j |B_j|^s : \{B_j\} \text{ is a collection of disjoint balls of radii at most } \delta \text{ with centers in } F_1 \right\}
\]
and $P^s_0(F) = \lim_{\delta \to 0} P^s_\delta(F)$. Finally in the definition of $P^s(F)$, the $s$-dimensional packing measure of the Borel set $F$, one needs to set
\[
P^s(F) = \inf\{\sum_i P^s_0(F_i) : F \subset \bigcup_i F_i\}.
\] (12)

We also recall that by a result of P. Humke and G. Petruska ([9]) the packing dimension of the typical continuous function equals two. Next we see that from the packing dimension point of view $UMT(f)$ is sufficiently large for the typical continuous function.

By considering typical restrictions of functions in $C[0,1]$ onto intervals $[a, b] \subset [0,1]$ with rational endpoints one can easily see that the graph of the typical continuous function $f \in C[0,1]$ restricted onto any interval $[a, b], (a < b)$ is of packing dimension two.

**Theorem 8.** For the typical continuous function $f \in C[0,1]$ the packing dimension of $UMT(f)$ equals two.
Proof. We will work in the relative topology of graph(f) for an f ∈ C[0, 1]. Assume that UMT(f) is a dense Gδ subset of graph(f) and P*(UMT(f)) < ∞ for an s < 2.

Next we show that there exists an interval [a, b] on which the graph of f is of packing dimension less or equal s. From (12) and Baire’s Category Theorem applied to the graph of f it follows that there exists an Fl in a countable covering of F for which P0n(Fl) < ∞ and Fl is dense in a portion of graph(f). From now on we assume that this Fl is fixed. Choose an interval [a, b] ⊆ [0, 1], (a < b) such that Fl is dense in the set S = graph(f)|[a,b]). Since Fl ∩ S is dense in S we have P0n(Fl ∩ S) = P0n(S). Thus P0n(S) < ∞, which implies P* (S) < ∞ and hence the packing dimension of S is less than two, but by the Hukum-Petruska result for the typical continuous function graph(f)|[a,b]) is of packing dimension two. This implies the statement of Theorem 8.

4 Brownian motion

In this section instead of working with C[0, 1] we will work with C[0, +∞] our definitions concerning micro-tangent sets generalize to this case the obvious way. We use the notation of [1] Chapter 7.

Assume that [W(t) : t ≥ 0] denotes the Brownian motion. By [1] 37.14, p. 505 if

\[ X_{n,k} = \max \left\{ \left| W\left( \frac{k + 1}{2^n} \right) - W\left( \frac{k}{2^n} \right) \right|, \left| W\left( \frac{k + 2}{2^n} \right) - W\left( \frac{k + 1}{2^n} \right) \right| \right\} \]

then \( P[X_{n,k} \leq \epsilon] \leq (2 \cdot 2^{n/2} \cdot \epsilon)^3 \) (where \( P[X_{n,k} \leq \epsilon] \) denotes the probability that \( X_{n,k} \leq \epsilon \)). Hence if \( Y_n = \min_{k \leq n} X_{n,k} \) then

\[ P[Y_n \leq \epsilon] \leq n \cdot 2^n (2 \cdot 2^{n/2} \epsilon)^3 \] (13)

Now we can formulate the main theorem of this section.

**Theorem 9.** For almost every Brownian motion path, \( W(t) \) from \( F \in WCMT(t) \), \( t > 0 \) it follows that \( F \subset S_0^\text{def} \{ (0; y) : |y| \leq 1 \} \). Therefore, CGLMT(W) = ∅ and UMT(W) = ∅ with probability one.

**Proof.** We want to show that for any \( \eta \in (0, 1) \) with probability one for the Brownian motion path at any \( t > 0 \) there exists \( \delta_{t, \eta} > 0 \) such that for any \( \delta \in (0, \delta_{t, \eta}) \)

\[ CENT(F(W, t, \delta)) \subset S_\eta^\text{def} \{ (x; y) : |x| \leq \eta, |y| \leq 1 \}. \] (14)

