Averages along the squares on the torus

Zoltán Buczolich[*], Department of Analysis, Eötvös Loránd University, Pázmány Péter Sétány 1/c, 1117 Budapest, Hungary
email: buczo@cs.elte.hu
www.cs.elte.hu/∼buczo

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Abstract

Answering a question raised by J-P. Conze we show that for any $x, \alpha \in \mathbb{T}$, $\alpha \notin \mathbb{Q}$ there exist $f \in L^1(\mathbb{T})$, $f \geq 0$ such that the averages

$$(\star) \qquad \frac{1}{N} \sum_{n=1}^{N} f(y + nx + n^2\alpha)$$

diverge for a.e. y. By Birkhoff's Ergodic Theorem applied on $\mathbb{T}^2$ for the transformation $(x, y) \mapsto (x + \alpha, y + 2x + \alpha)$ for almost every $x \in \mathbb{T}$ the averages $(\star)$ converge for a.e. y. We show that given $\alpha \notin \mathbb{Q}$ one can find $f \in L^1(\mathbb{T})$ for which the set $D_{\alpha,f} \stackrel{\text{def}}{=} \{x \in \mathbb{T} : (\star)$ diverges for a.e. y as $N \to \infty\}$ is of Hausdorff dimension one. We also show that for a polynomial $p(n)$ of degree two with integer coefficients the sequence $p(n)$ is universally L^1-bad.

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1 Introduction and statement of the main results

On the torus $\mathbb{T}^2$ we consider the ergodic transformation $T(x,y) = (x+\alpha, y+2x+\alpha)$ with $\alpha \notin \mathbb{Q}$. Suppose that $f \in L^1(\mathbb{T})$ and $\widetilde{f}(x,y) = f(y)$. Then $(\widetilde{f} \circ T^n)(x,y) = \widetilde{f}(x+n\alpha, y+2nx+n^2\alpha) = f(y+2nx+n^2\alpha)$ and by Birkhoff's Ergodic Theorem applied to $\widetilde{f}$ we obtain that for (Lebesgue) almost every (x,y)

$$\frac{1}{N}\sum_{n=1}^{N}(\widetilde{f} \circ T^n)(x,y) = \frac{1}{N}\sum_{n=1}^{N} f(y+2nx+n^2\alpha) \to \int_{\mathbb{T}^2} \widetilde{f} = \int_{\mathbb{T}} f.$$

By the main result of [1] there exists $f \in L^1(\mathbb{T})$ such that for $x = 0$ the averages $\frac{1}{N}\sum_{n=1}^{N} f(y + n \cdot 0 + n^2\alpha)$ do not converge almost everywhere. J-P. Conze raised during the problem session of the 2008 Chapel Hill workshop (see [3]) the following question: What are the values x for which the averages

$$\frac{1}{N}\sum_{n=1}^{N} f(y+nx+n^2\alpha)$$

diverge for almost every y.

It is clear from the ergodicity of $T(x,y) = (x+\alpha, y+2x+\alpha)$ that $\frac{1}{N}\sum_{n=1}^{N} f(y+nx+n^2\alpha)$ converges for almost every y for almost every (fixed) x.

Theorem 1. *If $x \in \mathbb{T}$ is independent from $\alpha \notin \mathbb{Q}$ then there exists $f \in L^1(\mathbb{T})$, $f \geq 0$ such that the averages*

$$\frac{1}{N}\sum_{n=1}^{N} f(y+nx+n^2\alpha) \tag{1}$$

diverge for almost every y.

The proof of this theorem is essentially a more sophisticated transference argument for the result [1] on the real line.

Theorem 2. *For any polynomial $p(n)$ of degree two with integer coefficients the sequence $p(n)$ is universally L^1-bad.*

Remark 1. Using our methods from [1] P. LaVictoire in [6] generalized the results of [1] to the sequence of powers (n^m), with $m \in \mathbb{N}$ (the main result of [1] is the case $m = 2$). Theorem 2 is a further step in the direction of determining the behavior of polynomial sequences $p(n)$.

