Occupation measure and level sets of the Weierstrass–Cellerier function

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Summary. We show that the occupation measure of the Weierstrass–Cellerier function $\mathcal{W}(x) = \sum_{n=0}^{\infty} 2^{-n} \sin(2\pi 2^n x)$ is purely singular. Using our earlier results we can deduce from this that almost every level set of $\mathcal{W}(x)$ is finite. These previous results and Besicovitch's projection theorem imply that for almost every c the occupation measure of $\mathcal{W}(x, c) = \mathcal{W}(x) + cx$ is purely singular. In this paper we verify that this result holds for *all* $c \in \mathbb{R}$, especially for $c = 0$. As it happens quite often it is not that easy to obtain from an almost everywhere true statement one that holds everywhere.

1 Introduction

I proposed for the annual Miklós Schweitzer Mathematical competition of the János Bolyai Mathematical Society in 2006 the following problem

Suppose that $f(x) = \sum_{n=0}^{\infty} 2^{-n}||2^n x||$, *where* $||x||$ *is the distance of* x *from the closest integer (that is,* f *is Takagi's function). What can we say for Lebesgue almost every* $y \in f(\mathbb{R})$ *about the cardinality of the level set*

$$L_y = \{x \in [0, 1] : f(x) = y\}?$$

The somewhat surprizing answer is that it is finite.

It is natural to study the same question for functions defined similarly to Takagi's function. In this paper we show that if in the above problem one uses the Weierstrass–Cellerier function

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$$\mathcal{W}(x)=\sum_{n=0}^{\infty}2^{-n}\sin(2\pi 2^{n}x) \qquad (1)$$

then the same result holds, that is, almost every level set of $\mathcal{W}(x)$ is finite. The level sets are sets of the form $\{x : \mathcal{W}(x) = y\}$ for a $y \in \mathbb{R}$. To answer the question about the cardinality of the level sets we have to study properties of occupation measures. If λ denotes the Lebesgue measure and μ denotes the occupation measure of $\mathcal{W}$ then

$$\mu(A)=\lambda\{x\in[0,1]:\mathcal{W}(x)\in A\}=\lambda(\mathcal{W}^{-1}(A)\cap[0,1]).$$

for a (Borel) measurable set $A \subset \mathbb{R}$. Occupation measures are studied in the theory of stochastic processes. For example, with probability one, the occupation measure of the Brownian motion is absolutely continuous with respect to the Lebesgue measure, it satisfies the local time condition. One can find lots of interesting facts about occupation measures in the survey paper [13] by D. Geman and J. Horowitz. Results for stochastic process are almost everywhere results. It is mentioned in [13] that it is difficult to apply the methods valid for random functions (with probability one) for non-random functions.

Level sets and occupation measures of some self affine functions were studied by J. Bertoin in [3] and [4]. These occupation measures either satisfy the local time condition, or are singular with respect to the Lebesgue measure.

Theorem 19 of our paper [7] implies that from the singularity of the occupation measure of $\mathcal{W}(x)$ it follows that almost every level set of $\mathcal{W}(x)$ is finite.

Using Theorem 13 of [7] one can decompose the graph of $\mathcal{W}(x)$ over $[0, 1]$ into two sets S^*_{irr} and S^*_{reg}. The set S^*_{reg} can be covered by the union of the graphs of countably many strictly monotone functions. The set S^*_{irr} is an irregular (or purely unrectifiable) 1-set. This means that $\mathcal{H}^1(S^*_{irr})$, the 1-dimensional Hausdorff measure of S^*_{irr}, is positive and finite, moreover, S^*_{irr} intersects every continuously differentiable curve in a set of $\mathcal{H}^1$-measure zero. By Besicovitch's projection theorem (see [5], or [9]) the projection of S^*_{irr} in almost all directions is of zero Lebesgue measure.

It also turns out in [7] that the projection of S^*_{irr} onto the x-axis is of measure one, while S^*_{reg} projects onto a set of measure zero. Hence the "occupation measure behavior" of $\mathcal{W}(x)$ over $[0, 1]$ is completely determined by the part of its graph belonging to S^*_{irr}. As it is explained in Section 5.1 of [7], Besicovitch's projection theorem implies that for almost every $c \in \mathbb{R}$ the occupation measure of the function $\mathcal{W}(x, c) = \mathcal{W}(x) + cx$ is purely singular. For these c if $S^*_{irr,c}$ denotes the irregular one set on the graph of $\mathcal{W}(x, c)$ over $[0, 1]$, the projection of $S^*_{irr,c}$ onto the y-axis is of zero Lebesgue measure, while its projection onto the x-axis is of measure one.

It is quite often rather difficult to obtain from an almost everywhere result, a result concerning a specific parameter value. My favorite example for this phenomenon is that it is known that for almost every $\theta > 1$ the sequence of the fractional part of θ^n is uniformly distributed in $[0,1]$ but no explicit example of a real number θ is known for which this sequence is uniformly distributed. For example the case $\theta = 3/2$ is a famous open problem.

In this paper we show that for all $c \in \mathbb{R}$ (with no exceptional values) the occupation measure of $\mathcal{W}(x,c)$ is purely singular. To deduce this result we have to consider a larger class of functions. This class was introduced in [7] and the main reason for the introduction of these functions was the fact that we need them to prove the singularity of the occupation measure of $\mathcal{W}(x)$.

We denote by $\mathcal{F}_{\mathcal{W}}$ the set of those twice continuously differentiable functions f_{-1} on $[0,1]$ for which for any trigonometric polynomial P the function $f_{-1} + P$ is piecewise strictly monotone, or constant. If a function is analytic on an open set $G \supset [0,1]$ then its restriction to $[0,1]$ belongs to $\mathcal{F}_{\mathcal{W}}$. Given $f_{-1} \in \mathcal{F}_{\mathcal{W}}$ we studied in Section 5 of [7] functions

$$f(x) = f_{-1}(x) + \sum_{n=0}^{\infty} 2^{-n} \sin(2\pi 2^n x) = f_{-1}(x) + \mathcal{W}(x). \tag{2}$$

If $f_{-1}(x) = 0$ identically then we obtain $\mathcal{W}(x)$, if $f_{-1}(x) = cx$ we obtain $\mathcal{W}(x,c)$.