This will imply that if \( F \in WCMT(t) \) then \( F \subset S_\eta \) for all \( \eta > 0 \), that is, \( F \subset S_0 = \cap_{\eta > 0} S_\eta \).
To verify (14) it is sufficient to show that for any \( t > 0 \) there exists \( \delta_{t,\eta} > 0 \) such that for any \( \delta \in (0, \delta_{t,\eta}) \) one can find \( t_- \in [t - \eta \delta, t] \) and \( t_+ \in [t, t + \eta \delta] \) satisfying

\[
|f(t_-) - f(t)| > \delta \quad \text{and} \quad |f(t_+) - f(t)| > \delta.
\]  
(15)

Set

\[
K_\eta = 32/\eta \quad \text{and} \quad \epsilon_n = K_\eta \cdot 2^{-n}.
\]  
(16)

From (13) applied with \( \epsilon = \epsilon_n \) it follows that

\[
P[Y_n \leq K_\eta 2^{-n}] \leq n \cdot 2^n (2^n / 2^{n/2} K_\eta \cdot 2^{-n})^3 = n \cdot 2^n (2K_\eta)^3 2^{-3n/2}.
\]

Thus \( \sum_n P[Y_n \leq K_\eta 2^{-n}] < \infty \) and by the Borel-Cantelli lemma with probability one we have \( Y_n \leq K_\eta 2^{-n} \) for only finitely many \( n \)’s for a Brownian motion path \( W \).

Assume that for \( W \) under consideration \( N_0 \) is chosen so that \( Y_n > K_\eta 2^{-n} \) if \( n > N_0 \). For a fixed \( t > 0 \) we can assume that \( N_0 \) is chosen to be so large that \( t \in (0, N_0/2) \).

Choose \( \delta_{t,\eta} > 0 \) such that

\[
2^{N_0} < \frac{4}{\eta \delta_{t,\eta}} \quad \text{and} \quad \delta_{t,\eta} < t.
\]  
(17)

Then for a \( \delta \in (0, \delta_{t,\eta}) \) choose \( n \) such that

\[
4 \cdot 2^{-n} < \eta \delta \leq 8 \cdot 2^{-n}.
\]  
(18)

This implies \( 4/\eta \delta_{t,\eta} < 4/\eta \delta < 2^n \) and by (17) we have \( n > N_0 \). By the definition of \( Y_n \) in any subinterval of length \( 4 \cdot 2^{-n} \) in \([0, n]\) one can choose two points \( t_1, t_2 \) such that

\[
|f(t_1) - f(t_2)| > K_\eta 2^{-n} \geq (K_\eta / 8) \eta \delta > 2 \delta
\]  
(19)

where the last inequality follows from (16). By using (18) from (19) it follows that one can find \( t_1, t_2 \) either in \([t - \eta \delta, t]\), or in \([t, t + \eta \delta]\) such that (19) holds. This implies (14).

\[\square\]

5 Specific functions

The behavior experienced at the Brownian motion is in a certain aspect the worst possible, the function was central graph like at no point. In this section we want to illustrate that there are other examples of non-differentiable functions for which one can find a lot of points where \( GLMT(f) \) and/or \( CGLMT(f) \) is non-trivial. To illustrate the applicability of micro tangent sets here we discuss two such examples. (Of course, exact determination of the micro tangent properties of other functions and classes of functions can be subject of further research.)
First example is Takagi’s function, $T(x)$. Let $\Phi(x) = \text{dist}(x, \mathbb{Z})$ and set

$$T(x) = \sum_{n=0}^{\infty} 2^{-n} \Phi(2^n x).$$

It is one of the well-known examples of nowhere differentiable functions, however its Hölder spectrum is very simple; it is a monofractal, [10] Section 6.

**Theorem 10.** Almost every $x_0 \in \mathbb{R}$ is a graph like micro tangent point of Takagi’s function, $T(x)$. In fact, this function is “micro-self similar” in the sense that if we take $g = T\big|_{-1,1}$ then $\text{graph}(g) \in f_{MT}(x_0)$ for almost every $x_0 \in \mathbb{R}$. (We remind the reader that $f_{MT}(x_0)$ was defined in Definition 1.)