Corollary 3. *If $x \in \mathbb{T}$ and $\alpha \notin \mathbb{Q}$ are dependent over $\mathbb{Q}$ then there exists $f \in L^1(\mathbb{T})$, $f \geq 0$ such that the averages* (1) *diverge almost everywhere. This implies that Theorem 1 holds for any x.*

Indeed, suppose $k_1x + k_2\alpha = 0$, $k_1, k_2 \in \mathbb{Z}$, $k_1^2 + k_2^2 \neq 0$. From $\alpha \notin \mathbb{Q}$ it follows that $k_1 \neq 0$. If $x = 0$ then the main result of [1] applies. Therefore, we need to deal with the case when $k_1 \neq 0$, $k_2 \neq 0$ and $x \neq 0$. Then $x = -\frac{k_2}{k_1}\alpha$ and have to consider the averages $\frac{1}{N}\sum_{n=1}^{N} f(y - \frac{k_2}{k_1}\alpha n + n^2\alpha) = \frac{1}{N}\sum_{n=1}^{N} f(\frac{1}{k_1}(k_1 y - k_2 n\alpha + k_1 n^2\alpha)) = \frac{1}{N}\sum_{n=1}^{N} g(y' + (k_1 n^2 - k_2 n)\alpha)$ with $g \in L^1$, $y' = k_1 y$ and Theorem 2 is applicable.

Definition 1. Given $\alpha \in \mathbb{T}$ and $f \in L^1(\mathbb{T})$ let

$$D_{\alpha,f} = \Big\{x \in \mathbb{T} : \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(y + nx + n^2\alpha) \text{ does not exist for a.e. } y\Big\}.$$

The Hausdorff dimension of a set A will be denoted by $\dim_H A$.

Theorem 4. *For any irrational α there exists $f \in L^1(\mathbb{T})$ such that* $\dim_H D_{\alpha,f} = 1$.

The above theorem shows that though $D_{\alpha,f}$ for a fixed α is of zero Lebesgue measure it can be of Hausdorff dimension one.

2 Preliminary Results and Notation

From results in [1] it follows that given irrational α there exist functions $f \in L^1(\mathbb{T})$, $f \geq 0$ such that

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(y + n^2\alpha) = +\infty \tag{2}$$

for almost every y. (In Theorem 1 of [1] it is only verified that (n^2) is an L^1-universally bad sequence, which means that we have (2) on a set of positive measure of y's. Since $f \geq 0$ one can take $\widetilde{f}(x) = \sum_{k=0}^{\infty} 2^{-k} f(x - k\alpha) \in L^1$ and for $\widetilde{f}$ we have (2) almost everywhere.)

By $|U|$, and $\lambda(U)$ we denote the diameter and Lebesgue Measure of the set U.

Free $\mathbb{Z}^2$ actions on Lebesgue spaces are natural generalizations of independent rotations of the circle. Assume that x and α are independent over $\mathbb{Q}$. Consider the free $\mathbb{Z}^2$ action on $\mathbb{T}$ which is generated by $y \mapsto y + x$ and $y \mapsto y + \alpha$. Given M denote by $Q_M \subset \mathbb{Z}^2$ the square

$$Q_M \stackrel{\text{def}}{=} \{(n_1, n_2) : 1 \leq n_1, n_2 \leq M\}.$$

Observe that translated copies of Q_M form a partition of $\mathbb{Z}^2$, that is, Q_M is a *tiling* set in the sense of [5]. By Theorem 2 of [5] Rohlin's lemma is valid for the above free $\mathbb{Z}^2$ action and Q_M. This means the following:
For any $\epsilon > 0$ there is a measurable set A such that

(i) $\{A + n_1 x + n_2 \alpha : (n_1, n_2) \in Q_M\}$ are disjoint sets, and

(ii) $\lambda(\bigcup_{(n_1,n_2)\in Q_M} A + n_1 x + n_2 \alpha) > 1 - \epsilon$.

In general by a dynamical system $(X, \mathcal{S}, \mu, T)$ we mean an invertible measure preserving transformation acting on the probability measure space $(X, \mathcal{S}, \mu)$.

Given a Lebesgue measurable set A, periodic by p we put

$$\overline{\lambda}(A) = \frac{1}{p}\lambda(A \cap [0, p)) = \lim_{N\to\infty} \frac{\lambda(A \cap [-N, N])}{2N}.$$

We also use the Mass Distribution Principle, see for example [4], Chapter 4.

Theorem 5. *Let μ be a mass distribution (measure) on $F \subset \mathbb{R}$. Suppose that for some $s \geq 0$ there are numbers $c > 0$ and $\delta > 0$ such that $\mu(U) \leq c|U|^s$ for all sets U with $|U| \leq \delta$. Then $\mathcal{H}^s(F) \geq \mu(F)/c$ and $s \leq \dim F$.*