We also need in this paper the special case when

$$f_{-1}(x) = \gamma_0 x + S_{-\infty}(x) \stackrel{\text{def}}{=} \gamma_0 x + \sum_{n=-\infty}^{-1} \left(2^{-n} \sin(2\pi 2^n x) - 2\pi x\right) \tag{3}$$

with a $\gamma_0 \in \mathbb{R}$. In Section 2 we show that this $f_{-1} \in \mathcal{F}_{\mathcal{W}}$. Theorem 18 of [7] implies that the occupation measure of f is nonatomic if f is defined in (2) with an $f_{-1} \in \mathcal{F}_{\mathcal{W}}$. Arguing as in Section 5.1 of [7] one can see that for almost every $\gamma_0 \in \mathbb{R}$ the occupation measure is singular for f defined in (2) by using f_{-1} from (3).

This paper is organized the following way: in Section 2 some preliminary results are given. In Section 3 the main result about the singularity of the occupation measure of $\mathcal{W}(x,c)$ for all $c \in \mathbb{R}$ is proved based on two lemmas. The proof of one of these lemmas is quite technical and the details of the proof of Lemma 3 are given in Section 4.

2 Notation and Preliminary results

By $\lambda(A)$ we denote the Lebesgue measure of the set $A \subset \mathbb{R}$.

Set $\mathcal{W}_N(x) = \sum_{n=0}^{N} 2^{-n} \sin(2\pi 2^n x)$. We will use the following estimate

$$|\mathcal{W}_N''(x)| = \left| \sum_{n=0}^{N} -4\pi^2 2^n \sin(2\pi 2^n x) \right| < 8\pi^2 2^N. \tag{4}$$

For $N_1 \leq -1$ and $x \in [-2,2]$ we set

$$S_{N_1}(x) = \sum_{n=N_1}^{-1} (2^{-n}\sin(2\pi 2^n x) - 2\pi x) \text{ and } S_{-\infty}(x) = \lim_{N_1 \to -\infty} S_{N_1}(x). \quad (5)$$

Next we need some elementary estimates of the derivatives of the terms of S_{N_1}. If $n \leq -2$

$$\max_{x\in[-2,2]} |2\pi\cos(2\pi 2^n x) - 2\pi| = 2\pi \max_{x\in[-2,2]} |\cos(2\pi 2^n x) - 1| = \quad (6)$$

$$2\pi|\cos(4\pi 2^n) - 1| \leq 2\pi \max_{c\in[0,4\pi 2^n]} |4\pi 2^n \sin(c)| \leq 8\pi^2 2^n.$$

This implies that

$$S'_{N_1}(x) = \sum_{n=N_1}^{-1} (2\pi\cos(2\pi 2^n x) - 2\pi) \text{ converges uniformly on } [-2,2]. \quad (7)$$

Moreover, for $k \geq 2$ with $\phi(x) = \pm\sin(x)$, or $\pm\cos(x)$ we have

$$|S^{(k)}_{N_1}(x)| = |\sum_{n=N_1}^{-1} 2^{-n}(2\pi)^k (2^n)^k \phi(2\pi 2^n x)| \leq (2\pi)^k \sum_{n=N_1}^{-1} 2^n \leq (2\pi)^k. \quad (8)$$

From (7) and (8) it follows that $S_{-\infty}$ is a (real) analytic function on $[-2,2]$.

For $N \geq 0$ and $\gamma_0 \in \mathbb{R}$ set

$$\mathcal{W}_{-\infty}(x) = S_{-\infty}(x) + \mathcal{W}(x), \quad \mathcal{W}_{-\infty,N}(x) = S_{-\infty}(x) + \mathcal{W}_N(x) \text{ and}$$

$$\mathcal{W}_{-\infty}(x,\gamma_0) = \mathcal{W}_{-\infty}(x) + \gamma_0 \cdot x. \quad (9)$$

For $N \leq -1$ put $\mathcal{W}_{N,\infty}(x) \stackrel{\text{def}}{=} S_N(x) + \mathcal{W}(x)$.

Since $S_{-\infty}$ is analytic on an open interval containing $[0,1]$ it is in the class of functions $\mathcal{F}_{\mathcal{W}}$. By an argument analogous to that of Section 5.1 in [7] one can see by Besicovitch's projection theorem that for almost every $\gamma_0 \in \mathbb{R}$ the function $\mathcal{W}_{-\infty}(x,\gamma_0)$ has singular occupation measure. By Theorem 18 of [7] this occupation measure is nonatomic. These properties imply the following lemma:

Lemma 1. *For almost every $\gamma_0 \in \mathbb{R}$. Given $\psi > 0$ there exist y_1 and h_1 such that*

$$\lambda\{x \in [0,1] : \mathcal{W}_{-\infty}(x,\gamma_0) \in (y_1, y_1 + h_1)\} > \psi h_1. \quad (10)$$

Moreover, there exists $\epsilon_{0,\psi} > 0$ such that if $g \in C[0,1]$ and

$$|\mathcal{W}_{-\infty}(x,\gamma_0) - g(x)| < \epsilon_{0,\psi} \text{ for all } x \in [0,1] \quad (11)$$

then

$$\lambda\{x \in [0,1] : g(x) \in (y_1, y_1 + h_1)\} > \psi h_1. \quad (12)$$

In the sequel we suppose that γ_0 is chosen so that the occupation measure of $\mathcal{W}_{-\infty}(x,\gamma_0)$ is singular (with respect to the Lebesgue measure) and Lemma 1 holds for this γ_0.

The next lemma is a simple consequence of Lebesgue's density theorem and is related to Vitali's covering theorem see 2.8.17 in [12].

Lemma 2. *Suppose that $H \subset [0,1]$ is Lebesgue measurable and there exists $\rho > 0$ satisfying the following property. For every $x \in H$ there are arbitrarily small intervals J_x containing a measurable subset S_x such that $x \in J_x$ and $\lambda(S_x) > \rho\lambda(J_x)$. Then for almost every $x_0 \in H$ there exist sets S_x, $(x \in H)$ of arbitrarily small diameter such that $x_0 \in S_x$.*

Proof. Suppose there is $\epsilon > 0$ and a set $H' \subseteq H$ such that H' does not intersect the union of the sets of the form S_x with $x \in H$, $\operatorname{diam}(S_x) < \epsilon$. Suppose that x_0 is a Lebesgue density point of H'. Then by our assumptions there exists a sufficiently short J_{x_0} such that $\lambda(J_{x_0}) < \epsilon$, $\lambda(J_{x_0} \cap H') > (1-\rho)\lambda(J_{x_0})$ but this contradicts that $\lambda(S_{x_0}) > \rho\lambda(J_{x_0})$, $\operatorname{diam}(S_{x_0}) < \lambda(J_{x_0}) < \epsilon$, and $S_{x_0} \cap H' = \emptyset$. Therefore H' has no Lebesgue density points and this implies that it is of measure zero. □

3 Main Results

Definition 1. *For $N = 0, 1, ...$ and $x \in [0,1)$ denote by $P(x,N)$ the point $\frac{k}{2^N}$ satisfying $x \in [\frac{k}{2^N}, \frac{k+1}{2^N})$.*

Lemma 3. *For almost every $x_0 \in [0,1)$, $\{\mathcal{W}'_N(P(x_0,N)) : N \in \mathbb{N}\}$ is dense in $\mathbb{R}$.*

We postpone the proof of this lemma to Section 4.