**Proof.** By inspecting the graph of $T(x)$ (see Figure 1) and doing some elementary estimates of the first few terms of the sum defining $T(x)$ one can see that if $x_0 \in [0.49, 0.51]$ and $\delta = 0.25$ then

$$(x; T(x)) \in Q((x_0; T(x_0)), \delta) \text{ for } x \in [x_0 - \delta, x_0 + \delta].$$

(20)

It is well-known that for almost every $x \in [0, 1]$ infinitely often the number of zeros and ones in the dyadic expansion of $x$ is the same. (The corresponding symmetric random walk model, where an $n$'th digit 0 means a unit step in the negative and an $n$'th digit 1 means a unit step in the positive direction, by Pólya's theorem (p. 118 of [1]) is persistent, that is, the particle doing the random walk returns infinitely often to the origin.)
For an \( x \in \mathbb{R} \) we will consider the dyadic “expansion”

\[
x = [x] + \sum_{j=1}^{\infty} r_j(x)2^{-j}, \text{ where } r_j(x) \in \{0, 1\}
\]

and \([x]\) is the integer part of \( x \). Since we work with almost every \( x \) we can exclude the dyadic rationals and hence the \( r_j(x) \)'s are unique. Denote by \( X_\infty \) the set of those \( x \in \mathbb{R} \) for which infinitely often the number of zeros and ones in the above dyadic expansion is the same. To make this more precise, set \( E(x, 0) = 0 \), and if \( E(x, k) \) for a \( k \geq 0 \) is given then let \( E(x, k + 1) \) be the least \( n > E(x, k) \) for which

\[
\# \{ j : r_j(x) = 0, 1 \leq j \leq n \} = \# \{ j : r_j(x) = 1, 1 \leq j \leq n \}.
\]

For an \( \eta \in (0, 0.001) \) we will denote by \( X^{*, \eta}_\infty \) the set of those \( x \in X_\infty \) for which there are infinitely many \( k(j, x, \eta)'s \) \( (j = 1, 2, ...) \) such that

\[
\left| \text{dist} \left( x, 2^{-E(x,k(j,x,\eta))} \right) - \frac{1}{2} \cdot 2^{-E(x,k(j,x,\eta))} \right| < \eta \cdot 2^{-E(x,k(j,x,\eta))}.
\]

It is as an easy exercise to show that for any \( \eta \in (0, 0.001) \) almost every \( x \in X_\infty \) belongs to \( X^{*, \eta}_\infty \); by request of the referee of this paper a solution to this exercise is given after the proof of this theorem.

We claim that if \( x_0 \in X^{*, \eta}_\infty \) then there exists \( \tau^*_\eta \in [-4\eta, 4\eta] \) such that if \( g_\eta(x) \) is the restriction of \( T(x + \tau^*_\eta) - T(\tau^*_\eta) \) onto \([-1, 1]\] then \( \text{graph}(g_\eta) = \text{graph}(g_\eta) \cap Q^2 \subset f_M(T(x_0)) \).

Denote by \( I_j \) the interval of length \( l_j \equiv 2^{-E(x_0,k(j,x,\eta))} \) containing \( x_0 \) and with endpoints in \( I_j \mathbb{Z} \). If \( m_j \) equals the midpoint of \( I_j \) then by (22)

\[
|x_0 - m_j| < \eta \cdot l_j.
\]

Put \( T_N(x) = \sum_{n=0}^{N} 2^{-n} \Phi(2^nx) \) and \( T_N^\eta(x) = \sum_{n=N+1}^{\infty} 2^{-n} \Phi(2^nx) \). Set \( N_{j, \eta} = E(x_0, k(j, x_0, \eta)) + 1 \) and observe that \( T_{N_{j, \eta}}(x) \) is constant on \( I_j \) and \( T_{N_{j, \eta}}(x) \) is an \( l_j \)-times rescaled (in both \( x \) and \( y \) directions) copy of \( T(x) \).

Hence, from (20) and (23) it follows that setting \( \delta_j = 0.25l_j \) we have \( (x; T(x)) \in Q((x_0; T(x_0)), \delta_j) \) for \( x \in [x_0 - \delta_j, x_0 + \delta_j] \) and

\[
F(T, x_0, \delta_j) = \text{graph}(T(x + \tau_j) - T(\tau_j)) \cap Q^2
\]

where the translation vector \( \tau_j \in [-4\eta, 4\eta] \). By compactness there exists \( \tau^*_j \in [-4\eta, 4\eta] \) to which a suitable subsequence of \( \{\tau_j\} \) converges. Then for this subsequence \( F(T, x_0, \delta_j) \) converges to \( \text{graph}(g_\eta) \cap Q^2 \) in the Hausdorff metric. This implies that \( \text{graph}(g_\eta) \in f_M(T(x_0)) \), as we claimed.