3 Proofs of the main results

Proof of Theorem 1. In the proof of Theorem 1 of [1] (close to the end of [1]) given $p \in \mathbb{N}$ based on Theorem 8 of [1] a periodic transformation

$$T_p(x) = x + \frac{1}{\tau_p} \bmod 1 \text{ with } \tau_p \in \mathbb{N} \tag{3}$$

and an $f_p : [0,1) \to [0,+\infty)$ are constructed for which there exists a number $t_p > 0$ such that

$$\text{if } \widetilde{U}_p = \{x \in \mathbb{T} : \sup_{0<N} \frac{1}{N} \sum_{k=1}^{N} f_p(T_p^{k^2} x) > t_p\} \text{ then } \lambda(\widetilde{U}_p) > 1 - \frac{2}{p} \tag{4}$$

and

$$\frac{\int f_p d\lambda}{t_p} = \frac{\int |f_p| d\lambda}{t_p} < \frac{32}{4^p} \tag{5}$$

(we slightly altered the notation of [1]). (The reader should not be discouraged by our references to methods of the lengthy and technical paper [1] because fortunately we need to do some alterations of the not too involved parts of [1].)

By multiplying f_p by $\frac{p}{t_p}$ we can assume that $t_p = p$ and then (5) implies

$$\int f_p d\lambda < \frac{32p}{4^p}. \tag{6}$$

By choosing N_p sufficiently large from (4) with $t_p = p$ we deduce that

$$\text{if } U'_p = \{x \in \mathbb{T} : \sup_{0<N<N_p} \frac{1}{N} \sum_{k=1}^{N} f_p(T_p^{k^2} x) > p\} \text{ then } \lambda(U'_p) > 1 - \frac{2}{p}. \tag{7}$$

We will define the function f satisfying the claim of Theorem 1 as $f \stackrel{\text{def}}{=} \sum_{m=1}^{\infty} \overline{f}_m$. To define the functions $\overline{f}_m$ on $\mathbb{T}$ first consider the free $\mathbb{Z}^2$ action on $\mathbb{T}$ generated by the translations $y \mapsto y + x$ and $y \mapsto y + \alpha$ and apply the two dimensional Rohlin lemma with a sufficiently large M_m, $\epsilon_m = 2^{-m}$. Then the measurable sets $A_m + n_1 x + n_2 \alpha$, $1 \le n_1 \le M_m$ and $1 \le n_2 \le M_m$ are disjoint and

$$\lambda\Big(\bigcup_{1 \le n_1, n_2 \le M_m} A_m + n_1 x + n_2 \alpha\Big) > 1 - 2^{-m}. \tag{8}$$

Next with $p = 2^m$ we choose $T_p(x) = T_{2^m}(x) = x + \frac{1}{\tau_{2^m}} \mod 1$, with $\tau_{2^m} \in \mathbb{N}$ and $f_p = f_{2^m}$ such that we have (7), that is,

$$\text{if } U'_{2^m} = \{x \in \mathbb{T} : \sup_{0<N<N_{2^m}} \frac{1}{N} \sum_{k=1}^{N} f_{2^m}(T_{2^m}^{k^2} x) > 2^m\} \text{ then } \lambda(U'_{2^m}) > 1 - 2^{-m+1}. \tag{9}$$

Suppose $\widetilde{\varphi}_m : A_m \to [0, \frac{1}{\tau_{2^m}})$ is a measurable, invertible map and for any measurable set $S \subset A_m$ we have $\lambda(\widetilde{\varphi}_m(S)) = \frac{1}{\lambda(A_m)\tau_{2^m}} \lambda(S)$. For $1 \leq n_2 \leq M_m$ choose $r(n_2) \in \{0, ..., \tau_{2^m} - 1\}$ such that $n = t\tau_{2^m} + r(n_2)$ with $t \in \mathbb{Z}$. For $1 \leq n_1, n_2 \leq M_m$ define the measurable, invertible map $\varphi_{m,n_1,n_2} : A_m + n_1x + n_2\alpha \to [\frac{r(n_2)}{\tau_{2^m}}, \frac{r(n_2)+1}{\tau 2^m})$ by $\varphi_{m,n_1,n_2}(y) = \widetilde{\varphi}_m(y - n_1x - n_2\alpha) + \dfrac{r(n_2)}{\tau_{2^m}}$. For $y \in A_m + n_1x + n_2\alpha$ set $\overline{f}_m(y) = f_{2^m}(\varphi_{m,n_1,n_2}(y))$. Clearly, if $n_1 + n$ and $n_2 + n^2$ are not greater than M_m then

$$\overline{f}_m(y + nx + n^2\alpha) = f_{2^m}(T_{2^m}^{n^2} \varphi_{m,n_1,n_2}(y)). \tag{10}$$

If $y \notin \bigcup_{1 \leq n_1, n_2 \leq M_m} A_m + n_1x + n_2\alpha$ then set $\overline{f}_m(y) = 0$. We can suppose that M_m is an integer multiple of τ_{2^m}. Recalling (6) this also implies

$$\int \overline{f}_m d\lambda \leq \int f_{2^m} < \frac{32 \cdot 2^m}{4^{2^m}}. \tag{11}$$