Remark 1. The main difficulty in Lemma 3 is that we need density of $\{\mathcal{W}'_N(P(x_0,N))\}$ and not of $\{\mathcal{W}'_N(x_0)\}$. The referee of this paper suggested that we mention that as an alternative to our direct "elementary" approach one could also use results from Ergodic and Probability Theory to deal with properties of $\{\mathcal{W}'_N(x_0)\}$.

Indeed, $\mathcal{W}'_N(x) = \sum_{n=0}^{N} 2\pi\cos(2\pi 2^n x)$ is the Birkhoff sum, $\sum_{n=0}^{N} \overline{\phi}(T^n x)$ of $\overline{\phi} \overset{\text{def}}{=} 2\pi\cos(2\pi x)$ with respect to the ergodic transformation $Tx = \{2x\}$ defined on $X = \mathbb{T} = \mathbb{R}/\mathbb{Z}$. Since $\int_X \overline{\phi} = 0$ by recurrence results concerning co-cycles and random walks [1, Chapter 8, 8.1.2, 8.1.5], [2], the cocycle $\sum_{n=0}^{N} \overline{\phi}(T^n x)$ is recurrent, that is $\liminf_{N\to\infty} |\sum_{n=0}^{N} \overline{\phi}(T^n x)| = 0$. Using Lusin's theorem, a Lebesgue density argument and the continuous differentiability of $\overline{\phi}$ one can also verify that $\overline{\phi}$ is not a coboundary, that is, there is no Borel measurable map $h : X \to \mathbb{R}$ such that $\overline{\phi}(x) = h(x) - h(Tx)$. By using the argument of [1, Corollary 8.3.4] one can see that if $\overline{\phi}$ is not a coboundary then its group of persistencies $\Pi(\overline{\phi}) = \{a \in \mathbb{R} : \forall A \in \mathcal{B}, \lambda(A) > 0, \forall \epsilon > 0, \exists N \geq 1, \lambda(A \cap T^{-N} A \cap \{x : |\sum_{n=0}^{N-1} \overline{\phi}(T^n x) - a| < \epsilon\}) > 0\}$ is non-compact. By [1, Proposition 8.2.1] it is a closed subgroup of $(\mathbb{R}, +)$. Therefore, for almost every $x_0 \in X$, $\liminf_{N\to\infty} \mathcal{W}'_N(x_0) = \liminf_{N\to\infty} \sum_{n=0}^{N} \overline{\phi}(T^n x_0) = -\infty$ and $\limsup_{N\to\infty} \mathcal{W}'_N(x_0) = \limsup_{N\to\infty} \sum_{n=0}^{N} \overline{\phi}(T^n x_0) = +\infty$.

We also remark that in [11] by using Ergodic Theory the authors estimate the Hausdorff dimension of those points where the growth rate of $\max_{N\leq K} |\sum_{n=0}^{N} \overline{\phi}(T^n x)|$ is slower than the one expected from the Law of the Iterated Logarithm Theorem or from the Central Limit Theorem. These results are also applied in [11] to the Weierstrass-Cellerier function (under the name of Hardy function) to estimate the Hausdorff dimension of those points where the Weierstrass-Cellerier-(Hardy) function exhibits various local behavior.

Lemma 4. *For any $c \in \mathbb{R}$, $\psi > 0$ there exist N_1 and h_1 for which the following holds. For almost every $x_0 \in [0, 1]$ there exist infinitely many N_0's such that if $x_0 \in [\frac{k_0}{2^{N_0}}, \frac{k_0+1}{2^{N_0}})$ then there exists y_1 such that*

$$\lambda\{x \in [\frac{k_0}{2^{N_0}}, \frac{k_0}{2^{N_0}} + \frac{1}{2^{N_0+N_1+1}}) : \mathcal{W}(x, c) \in [y_1, y_1 + \frac{h_1}{2^{N_0+N_1+1}})\} > \psi \frac{h_1}{2^{N_0+N_1+1}}. \tag{13}$$

Proof. Choose γ_0 such that $\mathcal{W}_{-\infty}(x) + \gamma_0 x$ has singular occupation measure. This implies that the conclusion of Lemma 1 holds for γ_0. By Lemma 1 choose y'_1, h_1 and $\epsilon_{0,\psi}$, such that (10-12) hold. Suppose $x_0 \in [0, 1]$ is arbitrary and $x_0 \in [\frac{k_0}{2^{N_0}}, \frac{k_0+1}{2^{N_0}})$. Choose N'_1 such that for any N_0 and $N_1 \geq N'_1$ we have for any $x \in [\frac{k_0}{2^{N_0}}, \frac{k_0}{2^{N_0}} + \frac{1}{2^{N_0+N_1}}) \subset [0, 1)$

$$|\mathcal{W}'_{N_0}(\frac{k_0}{2^{N_0}}) - \mathcal{W}'_{N_0}(x)| \leq \max_{x'\in[0,1)} |\mathcal{W}''_{N_0}(x')| \frac{1}{2^{N_0+N_1}} \leq \tag{14}$$

(using (4))

$$8\pi^2 2^{N_0} \frac{1}{2^{N_0+N_1}} \leq \frac{8\pi^2}{2^{N'_1}} < \frac{\epsilon_{0,\psi}}{3}.$$

We also choose and fix $N_1 \geq N'_1$ such that for $x \in [0, 1]$

$$|\mathcal{W}_{-N_1,\infty}(x) - \mathcal{W}_{-\infty}(x)| \leq |\sum_{n=-\infty}^{-N_1-1} (2^{-n} \sin(2\pi 2^n x) - 2\pi x)| \leq \frac{\epsilon_{0,\psi}}{3}. \tag{15}$$

Set

$$\gamma = \gamma_0 - N_1 2\pi. \tag{16}$$

By Lemma 3 for almost every x_0 there exist infinitely many N_0 such that if $P(x_0, N_0) = \frac{k_0}{2^{N_0}}$ then $x_0 \in [\frac{k_0}{2^{N_0}}, \frac{k_0+1}{2^{N_0}})$ and

$$|\mathcal{W}'_{N_0}(\frac{k_0}{2^{N_0}}) - \gamma + c| < \frac{\epsilon_{0,\psi}}{3}. \tag{17}$$