Next, letting \( \eta_M = 1/M \), clearly \( \tau^*_j \rightarrow 0 \) and almost every \( x_0 \in \mathbb{R} \) belongs to \( X^{*, \eta}_\infty \). If \( x_0 \in X^{*, \eta}_\infty \) then one can easily choose a sequence \( \delta_j \rightarrow 0 \) such that \( F(T, x_0, \delta_j) \) converges in the Hausdorff metric to the graph of \( g(x) = T(x)[[-1,1]] = \lim_{M \rightarrow \infty} g_{M}(x) \). Hence \( \text{graph}(g) \) belongs to \( f_M(T(x_0)) \).
Solution to the exercise from the proof of Theorem 10.

Set

\[ \Psi_n = \left\{ x : \left| \text{dist}(x, 2^{-n} \mathbb{Z}) - \frac{1}{2}2^{-n} \right| < \eta 2^{-n}, \text{ and } \exists k \in \mathbb{N}, \ E(x, k) = n \right\} . \]

If \( E(x, k) = n \) and the interval \( I = [k2^{-n}, (k+1)2^{-n}] \) contains \( x \) then

\[ \lambda(I \cap \Psi_n) / \lambda(I) = 2\eta. \] (24)

Set \( L_m = \bigcup_{n=m}^{\infty} \Psi_n \). From (24) it follows that the upper Lebesgue density of \( L_m \) at any \( x \in X_\infty \) is positive and hence by Lebesgue’s density theorem almost every \( x \in X_\infty \) is a density point of \( L_m \), which implies \( \lambda(X_\infty \setminus L_m) = 0 \). Therefore, \( \lambda(X_\infty \setminus \bigcap_{m=1}^{\infty} L_m) = 0 \) and \( X^*_\infty, \eta = \bigcap_{m=1}^{\infty} L_m \) is of full measure and almost every \( x \in X_\infty \) belongs to \( X^*_\infty, \eta \).

Our final example will be one of the simplest cases of Weierstrass’s nowhere differentiable function. Probably the most famous detailed study of this type of functions is Hardy’s paper [8]. We will take \( \Psi(x) = \sin(2\pi x) \) and consider

\[ \mathcal{W}(x) = \sum_{n=0}^{\infty} 2^{-n} \Psi(2^n x). \]

A similar, but a little more complicated argument works if one takes \( \Psi(x) = \cos(2\pi x) \). For the partial and tail sums we will again use the notation

\[ \mathcal{W}_N(x) = \sum_{n=0}^{N} 2^{-n} \Psi(2^n x), \text{ and } \mathcal{W}_N^*(x) = \sum_{n=N+1}^{\infty} 2^{-n} \Psi(2^n x). \]

**Theorem 11.** Almost every \( x_0 \in \mathbb{R} \) is a central graph like micro tangent point of Weierstrass’s function, \( \mathcal{W}(x) \).

To prove this theorem we need the following lemma, which seems to be quite natural.

**Lemma 12.** For almost every \( x_0 \in \mathbb{R} \) we can find a strictly monotone increasing sequence \( \{N(j, x_0)\}_{j=1}^{\infty} \) such that \( \mathcal{W}_{N(j, x_0)}(x_0) \) and \( \mathcal{W}_{N(j, x_0)}^*(x_0) \) are of opposite sign, which implies

\[ |\mathcal{W}^{\eta}_{N(j, x_0)}(x_0)| = \left| \sum_{n=0}^{N(j, x_0)} 2\pi \cos(2\pi 2^n x_0) \right| \leq 2\pi. \] (25)

First we will prove Theorem 11 based on this lemma and finally we will provide a proof of Lemma 12.

**Proof.** (Theorem 11.) Since \( \mathcal{W}^{\eta}_{N(j, x_0)}(x) = \sum_{n=0}^{N(j, x_0)} -4\pi 2^n \sin(2\pi 2^n x) \) we have

\[ |\mathcal{W}^{\eta}_{N(j, x_0)}(x)| < 8\pi^2 2^{N(j, x_0)}. \]

Hence, setting \( \delta_j = 2^{-N(j, x_0)} -1 \) and \( I_j = [x_0 - \delta_j, x_0 + \delta_j] \), by using Lagrange’s mean value theorem and (25) we obtain

\[ |\mathcal{W}^{\eta}_{N(j, x_0)}(x)| < 5\pi^2 \text{ for } x \in I_j. \] (26)