Denote by Y_m the set of those $y \in \bigcup_{1 \leq n_1, n_2 \leq M_m} A_m + n_1x + n_2\alpha$ for which $y \in A_m + n_1(y)x + n_2(y)\alpha$ and $1 \leq n_1(y) \leq M_m - N_{2^m}$ and $1 \leq n_2(y) \leq M_m - N_{2^m}^2$. Clearly, the larger M_m the closer the measure of Y_m to the measure of $\bigcup_{1 \leq n_1, n_2 \leq M_m} A_m + n_1x + n_2\alpha$.

Next we define $U_{2^m} \subset \bigcup_{1 \leq n_1, n_2 \leq M_m} A_m + n_1x + n_2\alpha$ by the following property:

$$\text{for any } 1 \leq n_1, n_2 \leq M_m, \quad U_{2^m} \cap A_m + n_1x + n_2\alpha \stackrel{\text{def}}{=} \varphi^{-1}_{m,n_1,n_2}(U'_{2^m} \cap [\frac{r(n_2)}{\tau_{2^m}}, \frac{r(n_2)+1}{\tau_{2^m}})). \tag{12}$$

This way by (9) and (10)

$$\text{if } y \in Y_m \cap U_{2^m} \text{ then } \sup_{0<N<N_{2^m}} \frac{1}{N} \sum_{n=1}^{N} \overline{f}_m(y + nx + n^2\alpha) > 2^m. \tag{13}$$

By (9) if M_m is sufficiently large

$$\lambda(Y_m \cap U_{2^m}) > 1 - 2^{-m+2}. \tag{14}$$

By (11), $f = \sum_{m=1}^{\infty} \overline{f}_m \in L^1(\mathbb{T})$. By (13), (14) and by the Borel–Cantelli lemma for almost every $y \in \mathbb{T}$

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(y + nx + n^2\alpha) = +\infty.$$

□

Proof of Theorem 2. Suppose we have a polynomial of degree two with integer coefficients: $a_2n^2 + a_1n + a_0$ with $a_2 \neq 0$. We work with invertible transformations so we also suppose $a_2 > 0$, since otherwise we could take T^{-1} instead of T. Since $T^{a_2n^2+a_1n+a_0}x = T^{a_2n^2+a_1n}(T^{a_0}x)$ it is sufficient to show that $(a_2n^2 + a_1n)$ is L^1-universally bad. Indeed, if $\mu(A) > 0$ and $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^{a_2n^2+a_1n}x)$ fails to exist for $x \in A$ then $\mu(T^{-a_0}A) > 0$ and $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^{a_2n^2+a_1n+a_0}x)$ fails to exist for $x \in T^{-a_0}A$. Moreover, $(a_2n^2 + a_1n)4a_2 = (2a_2n + a_1)^2 - a_1^2$. Hence if we can show that $(2a_2n + a_1)^2$ is L^1-universally bad, then we also verified that $(a_2n^2 + a_1n)4a_2$ is also universally bad. If we know that $(a_2n^2 + a_1n)4a_2$ is L^1-universally bad then by the Conze principle and the Banach principle of Sawyer (see [2], [8], or [7]), for any $C < \infty$ there exists a system $(X, \mathcal{S}, \mu, T)$ and $f \in L^1(\mu)$ and $t \in \mathbb{R}$ such that

$$\mu\left(\left\{x : \sup_{N\geq 1} \left|\frac{1}{N} \sum_{n=1}^{N} f(T^{4a_2(a_2n^2+a_1n)}x)\right| > t\right\}\right) > \frac{C}{t} \int |f| d\mu. \tag{15}$$

Considering the system $(X, \mathcal{S}, \mu, T^{4a_2})$ from (15) one can deduce again by the Conze principle and the Banach principle of Sawyer that the sequence $(a_2n^2 + a_1n)$ is also L^1-universally bad. Hence we can reduce our problem to verifying that for $a_2 > 0$ the sequence $(2a_2n + a_1)^2$ is L^1-universally bad. For ease of notation it is sufficient to verify that the sequence $(a_2n + a_1)^2$ is L^1 universally bad for any $a_1, a_2 \in \mathbb{Z}$, $a_2 > 0$. This can be done by using some methods from [1]. Next we explain this.

By Definition 4 of [1] any finite subset $\mathcal{P}$ of $\mathbb{N}$ has *sufficiently large complement*, since there are infinitely many primes relatively prime to any number in $\mathcal{P}$.