Choose and fix one N_0 satisfying the above assumption. Clearly, for $x \in [\frac{k_0}{2^{N_0}}, \frac{k_0}{2^{N_0}} + \frac{1}{2^{N_0+N_1+1}})$

$$\mathcal{W}(x,c) = \left(\mathcal{W}_{N_0}(x) + cx - \gamma(x - \frac{k_0}{2^{N_0}})\right) +$$

$$\left(\sum_{n=-N_1}^{\infty} 2^{-n} 2^{-N_0-N_1-1} \sin(2\pi 2^n 2^{N_0+N_1+1}(x - \frac{k_0}{2^{N_0}})) + \gamma(x - \frac{k_0}{2^{N_0}})\right) =$$

$$\mathcal{V}_1(x) + \mathcal{V}_2(x).$$

By (17), $|\mathcal{V}'_1(\frac{k_0}{2^{N_0}})| < \frac{\epsilon_{0,\psi}}{3}$ and by (14), $|\mathcal{V}'_1(\frac{k_0}{2^{N_0}}) - \mathcal{V}'_1(x)| < \frac{\epsilon_{0,\psi}}{3}$ for all $x \in [\frac{k_0}{2^{N_0}}, \frac{k_0}{2^{N_0}} + \frac{1}{2^{N_0+N_1+1}})$. Therefore, by the Lagrange Mean Value Theorem

$$|\mathcal{V}_1(\frac{k_0}{2^{N_0}}) - \mathcal{V}_1(x)| < \frac{2\epsilon_{0,\psi}}{3} \frac{1}{2^{N_0+N_1+1}} \tag{18}$$

holds for all $x \in [\frac{k_0}{2^{N_0}}, \frac{k_0}{2^{N_0}} + \frac{1}{2^{N_0+N_1+1}})$. On the other hand, using (16)

$$\mathcal{V}_2(x) = 2^{-N_0-N_1-1}\left(\sum_{n=-N_1}^{-1} \left(2^{-n} \sin(2\pi 2^n 2^{N_0+N_1+1}(x - \frac{k_0}{2^{N_0}})) - \right.\right. \tag{19}$$

$$\left. 2\pi 2^{N_0+N_1+1}(x - \frac{k_0}{2^{N_0}})\right) + \sum_{n=0}^{\infty} 2^{-n} \sin(2\pi 2^n 2^{N_0+N_1+1}(x - \frac{k_0}{2^{N_0}})) +$$

$$\left.\gamma_0 2^{N_0+N_1+1}(x - \frac{k_0}{2^{N_0}})\right) = 2^{-N_0-N_1-1}\left(\mathcal{W}_{-N_1,\infty}(2^{N_0+N_1+1}(x - \frac{k_0}{2^{N_0}})) + \right.$$

$$\left.\gamma_0 2^{N_0+N_1+1}(x - \frac{k_0}{2^{N_0}})\right).$$

Let $u = 2^{N_0+N_1+1}(x - \frac{k_0}{2^{N_0}})$, $u \in [0,1)$. Then we have

$$\mathcal{W}(\frac{k_0}{2^{N_0}} + 2^{-N_0-N_1-1}u, c) = \mathcal{V}_1(\frac{k_0}{2^{N_0}}) + \mathcal{V}_1(\frac{k_0}{2^{N_0}} + 2^{-N_0-N_1-1}u) - \tag{20}$$

$$\mathcal{V}_1(\frac{k_0}{2^{N_0}}) + 2^{-N_0-N_1-1}(\mathcal{W}_{-N_1,\infty}(u) + \gamma_0 u) =$$

$$\mathcal{V}_1(\frac{k_0}{2^{N_0}}) + 2^{-N_0-N_1-1}\Bigg(2^{N_0+N_1+1}\Bigg(\mathcal{V}_1(\frac{k_0}{2^{N_0}} + 2^{-N_0-N_1-1}u) -$$

$$\mathcal{V}_1(\frac{k_0}{2^{N_0}})\Bigg) + \mathcal{W}_{-N_1,\infty}(u) + \gamma_0 u\Bigg) = \mathcal{V}_1(\frac{k_0}{2^{N_0}}) + 2^{-N_0-N_1-1}g(u). \tag{21}$$

By (18) we have for all $u \in [0,1)$

$$\Big|2^{N_0+N_1+1}\Big(\mathcal{V}_1(\frac{k_0}{2^{N_0}}) - \mathcal{V}_1(\frac{k_0}{2^{N_0}} + 2^{-N_0-N_1-1}u)\Big)\Big| < \frac{2\epsilon_{0,\psi}}{3}.$$

Hence (15) implies that

$$|g(u) - \mathcal{W}_{-\infty}(u,\gamma_0)| < \epsilon_{0,\psi}. \tag{22}$$

This can be interpreted as approximate repeated similarity property. The function $\mathcal{W}(x,c)$ infinitely often (for infinitely many N_0's) approximates very well on intervals of the form $[\frac{k_0}{2^{N_0}}, \frac{k_0}{2^{N_0}} + \frac{1}{2^{N_0+N_1+1}})$ a rescaled and translated copy of $\mathcal{W}_{-\infty}(x,\gamma_0)$ see (20-22). This explains why we need the functions $\mathcal{W}_{-\infty}(x,\gamma_0)$ and the class of functions $\mathcal{F}_{\mathcal{W}}$. This property is not a self similarity property since not $\mathcal{W}(x,c)$, but $\mathcal{W}_{-\infty}(x,\gamma_0)$ is approximated by these rescaled and translated copies. (This property can also be interpreted by using the concept of micro tangent sets from [6] and [8]. Since as we zoom in the graph of $\mathcal{W}(x,c)$ at the point $(x_0, \mathcal{W}(x_0,c))$ at certain scales we see something very close to a translated copy of $\mathcal{W}_{-\infty}(x,\gamma_0)$.)

By (22) and by our choice of y_1' and h_1 we have the following version of (12)

$$\lambda\{u \in [0,1) : g(u) \in (y_1', y_1' + h_1)\} > \psi h_1. \tag{23}$$

Set $y_1 = \mathcal{V}_1(\frac{k_0}{2^{N_0}}) + 2^{-N_0-N_1-1}y_1'$. By (20-21) and (23) we have

$$\lambda\{u \in [0,1) : \mathcal{W}(\frac{k_0}{2^{N_0}} + 2^{-N_0-N_1-1}u, c) \in [y_1, y_1 + 2^{-N_0-N_1-1}h_1)\} > \psi h_1. \tag{24}$$

After the substitution $x = \frac{k_0}{2^{N_0}} + 2^{-N_0-N_1-1}u$ we obtain (13) and this proves the lemma. □

Based on Lemma 4 it is easy to verify the main result of this paper.