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The “$Q^2$ rescaled” copies of the partial and tail sums of $W$ will be denoted by

$$U_{N,j,x_0}(x) = \frac{1}{\delta_j} \left( W_{N,j,x_0}(\delta_j x + x_0) - W_{N,j,x_0}(x_0) \right)$$

and

$$U^*_N(j,x_0)(x) = \frac{1}{\delta_j} \left( W^*_N(j,x_0)(\delta_j x + x_0) - W^*_N(j,x_0)(x_0) \right).$$

Clearly,

$$W_{x_0, \delta_j}(x) \equiv \frac{1}{\delta_j} (W(\delta_j x + x_0) - W(x_0)) = U_{N,j,x_0}(x) + U^*_N(j,x_0)(x)$$

and

$$F(W,x_0,\delta_j) = \text{graph}(W_{x_0,\delta_j}) \cap Q^2.$$

From (26) it follows that

$$|U'_{N,j,x_0}(x)| \leq 5\pi^2 \text{ for } x \in [-1, 1]. \quad (27)$$

For each $j$ there exists $\tau_j \in [-1, 1]$ such that $U^*_N(j,x_0)(x) = W(x + \tau_j) - W(\tau_j).

By $U_{N,j,x_0}(0) = 0$ and (27) the family of functions $U_{N,j,x_0}(x)$ is uniformly bounded and equicontinuous so by the Arzela–Ascoli theorem (see, for example, [7] 1.6.9 p. 37) there exists a subsequence $\{U_{N,j_k,x_0}(x)\}$ which uniformly converges to a function $U_{x_0}(x)$. From (27) it also follows that

$$|U_{x_0}(x) - U_{x_0}(y)| \leq 5\pi^2 \text{ for } x, y \in \mathbb{R} \quad (28)$$

By turning to a subsequence, if necessary, we can also assume that $\tau_{j_k} \to \tau^* \in [-1, 1]$. Hence,

$$W_{x_0,\delta_j}(x) \text{ converges uniformly to } g(x) \equiv U_{x_0}(x) + W(x + \tau^*) - W(\tau^*). \quad (29)$$

Since $W$ is nowhere differentiable by (28) there is no interval on which $g$ is constant. Local extrema of $g$ on the boundary of $Q^2$ might cause some minor problems, this is why we introduce $g_1$ below.

It follows also from (28) and (29) that we can choose $g_1 \in C[-1, 1]_0$ such that

- $CENT(\text{graph}(g)) \subseteq CENT(\text{graph}(g_1)) \subseteq cl \left( CENT(\text{int}(Q^2) \cap \text{graph}(g)) \right)$,

- $CENT(\text{graph}(g_1)) \in W_{\text{CMT}}(x_0)$, and

- $|g_1(x)| > 1$ if $(x; g_1(x)) \notin CENT(\text{graph}(g_1))$.  

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This shows that $x_0$ is a central graph like micro tangent point of $f$.  

Finally we prove Lemma 12.

**Proof.** (Lemma 12) Denote by $X_{\pm}$ the set of those $x$’s in $\mathbb{R}$ for which the sequence $\{W_N^e(x)\}_{N=1}^{\infty}$ changes its sign infinitely often. We need to show that $\lambda(X_{\pm}) = 0$, where we use the notation $A^c$ for the complement of $A \subset \mathbb{R}$.

Proceeding towards a contradiction assume that $\lambda(X_{\pm}) > 0$. Set $\lambda_{\pm} = \lambda(X_{\pm} \cap [0,1])$. Since for all $N$’s $W_N^e$ is periodic by one we have $\lambda_{\pm} > 0$.

For $x \in \mathbb{R}$ we use the dyadic expansion of the form (21). Denote by $X_{0,\infty}$ the set of those $x$ in $\mathbb{R} \setminus \mathbb{Q}$ for which we have arbitrarily long blocks of 0’s in the sequence $\{r_j(x)\}_{j=1}^{\infty}$. It is well-known (and not difficult to see) that $\lambda(X_{0,\infty}) = 0$.

Clearly, $\{W_N^e(x)\}_{N=1}^{\infty}$ is not bounded if $x \in X_{0,\infty}$.