We also need to recall the definition of a $K-M$-family living on $\Lambda=\mathbb{R}$, which is a special case of Definition 6 of [1]. The definition of $M-0.99$ distribution can be found in Definition 6 of [1] and we do not need more details of this definition in this paper. For detailed calculations related to this distribution we will refer to corresponding sections of [1].

Definition 2. A $K-M$ family on $\mathbb{R}$ with input parameters $\delta>0$, Ω, $\Gamma>1$, $A\in\mathbb{N}$, $\mathcal{P}$ with output objects τ, f_h, X_h $(h=1,...,K)$; E_δ, $\omega(x)$, $\alpha(x)$ and $\tau(x)$ is a system satisfying:

(i) There exist a period τ, functions $f_h:\mathbb{R}\to[0,\infty)$, pairwise independent $M-0.99$-distributed on Λ "random" variables $X_h:\mathbb{R}\to\mathbb{R}$, for $h=1,...,K$, and a set E_δ such that all these objects are periodic by the integer τ.

(ii) We have $\overline{\lambda}(E_\delta)<\delta$.

For all $x\notin E_\delta$, there exist integers $\omega(x)>\alpha(x)>A$, $\tau(x)<\tau$ such that $\omega^2(x)<\tau$, $\frac{\omega(x)}{\alpha(x)}>\Omega\cdot\tau(x)$; moreover if $\alpha(x)\le n<n+m\le\omega(x)$ and $\tau(x)|m$, then for all $h=1,...,K$,

$$\frac{1}{m}\sum_{k=n}^{n+m-1}f_h(x+k^2)>X_h(x). \tag{16}$$

(iii) For all $p\in\mathcal{P}$, $(\tau(x),p)=1$, $(\tau,p)=1$.

(iv) For all $x\in\Lambda\setminus E_\delta$, for all $h\in\{1,...,K\}$

$$f_h(x+j+\tau(x))=f_h(x+j) \tag{17}$$

whenever $\alpha^2(x)\le j<j+\tau(x)\le\omega^2(x)$.

(v) Finally, for $h=1,...,K$

$$\frac{1}{\tau}\int_0^\tau f_h=\overline{\int}f_h<\Gamma\cdot\gamma'\cdot 2^{-M+1}. \tag{18}$$

The most difficult part of [1] is the verification of the following lemma (Lemma 5 of [1]). Fortunately in this paper we do not need any details of its proof.

Lemma 6. *For each positive integer K and parameters $\delta > 0$, $\Omega, \Gamma > 1$, $A \in \mathbb{N}$, and $\mathcal{P} \subset \mathbb{N}$ such that $\mathcal{P}$ has sufficiently large complement there exists a $K - M$ family living on $\mathbb{R}$ with these parameters.*

Many parameters in Lemma 6 are quite technical and we only need them to prove the existence of $K-M$ families by a rather involved double induction process.

For our current purpose we suppose that $\mathcal{P} = \{a_2\}$. Then $\mathcal{P}$ has sufficiently large complement. The other property we need is that we can choose $\Omega = 2000a_2$ and $\alpha(x) > A \geq |a_1| + 1$. Then by (ii)

$$\frac{\omega(x)}{\alpha(x)} > 2000a_2\tau(x) \text{ and hence } \omega(x) > a_2(2\alpha(x) + 1000\tau(x)\alpha(x) + a_1) \quad (19)$$

and by (iii), $(\tau(x), a_2) = 1$. Hence for any k the numbers

$$a_2k + a_1,\ a_2(k+1) + a_1, ..., a_2(k + \tau(x) - 1) + a_1 \text{ hit each residue class modulo } \tau(x) \text{ exactly once.} \quad (20)$$

Moreover, from $k \geq 2\alpha(x)$ it follows that $a_2k + a_1 \geq \alpha(x)$.

Then using the "local periodicity property" of f_h given by (17) one can infer that from (16) and (19) it follows that for all $h = 1, ..., K$

$$\frac{1}{1000 \cdot \alpha(x)\tau(x)} \sum_{k=2\alpha(x)}^{2\alpha(x)+1000\alpha(x)\tau(x)-1} f_h(x + (a_2k + a_1)^2) > X_h(x). \quad (21)$$

With the above adjustments at the beginning of Theorem 8 of [1] one can repeat the rest of this proof and prove its following version:

Theorem 7. *Given $\delta > 0$, M and K there exist $\tau_0 \in \mathbb{N}$, $\overline{E}_\delta \subset [0,1)$, a measurable transformation $T : [0,1) \to [0,1)$, $T(x) = x + \frac{1}{\tau_0}$ modulo 1, $f : [0,1) \to [0,+\infty)$, $\overline{X}_h$, $h = 1, ..., K$ which are pairwise independent $M-0.99$-distributed random variables defined on $[0,1)$ equipped with the Lebesgue measure, λ, such that $\lambda(\overline{E}_\delta) < \delta$, for all $x \in [0,1) \setminus \overline{E}_\delta$ there exists N_x satisfying*

$$\frac{1}{N_x} \sum_{k=1}^{N_x} f(T^{(a_2k+a_1)^2}(x)) > \sum_{h=1}^{K} \overline{X}_h(x), \quad (22)$$

and $\int_{[0,1)} f d\lambda < K \cdot 2^{-M+2}$.

Based on Theorem 7 one can repeat the proof of Theorem 1 of [1] which can be found on the last two pages of [1]. Therefore, the sequence $(a_2k + a_1)^2 = n_k$ is L^1-universally bad.

□

Proof of Theorem 4. We can start to argue as in the proof of Theorem 1 of this paper. Given $p \in \mathbb{N}$ with a periodic transformation $T_p(x) = x + \frac{1}{\tau_p}$ mod 1 of $\mathbb{T}$ we can select a function $f_p \geq 0$ such that (6) and (7) hold with a suitable N_p. We also extend the definition of f_p onto $\mathbb{R}$ so that it is periodic by 1.

Since f_p is measurable by Luzin's theorem we can select an exceptional set $\Xi_p \subset \mathbb{R}$, periodic by 1 such that

$$\lambda(\Xi_p \cap [0,1)) < \frac{1}{20 \cdot 2^p N_p^2} \tag{23}$$

and f_p is continuous on $\mathbb{R} \setminus \Xi_p$. Without limiting generality we can suppose that Ξ_p is open and hence $[0,1] \setminus \Xi_p$ is compact. Since f_p is periodic by 1 it is uniformly continuous on $\mathbb{R} \setminus \Xi_p$ and we select $\delta_p > 0$ such that

$$\text{for } y_1, y_2 \in \mathbb{R},\ |y_1 - y_2| < \delta_p \text{ we have } |f_p(y_1) - f_p(y_2)| < 1. \tag{24}$$

Choose

$$0 < \epsilon_p < \min\{\frac{\delta_p}{2N_p^2}, \frac{1}{2\tau_p}\}. \tag{25}$$

Put $B_0 = 1$. Choose $B_p > p$ such that

$$B_p \Big(\frac{1}{B_p}\Big)^{1+\frac{1}{p}} < \epsilon_p, \text{ and } \Big|\{B_p\alpha\} - \frac{1}{\tau_p}\Big| < \epsilon_p. \tag{26}$$

Let

$$X_p \stackrel{\text{def}}{=} \bigcup_{l=0}^{B_p - 1} \Big[\frac{l}{B_p}, \frac{l}{B_p} + \Big(\frac{1}{B_p}\Big)^{1+\frac{1}{p}}\Big]. \tag{27}$$

Suppose $x \in X_p$, $n \leq N_p$. Then by (26) and (27)

$$||B_p(nx + n^2\alpha) - \frac{n^2}{\tau_p}|| < N_p^2 2\epsilon_p < \delta_p. \tag{28}$$

(Above by $||z||$, $z \in \mathbb{R}$ we denoted the distance of z from the closest integer.)

For $y \in \mathbb{R}$ we put

$$\overline{f}_p(y) = f_p(B_p y). \tag{29}$$

We denote by $Y_{p,x,\alpha}$ the set of those $y \in \mathbb{R}$ for which

$$y + B_p(nx + n^2\alpha) \notin \Xi_p \text{ and } y + \frac{n^2}{\tau_p} \notin \Xi_p \text{ for } n = 0, ..., N_p. \tag{30}$$

From (23) and the periodicity of Ξ_p by 1 it follows that

$$\lambda(Y_{p,x,\alpha} \cap [0,1)) > 1 - \frac{1}{10 \cdot 2^p}. \tag{31}$$

Set $f(y) = \sum_{p=1}^{\infty} \overline{f}_p(y)$. By (6) the series in the definition of f converges almost everywhere. Since the functions $\overline{f}_p$ are nonnegative $f \geq \overline{f}_p$ for all p.

$$\text{Set } X^* = \bigcap_{M=1}^{\infty} \bigcup_{p=M}^{\infty} X_p. \tag{32}$$