Theorem 1. *For all $c \in \mathbb{R}$ the occupation measure of the function $\mathcal{W}(x,c)$ is purely singular.*

Remark 2. When $c = 0$ this means that the occupation measure of the Weierstrass-Cellerier function is purely singular. By results of [7] this also implies that almost every level set of this function is finite.

Proof (Theorem 1). Suppose $c \in \mathbb{R}$ is fixed. For any $\psi > 0$ denote by X_ψ' the set of those x_0 for which Lemma 4 holds. Hence $\lambda(X_\psi') = 1$ and we can choose N_1, h_1 such that for all $x_0 \in X_\psi'$ there are infinitely many N_0's for which if the interval $J_{x_0} = [\frac{k_0}{2^{N_0}}, \frac{k_0+1}{2^{N_0}})$ contains x_0 then there is y_1 such that if

$$S_{x_0} = \{x \in [\frac{k_0}{2^{N_0}}, \frac{k_0}{2^{N_0}} + \frac{1}{2^{N_0+N_1+1}}) : \mathcal{W}(x,c) \in [y_1, y_1 + \frac{h_1}{2^{N_0+N_1+1}})\}$$

then we have (13), that is,

$$\lambda(S_{x_0}) > \psi \frac{h_1}{2^{N_1+1}} \lambda(J_{x_0}) = \psi \frac{h_1}{2^{N_0+N_1+1}}.$$

Using Lemma 2 with $\rho = \psi h_1 2^{-(N_1+1)}$ we can select a subset $X_\psi \subset X'_\psi$ such that $\lambda(X_\psi) = 1$ and for any $x \in X_\psi$ there are infinitely many N_0's and y_1's such that x belongs to a set of the form S_{x_0} defined above for an $x_0 \in X_\psi$. This implies that if $x_1 \in X_\psi$ then there exist infinitely many N's and y_1's such that $\mathcal{W}(x_1, c) \in [y_1, y_1 + \frac{h_1}{2^N})$ and

$$\mu([y_1, y_1 + \frac{h_1}{2^N})) = \lambda\{x \in [0,1) : \mathcal{W}(x,c) \in [y_1, y_1 + \frac{h_1}{2^N})\} > \psi \frac{h_1}{2^N}. \qquad (25)$$

Set $X_\infty = \cap_{K=1}^{\infty} X_K$. Then $\lambda(X_\infty) = 1$ and hence $\mu(\mathcal{W}(X_\infty, c)) = 1$. On the other hand, for any $y \in \mathcal{W}(X_\infty, c)$ we can choose $x \in X_\infty$ such that $\mathcal{W}(x,c) = y$ and for any $\psi = K$ there are infinitely many N's and y_1's such that

$$y \in [y_1, y_1 + \frac{h_1}{2^N}) \text{ and } \mu([y_1, y_1 + \frac{h_1}{2^N})) > K \frac{h_1}{2^N}.$$

This implies that $\lambda(\mathcal{W}(X_\infty, c)) = 0$ and hence μ is singular with respect to λ. □

4 Proof of Lemma 3

Before proving Lemma 3 we need some auxiliary results.

Lemma 5. *For any $\epsilon_1 > 0$ there are integers N', k_1 and k_2 such that*

$$0 < \mathcal{W}'_{-\infty,N'}(\frac{k_1}{2^{N'}}) - \mathcal{W}'_{-\infty,N'}(\frac{k_2}{2^{N'}}) < \epsilon_1, \quad 0 < k_1 < k_2 < 2^{N'}. \qquad (26)$$

Proof. For any $n \le 0$ and $N' > 2$

$$0 < \cos(2\pi 2^n(\frac{1}{4} - \frac{1}{2^{N'}})) - \cos(2\pi 2^n(\frac{1}{4} + \frac{1}{2^{N'}})) < 2\pi 2^n \frac{2}{2^{N'}}. \qquad (27)$$

For $n > 0$

$$\cos(2\pi 2^n(\frac{1}{4} - \frac{1}{2^{N'}})) - \cos(2\pi 2^n(\frac{1}{4} + \frac{1}{2^{N'}})) = 0. \qquad (28)$$

Choose k_1, k_2 such that $\frac{k_1}{2^{N'}} = \frac{1}{4} - \frac{1}{2^{N'}}$ and $\frac{k_2}{2^{N'}} = \frac{1}{4} + \frac{1}{2^{N'}}$. Then by (27) and (28)

$$0 < \mathcal{W}'_{-\infty,N'}(\frac{k_1}{2^{N'}}) - \mathcal{W}'_{-\infty,N'}(\frac{k_2}{2^{N'}}) =$$

$$\sum_{n=-\infty}^{N'} 2\pi\Big(\cos(2\pi 2^n(\frac{1}{4}-\frac{1}{2^{N'}}))-\cos(2\pi 2^n(\frac{1}{4}+\frac{1}{2^{N'}}))\Big)=$$

$$\sum_{n=-\infty}^{0} 2\pi\Big(\cos(2\pi 2^n(\frac{1}{4}-\frac{1}{2^{N'}}))-\cos(2\pi 2^n(\frac{1}{4}+\frac{1}{2^{N'}}))\Big)<$$

$$\sum_{n=-\infty}^{0} 4\pi^2\frac{2}{2^{N'}}2^n=\frac{16\pi^2}{2^{N'}}.$$

Therefore, if $16\pi^2/2^{N'}<\epsilon_1$ then we have (26). □

Lemma 6. *Given $\epsilon_0>0$ there exist N', q' and $0\le k'<2^{N'}$ such that for any $\gamma'\in[0,2\pi)$ there exists $1\le t'\le q'$, $t'\in\mathbb{N}$ such that*

$$\left|2\pi\left\{\frac{t'}{2\pi}\mathcal{W}'_{-\infty,N'}\Big(\frac{k'}{2^{N'}}\Big)\right\}-\gamma'\right|<\epsilon_0. \tag{29}$$

(Here $\{.\}$ denotes the fractional part.)