Set

$$X_a = \{x \in \mathbb{R} : \{W_N^e(x)\}_{N=1}^{\infty} \text{ is bounded from above} \}.$$ 

By periodicity for any $k \in \mathbb{N}$ from $x \in X_a$ it follows that $x \pm 2^{-k}$ is also in $X_a$.

Hence $X_a$ is periodic by $2^{-k}$ for all $k \in \mathbb{N}$. Clearly, $X_a$ is measurable and it is a consequence of the Lebesgue density theorem that there is a zero one-law, that is, $\lambda(X_a) = 0$ or $\lambda(X_a^c) = 0$. Similarly, letting

$$X_b = \{x \in \mathbb{R} : \{W_N^e(x)\}_{N=1}^{\infty} \text{ is bounded from below} \}$$

one can see that $\lambda(X_b) = 0$ or $\lambda(X_b^c) = 0$.

From $\lambda(X_{0,\infty}^c) = 0$ it follows that $\lambda(X_b^c) = 0$ and $\lambda(X_b) = 0$ is impossible. If $\lambda(X_a) = 0$ and $\lambda(X_b) = 0$ then $\lambda(X_{\pm}) = 0$ and this contradicts $\lambda_{\pm} > 0$.

Assume

$$\lambda(X_a) = 0 \text{ and } \lambda(X_b) = 0 \quad (30)$$

(a similar argument works if $\lambda(X_a) = 0$ and $\lambda(X_b) = 0$). Our goal is again to obtain a contradiction.

Set

$$X^K = \{x \in \mathbb{R} : W_N^e(x) > -K \text{ for all } N \in \mathbb{N} \}.$$ 

Then $X^K$ is periodic by one, measurable, $\bigcup_{K=1}^{\infty} X^K = X_a$ and $\lambda(X_b \cap [0,1]) = 1$.

Hence, there exists $K$ such that $\lambda(X^K \cap [0,1]) > 0.9$. For $j = 0, 1, 2$ put

$$X^K_{b,j} = \{x \in \mathbb{R} : x - \frac{j}{3} \in X^K \}.$$ 

Then $X^K_{b,j}$ is periodic by one and $\lambda(X^K_{b,j} \cap [0,1]) > 0.9$. Set $Y = \cap_{j=1}^{3} X^K_{b,j}$. Then $Y$ is also periodic by one $\lambda(Y \cap [0,1]) > 0.7$ and $W_N^e(x - (j/3)) > -K$ for every $x \in Y$ and $N \in \mathbb{N}$. Recall that $\sum_{j=0}^{2} \cos(2\pi(\theta - (-1)^n(j/3))) = 0$ for any $n \in \mathbb{N}$ and $\theta \in \mathbb{R}$. Therefore,

$$\sum_{j=0}^{2} W_N^e(x - \frac{j}{3}) = \sum_{j=0}^{2} \sum_{n=0}^{N} 2\pi \cos \left(2\pi 2^n \left(x - \frac{j}{3}\right)\right) =$$

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$$
\sum_{n=0}^{N} 2\pi \sum_{j=0}^{2} \cos \left(2\pi 2^n x - 2\pi \frac{(3-1)^n}{3}j \right) = \sum_{n=0}^{N} 2\pi \sum_{j=0}^{2} \cos \left(2\pi 2^n x - 2\pi \frac{(-1)^n}{3}j \right) = 0.
$$

Hence, for \( x \in Y \) we have

$$
\mathcal{W}_N(x) = -\left( \mathcal{W}_N \left( x - \frac{1}{3} \right) + \mathcal{W}_N \left( x - \frac{2}{3} \right) \right) < 2K.
$$

This would imply \( Y \subset X_a \) and \( \lambda(Y) \neq 0 \) which contradicts (30).

\[\Box\]

For further information about distribution of values of trigonometric polynomials considered in Lemma 12 we refer to [11] and [12], one can prove this lemma based on these results as well, but the treatment given here seemed to be more elementary.

The author would like to thank S. Konyagin for his suggestion of a simplified version of the proof of Lemma 12 and pointing out references [11], [12] and [13].

Our original “real analysis” version of the proof was based on the idea that if the sequence \( \mathcal{W}_N(x) \) is not changing for almost every \( x \) infinitely often its sign then \( \mathcal{W}(x) \) would equal \(+\infty\), or \(-\infty\) almost everywhere, which contradicts [19] Ch. IX. (4.4) Theorem (the first version of this result, valid for continuous functions, is due to N. N. Luzin [13]).

References