We will see later that the limsup set X^* is of Hausdorff dimension one. Suppose $x \in X^*$. Then there are infinitely many p's such that $x \in X_p$. Fix one such p. Since f_p is periodic by 1 using (3) and (7) we have after a slight change of notation

$$\text{if } U'_p = \{y \in \mathbb{R} : \sup_{0<N<N_p} \frac{1}{N} \sum_{n=1}^{N} f_p(y + \frac{n^2}{\tau_p}) > p\} \text{ then } \lambda(U'_p \cap [0,1)) > 1 - \frac{2}{p}. \tag{33}$$

Choose $y \in Y_{p,x,\alpha}$. We can apply (30) and using (24), (28) we obtain that for $n = 1, ..., N_p$ we have

$$|f_p(y + B_p(nx + n^2\alpha)) - f_p(y + \frac{n^2}{\tau_p})| < 1. \tag{34}$$

From (31), (33) and (34) we obtain

$$\text{if } U''_p = \{y \in \mathbb{R} : \sup_{0<N<N_p} \frac{1}{N} \sum_{n=1}^{N} f_p(y + B_p(nx + n^2\alpha)) > p - 1\} \tag{35}$$

$$\text{then } \lambda(U''_p \cap [0,1)) > 1 - \frac{2}{p} - \frac{1}{10 \cdot 2^p}.$$

Since f_p is periodic by 1 the set U_p'' is also periodic by 1 and hence $U_p \stackrel{\text{def}}{=} \frac{1}{B_p} U_p''$ is periodic by $\frac{1}{B_p}$ and by $B_p \frac{1}{B_p} = 1$ as well. Moreover, it satisfies

$$\lambda(U_p \cap [0,1)) > 1 - \frac{2}{p} - \frac{1}{10 \cdot 2^p}. \tag{36}$$

Therefore, for $y \in U_p$, $B_p y \in U_p''$ and

$$\sup_{0<N<N_p} \frac{1}{N} \sum_{n=1}^{N} f_p(B_p(y + nx + n^2\alpha)) > p - 1.$$

This and (29) imply

$$\sup_{0<N<N_p} \frac{1}{N} \sum_{n=1}^{N} f(y + nx + n^2\alpha) \geq \tag{37}$$

$$\sup_{0<N<N_p} \frac{1}{N} \sum_{n=1}^{N} \overline{f}_p(y + nx + n^2\alpha) > p - 1.$$

Since any $x \in X^*$ belongs to infinitely many X_p by (36) and (37) for almost every $y \in \mathbb{T}$ we have

$$\sup_{0<N} \frac{1}{N} \sum_{n=1}^{N} f(y + nx + n^2\alpha) = +\infty.$$

Next we prove that $\dim_H X^* = 1$.

It is sufficient to verify that for any $\epsilon > 0$

$$\dim_H(X^*) > 1 - \epsilon. \tag{38}$$

We are going to select a subsequence $p_k \to \infty$ and put $X_\epsilon^* = \bigcap_{k=1}^{\infty} X_{p_k}$. We will show that

$$\dim_H(X_\epsilon^*) = \dim_H(\bigcap_{k=1}^{\infty} X_{p_k}) > 1 - \epsilon. \tag{39}$$

Since by (32), $X_\epsilon^* \subset X^*$ this implies (38). In fact we will define sets $\widehat{X}_{p_k} \subset X_{p_k}$ and verify that $\dim_H(\bigcap_{k=1}^{\infty} \widehat{X}_{p_k}) > 1 - \epsilon$.

We select p_1 such that

$$p_1 > \frac{1}{\epsilon}, \text{ which implies } \Big(1 + \frac{1}{p_1}\Big)(1-\epsilon) < 1. \tag{40}$$

Then

$$\bigg(\Big(\frac{1}{B_{p_1}}\Big)^{1+\frac{1}{p_1}}\bigg)^{1-\epsilon} > \frac{1}{B_{p_1}}. \tag{41}$$

We set $\widehat{X}_{p_1} = X_{p_1} \subset [0,1)$. By (27) the set $\widehat{X}_{p_1}$ consists of intervals

$$I_1(l) = \Big[\frac{l}{B_{p_1}}, \frac{l}{B_{p_1}} + \Big(\frac{1}{B_{p_1}}\Big)^{1+\frac{1}{p_1}}\Big] \text{ with } l \in L_1 \stackrel{\text{def}}{=} \{0, ..., B_{p_1} - 1\}. \tag{42}$$

To apply the Mass Distribution Principle (Theorem 5) we also start to define our mass distribution μ:

$$\text{for } l \in L_1 \text{ set } \mu(I_1(l)) = \frac{1}{B_{p_1}}. \tag{43}$$

Then

$$\mu(\widehat{X}_{p_1}) = 1 \tag{44}$$

and by (41)

$$\mu(I_1(l)) = \frac{1}{B_{p_1}} < |I_1(l)|^{1-\epsilon} = \bigg(\Big(\frac{1}{B_{p_1}}\Big)^{1+\frac{1}{p_1}}\bigg)^{1-\epsilon}. \tag{45}$$