Proof. If there exists N', k' such that $\theta\stackrel{\text{def}}{=}\frac{1}{2\pi}\mathcal{W}'_{-\infty,N'}(\frac{k'}{2^{N'}})\notin\mathbb{Q}$ then $\{t'\theta\}$, $t'\in\mathbb{N}$ is dense in $[0,1)$ and hence we can choose a suitable q'. Suppose that for all N' and k', $\frac{1}{2\pi}\mathcal{W}'_{-\infty,N'}(\frac{k'}{2^{N'}})\in\mathbb{Q}$. By Lemma 5 used with $\epsilon_1=\epsilon_0^2/4\pi^2$ we choose N', k_1 and k_2 such that if $\frac{1}{2\pi}\mathcal{W}'_{-\infty,N'}(\frac{k_1}{2^{N'}})=\frac{p_1}{q_1}\in\mathbb{Q}$ and $\frac{1}{2\pi}\mathcal{W}'_{-\infty,N'}(\frac{k_2}{2^{N'}})=\frac{p_2}{q_2}\in\mathbb{Q}$ with $0<q_1$, $0<q_2$, $(p_1,q_1)=1$, and $(p_2,q_2)=1$ then

$$0<\frac{p_1}{q_1}-\frac{p_2}{q_2}<\frac{\epsilon_0^2}{4\pi^2}.$$

This implies that $\frac{1}{q_1q_2}<\frac{\epsilon_0^2}{4\pi^2}$. Hence, either $\frac{1}{q_1}<\frac{\epsilon_0}{2\pi}$, or $\frac{1}{q_2}<\frac{\epsilon_0}{2\pi}$. If $\frac{1}{q_1}<\frac{\epsilon_0}{2\pi}$ then we set $q'=q_1$ and $k'=k_1$, otherwise we put $q'=q_2$ and $k'=k_2$. Therefore, we have $\{\frac{1}{2\pi}\mathcal{W}'_{-\infty,N'}(\frac{k'}{2^{N'}})\}=\frac{p'}{q'}$ with a $p'<q'$, $(p',q')=1$ and $\frac{1}{q'}<\frac{\epsilon_0}{2\pi}$. Using that p' and q' are relatively prime we obtain that the set consisting of the numbers $\{t'\frac{p'}{q'}\}$, $t'=1,...,q'$ equals the set consisting of the numbers l/q', $l=0,...,q'-1$ and hence we have (29) for any $\gamma'\in[0,2\pi)$. □

Proof (Lemma 3). Suppose $c\in\mathbb{R}$ and $0<\epsilon_1<1$ are given. We want to show that for almost every $x_0\in[0,1)$ there exist infinitely many N such that

$$|\mathcal{W}'_N(P(x_0,N))-c|<\epsilon_1. \tag{30}$$

Set $c_0=2\pi\{\frac{c}{2\pi}\}$ and $D_c=\lfloor\frac{c}{2\pi}\rfloor$. Denote by $X_{\pm\infty}$ the set of those x_0's for which $\liminf_{N\to\infty}\mathcal{W}'_N(x_0)=-\infty$ and $\limsup_{N\to\infty}\mathcal{W}'_N(x_0)=+\infty$. It follows from results of [6], or from the results mentioned in Remark 1, that almost every $x_0\in[0,1)$ belongs to $X_{\pm\infty}$. Apply Lemma 6 with $\epsilon_0=\frac{\epsilon_1}{32}$ to obtain N', q' and $0\le k'<2^{N'}$. Without limiting generality we can also suppose that

$$q' > \max\{|D_c| + 1, 4\pi\}. \tag{31}$$

Choose N_1' such that for $N_1 \geq N_1' \geq 4$

$$\frac{8\pi^2}{2^{N_1}} < \frac{\epsilon_1}{8q'}. \tag{32}$$

Put

$$\mathcal{W}_{N_1,N'}(x) = S_{-N_1}(x) + \mathcal{W}_{N'}(x), \text{ and } N'' = N' + N_1 + 1.$$

Then $\mathcal{W}_{-\infty,N'}(x) = \lim_{N\to\infty} \mathcal{W}_{N_1,N'}(x)$ and $q'N'' > 4\pi > 1$. We also suppose that for any $N_1 > N_1'$ and $N' > 0$ we have

$$|\mathcal{W}'_{-\infty,N'}(x) - \mathcal{W}'_{N_1,N'}(x)| = |S'_{-\infty}(x) - S'_{-N_1}(x)| = \tag{33}$$

$$|\sum_{n=-\infty}^{-N_1-1} (2\pi\cos(2\pi 2^n x) - 2\pi)| < \frac{\epsilon_1}{64\pi q'}.$$

Since $|\mathcal{W}'_{N+1}(x_0) - \mathcal{W}'_N(x_0)| = |2\pi\cos(2\pi 2^{N+1}x_0)| \leq 2\pi$ for any N, and $x_0 \in X_{\pm\infty}$ there are infinitely many N_0's such that

$$-101q'N'' < \frac{1}{2\pi}\mathcal{W}'_{N_0}(x_0) < -100q'N''. \tag{34}$$

Fix one such N_0 and choose k_0 such that $\frac{k_0}{2^{N_0}} = P(x_0, N_0)$. Then by (4), $q'N'' > 4\pi$ and the Lagrange Mean Value Theorem

$$-102q'N'' < \frac{1}{2\pi}\mathcal{W}'_{N_0}(\frac{k_0}{2^{N_0}}) < -99q'N''. \tag{35}$$

If $2\pi\{\frac{1}{2\pi}\mathcal{W}'_{N_0}(\frac{k_0}{2^{N_0}})\} \leq c_0$ then set $D_0 = \lfloor\frac{1}{2\pi}\mathcal{W}'_{N_0}(\frac{k_0}{2^{N_0}})\rfloor$ and

$$c'_{\text{corr}} = \mathcal{W}'_{N_0}\left(\frac{k_0}{2^{N_0}}\right) - 2\pi D_0 = 2\pi\left\{\frac{1}{2\pi}\mathcal{W}'_{N_0}\left(\frac{k_0}{2^{N_0}}\right)\right\}, \tag{36}$$

otherwise set $D_0 = \lceil\frac{1}{2\pi}\mathcal{W}'_{N_0}(\frac{k_0}{2^{N_0}})\rceil$ and

$$c'_{\text{corr}} = \mathcal{W}'_{N_0}\left(\frac{k_0}{2^{N_0}}\right) - 2\pi D_0 = 2\pi\left(\left\{\frac{1}{2\pi}\mathcal{W}'_{N_0}\left(\frac{k_0}{2^{N_0}}\right)\right\} - 1\right). \tag{37}$$

By (35) we have

$$-103q'N'' < D_0 < -98q'N''. \tag{38}$$

We also have $0 \leq c_{\text{corr}} \overset{\text{def}}{=} c_0 - c'_{\text{corr}} < 2\pi$. If $\frac{\epsilon_1}{16\pi} \leq \frac{c_{\text{corr}}}{2\pi} \leq 1 - \frac{\epsilon_1}{16\pi}$ then set $\overline{c}_{\text{corr}} = c_{\text{corr}}$, if $0 \leq \frac{c_{\text{corr}}}{2\pi} < \frac{\epsilon_1}{16\pi}$ then set $\overline{c}_{\text{corr}} = \frac{\epsilon_1}{16\pi} \cdot 2\pi = \frac{\epsilon_1}{8}$. Finally, if $1 - \frac{\epsilon_1}{16\pi} \leq \frac{c_{\text{corr}}}{2\pi} < 1$ then set $\overline{c}_{\text{corr}} = 2\pi(1 - \frac{\epsilon_1}{16\pi}) = 2\pi - \frac{\epsilon_1}{8}$. For any of the previous cases we have $|c_{\text{corr}} - \overline{c}_{\text{corr}}| \leq \frac{\epsilon_1}{8}$. To obtain N', q' and k' Lemma 6 was used with $\epsilon_0 = \frac{\epsilon_1}{32}$ by (29) we can choose $t' \leq q'$, $t' \in \mathbb{N}$ such that

$$\left|2\pi\left\{\frac{t'}{2\pi}\mathcal{W}'_{-\infty,N'}(\frac{k'}{2^{N'}})\right\}-\overline{c}_{\text{corr}}\right|<\frac{\epsilon_1}{32}. \tag{39}$$

By $|c_{\text{corr}}-\overline{c}_{\text{corr}}|\leq\frac{\epsilon_1}{8}$ and $\frac{\epsilon_1}{8}\leq\overline{c}_{\text{corr}}\leq 2\pi-\frac{\epsilon_1}{8}$ we also have

$$\left|2\pi\left\{\frac{t'}{2\pi}\mathcal{W}'_{-\infty,N'}(\frac{k'}{2^{N'}})\right\}-c_{\text{corr}}\right|<\frac{\epsilon_1}{4}\text{ and} \tag{40}$$

$$\frac{\epsilon_1}{16}<2\pi\left\{\frac{t'}{2\pi}\mathcal{W}'_{-\infty,N'}(\frac{k'}{2^{N'}})\right\}<2\pi-\frac{\epsilon_1}{16},$$

that is,

$$\frac{\epsilon_1}{32\pi}<\left\{\frac{t'}{2\pi}\mathcal{W}'_{-\infty,N'}(\frac{k'}{2^{N'}})\right\}<1-\frac{\epsilon_1}{32\pi}. \tag{41}$$

This, $t'\leq q'$, (33) and (40) imply that

$$\left|2\pi\left\{\frac{t'}{2\pi}\mathcal{W}'_{N_1,N'}(\frac{k'}{2^{N'}})\right\}-c_{\text{corr}}\right|<\frac{\epsilon_1}{3}. \tag{42}$$

We will fix an integer $D_1>q'N''$ later.
Set $N'_0=N_0+D_1+t'(N'+1)$, $x'_0=\frac{k_0}{2^{N_0}}$, and

$$x'_l=\frac{k_0}{2^{N_0}}+\sum_{t=1}^{l}\frac{k'}{2^{N_0+tN''}}\text{ for }l=1,...,t'. \tag{43}$$

Then

$$\mathcal{W}'_{N'_0}(x'_{t'})=\sum_{n=0}^{N_0}2\pi\cos(2\pi 2^n x'_{t'})+\sum_{t=1}^{t'}\sum_{n=N_0+(t-1)N''+1}^{N_0+tN''}2\pi\cos(2\pi 2^n x'_{t'})+ \tag{44}$$

$$\sum_{n=N_0+t'N''+1}^{N'_0}2\pi\cos(2\pi 2^n x'_{t'})=\sum_{n=0}^{N_0}2\pi\cos(2\pi 2^n x'_{t'})+$$

(recall that $N''=N'+1+N_1$)

$$\left(\sum_{t=1}^{t'}\sum_{n=N_0+(t-1)N''+1}^{N_0+tN''}2\pi\cos(2\pi 2^n(x'_{t'}-x'_{t-1}))\right)+2\pi(D_1-t'N_1)\overset{\text{def}}{=}$$

$$\mathcal{S}_0+\sum_{t=1}^{t'}\mathcal{S}_t+2\pi(D_1-t'N_1).$$

Recalling $0\leq k'<2^{N'}$ and $N''=N'+N_1+1$ by (4), (32) and (43) we obtain

$$|\mathcal{S}_0-\mathcal{W}'_{N_0}(x'_0)|=|\mathcal{W}'_{N_0}(x'_{t'})-\mathcal{W}'_{N_0}(x'_0)|\leq 8\pi^2 2^{N_0}(x'_{t'}-x'_0)< \tag{45}$$

$$8\pi^2 2^{N_0}\frac{1}{2^{N_0+N_1}} = 8\pi^2\frac{1}{2^{N_1}} < \frac{\epsilon_1}{8q'}.$$

Set $N''_t \overset{\text{def}}{=} N_0 + (t-1)N'' + N_1 + 1$. We have

$$|\mathcal{S}_t - (\mathcal{W}'_{N_1,N'}(\frac{k'}{2^{N'}}) + 2\pi N_1)| \leq \tag{46}$$

$$\sum_{n=-N_1}^{N'} 2\pi|\cos(2^{N''_t}2^n 2\pi(x'_{t'} - x'_{t-1})) - \cos(2^n 2\pi(\frac{k'}{2^{N'}}))| \leq$$

$$\sum_{n=-N_1}^{N'} 2^n 4\pi^2 |2^{N''_t}(x'_{t'} - x'_{t-1}) - \frac{k'}{2^{N'}}|. \tag{47}$$

We rewrite

$$2^{N''_t}(x'_{t'} - x'_{t-1}) = 2^{N_0+(t-1)N''+N_1+1}(\sum_{l=t}^{t'}\frac{k'}{2^{N_0+lN''}}) =$$

$$2^{N_0+(t-1)N''+N_1+1}(\frac{k'}{2^{N_0+tN''}} + \sum_{l=t+1}^{t'}\frac{k'}{2^{N_0+lN''}}) =$$

$$2^{N_1+1}\frac{k'}{2^{N''}}(1+\sum_{l=1}^{t'-t}\frac{1}{2^{lN''}}) = 2^{N_1+1}\frac{k'}{2^{N'+N_1+1}}(1+\sum_{l=1}^{t'-t}\frac{1}{2^{lN''}}) = \frac{k'}{2^{N'}}+\frac{k'}{2^{N'}}\sum_{l=1}^{t'-t}\frac{1}{2^{lN'}}.$$

Therefore, recalling that $k' < 2^{N'}$ and using (32) we infer

$$|2^{N''_t}(x'_{t'} - x'_{t-1}) - \frac{k'}{2^{N'}}| \leq \frac{k'}{2^{N'}}\sum_{l=1}^{t'-t}\frac{1}{2^{lN''}} < \frac{1}{2^{N''}}\sum_{l=0}^{\infty}\frac{1}{2^{lN''}} \leq \tag{48}$$

$$\frac{2}{2^{N''}} = \frac{2}{2^{N'+N_1+1}} = \frac{1}{2^{N'+N_1}} < \frac{1}{2^{N'}}\cdot\frac{\epsilon_1}{8q'8\pi^2}.$$

Now we can continue (47) to have for all $t = 1, ..., t'$

$$|\mathcal{S}_t - (\mathcal{W}'_{N_1,N'}(\frac{k'}{2^{N'}}) + 2\pi N_1)| \leq 4\pi^2\frac{\epsilon_1}{2^{N'}8q'8\pi^2}\sum_{n=-N_1}^{N'} 2^n < \frac{\epsilon_1}{8q'}. \tag{49}$$

Using (44), (45) and (49) for all $t \in \{1, ..., t'\}$ we obtain

$$|\mathcal{W}'_{N'_0}(x'_{t'}) - \mathcal{W}'_{N_0}(x'_0) - (t'\mathcal{W}'_{N_1,N'}(\frac{k'}{2^{N'}}) + 2\pi t' N_1) - 2\pi(D_1 - t'N_1)| < \tag{50}$$

$$\frac{\epsilon_1}{8q'} + t'\frac{\epsilon_1}{8q'} < \frac{\epsilon_1}{4}.$$

Recalling that $c'_{\text{corr}} = \mathcal{W}'_{N_0}(\frac{k_0}{2^{N_0}}) - 2\pi D_0 = \mathcal{W}'_{N_0}(x'_0) - 2\pi D_0$ and simplifying we infer from (50)

$$|\mathcal{W}'_{N'_0}(x'_{t'}) - t'\mathcal{W}'_{N_1,N'}(\frac{k'}{2^{N'}}) - c'_{\text{corr}} - 2\pi(D_1 + D_0)| < \frac{\epsilon_1}{4}. \tag{51}$$

Recall that $c'_{\text{corr}} = c_0 - c_{\text{corr}}$. We can rewrite (51) the following way

$$\frac{\epsilon_1}{4} > \Bigg|\mathcal{W}'_{N'_0}(x'_{t'}) - 2\pi\Bigg\{\frac{t'}{2\pi}\mathcal{W}'_{N_1,N'}(\frac{k'}{2^{N'}})\Bigg\} + c_{\text{corr}} - c_0 - \tag{52}$$

$$2\pi\Bigg\lfloor\frac{t'}{2\pi}\mathcal{W}'_{N_1,N'}(\frac{k'}{2^{N'}})\Bigg\rfloor - 2\pi(D_1 + D_0)\Bigg|.$$

Set $D_2 = \lfloor\frac{t'}{2\pi}\mathcal{W}'_{N_1,N'}(\frac{k'}{2^{N'}})\rfloor$. Then using (42) and (52) we infer

$$\frac{\epsilon_1}{4} + \frac{\epsilon_1}{3} > |\mathcal{W}'_{N'_0}(x'_{t'}) - c_0 - 2\pi(D_0 + D_1 + D_2)|. \tag{53}$$

Now,

$$|D_2| < 1 + \frac{t'}{2\pi}\left(\Big(\sum_{n=-N_1}^{N'} 2\pi|\cos(2\pi 2^n \frac{k'}{2^{N'}})|\Big) + 2\pi N_1\right) <$$

$$1 + t'(N' + 2N_1 + 1) < 3q'(N' + N_1 + 1) = 3q'N''.$$

Using (31) and (38) we can choose $D_1 > q'N''$, so that, $D_1 < 110 \cdot q'N''$ and

$$D_0 + D_1 + D_2 = D_c. \tag{54}$$

By (53) we obtain that $|\mathcal{W}'_{N'_0}(x'_{t'}) - c| < \epsilon_1$. We have

$$N_0 + t'N'' \le N_0 + q'N'' < N'_0 = N_0 + D_1 + t'(N' + 1) \le N_0 + 111q'N''. \tag{55}$$

Hence, there is $k'' \in \mathbb{Z}$ such that $x'_{t'} = \frac{k''}{2^{N'_0}}$. Now, recalling $0 \le k' < 2^{N'}$ and $N_1 \ge 4$ we infer

$$\frac{k''}{2^{N'_0}} = x'_{t'} = \frac{k_0}{2^{N_0}} + \sum_{t=1}^{t'} \frac{k'}{2^{N_0 + tN''}} < \frac{k_0}{2^{N_0}} + \frac{1}{2^{N_0}} \sum_{t=1}^{\infty} \frac{k'}{2^{t(N'+N_1+1)}} <$$

$$\frac{k_0}{2^{N_0}} + \frac{1}{2 \cdot 2^{N_0}} \sum_{t=1}^{\infty} \frac{1}{2^{tN_1}} < \frac{k_0}{2^{N_0}} + \frac{1}{2} \cdot \frac{1}{2^{N_0}}.$$

Therefore, we have $[\frac{k''}{2^{N'_0}}, \frac{k''+1}{2^{N'_0}}) \subset [\frac{k_0}{2^{N_0}}, \frac{k_0+1}{2^{N_0}})$. It is also clear that, for $x \in [\frac{k''}{2^{N'_0}}, \frac{k''+1}{2^{N'_0}})$,

$$P(x, N'_0) = \frac{k''}{2^{N'_0}} \text{ and } |\mathcal{W}'_{N'_0}(P(x, N'_0)) - c| < \epsilon_1. \tag{56}$$

Hence, by (55) for any $x_0 \in X_{\pm\infty}$ there exist infinitely many N_0's such that in $(x_0 - \frac{1}{2^{N_0}}, x_0 + \frac{1}{2^{N_0}})$ there exists an interval of length longer than $\frac{1}{2^{111q'N''}} \cdot \frac{1}{2^{N_0}}$ such that for any x in this interval (56) holds. By Lemma 2 this implies that for almost every $x \in X_{\pm\infty}$, that is, for almost every $x \in [0,1]$, there exist infinitely many N_0''s such that (56) holds.

Repeating the above procedure for all rational $c \in \mathbb{R}$ and $\epsilon_1 = \frac{1}{n}$ we obtain the statement of Lemma 3. □

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