Suppose we have selected p_{k-1} and $\widehat{X}_{p_{k-1}} \subset X_{p_{k-1}} \subset [0,1)$ consisting of the intervals

$$I_{k-1}(l) = \Big[\frac{l}{B_{p_{k-1}}}, \frac{l}{B_{p_{k-1}}} + \Big(\frac{1}{B_{p_{k-1}}}\Big)^{1+\frac{1}{p_{k-1}}}\Big] \text{ with } l \in L_{k-1} \subset \{0, ..., B_{p_{k-1}} - 1\}. \tag{46}$$

We also have

$$\mu(\widehat{X}_{p_{k-1}}) = 1 \text{ and } \mu(I_{k-1}(l)) < |I_{k-1}(l)|^{1-\epsilon} = \tag{47}$$

$$\Big(\frac{1}{B_{p_{k-1}}}\Big)^{(1-\epsilon)(1+\frac{1}{p_{k-1}})} \text{for } l \in L_{k-1}.$$

We select $p_k > p_{k-1} > p_1$ such that

$$(B_{p_{k-1}})^{\epsilon(1+\frac{1}{p_{k-1}})} < \frac{B_{p_k}}{2}\Big(\frac{1}{B_{p_k}}\Big)^{(1-\epsilon)(1+\frac{1}{p_k})} \text{ and} \tag{48}$$

$$\frac{1}{B_{p_k}} < \frac{1}{1000}\Big(\frac{1}{B_{p_{k-1}}}\Big)^{1+\frac{1}{p_{k-1}}}. \tag{49}$$

For $l \in \mathbb{Z}$ set

$$I_k(l) = \Big[\frac{l}{B_{p_k}}, \frac{l}{B_{p_k}} + \Big(\frac{1}{B_{p_k}}\Big)^{1+\frac{1}{p_k}}\Big]. \tag{50}$$

Denote by L_k the set of those $l \in \mathbb{Z}$ for which $I_k(l) \subset \widehat{X}_{p_{k-1}} \subset [0,1)$ and set $\widehat{X}_{p_k} = \bigcup\limits_{l\in L_k} I_k(l)$. From (27) it follows that $\widehat{X}_{p_k} \subset X_{p_k} \subset [0,1)$. Suppose $l' \in L_{k-1}$, that is, $I_{k-1}(l') \subset \widehat{X}_{p_{k-1}}$. Set

$$\eta_{k,l'} \stackrel{\text{def}}{=} \#\{l \in \mathbb{Z} : I_k(l) \subset I_{k-1}(l')\}. \tag{51}$$

Using (42), (46), (49) and (50) we can estimate $\eta_{k,l'}$ to obtain

$$\eta_{k,l'} \geq B_{p_k}|I_{k-1}(l')| - 2 = B_{p_k}\Big(\frac{1}{B_{p_{k-1}}}\Big)^{1+\frac{1}{p_{k-1}}} - 2 > \frac{1}{2}B_{p_k}\Big(\frac{1}{B_{p_{k-1}}}\Big)^{1+\frac{1}{p_{k-1}}}. \tag{52}$$

If $I_k(l) \subset I_{k-1}(l')$ then set

$$\mu(I_k(l)) = \frac{1}{\eta_{k,l'}}\mu(I_{k-1}(l')). \tag{53}$$

This implies $\mu(\bigcup\limits_{\{l:I_k(l)\subset I_{k-1}(l')\}} I_k(l)) = \mu(I_{k-1}(l'))$ and

$$\mu(\widehat{X}_{p_k}) = \mu(\cup_{l\in L_k} I_k(l)) = \mu(\widehat{X}_{p_{k-1}}) = 1. \tag{54}$$

Moreover, by (47), (52) and (53)

$$\mu(I_k(l)) = \frac{\mu(I_{k-1}(l'))}{\eta_{k,l'}} < \tag{55}$$

$$\frac{2}{B_{p_k}} \cdot \frac{(B_{p_{k-1}})^{1+\frac{1}{p_{k-1}}}}{(B_{p_{k-1}})^{(1-\epsilon)(1+\frac{1}{p_{k-1}})}} = \frac{2}{B_{p_k}} \cdot (B_{p_{k-1}})^{\epsilon(1+\frac{1}{p_{k-1}})} = \star \tag{56}$$

(using (48) we infer)

$$\star < \Big(\frac{1}{B_{p_k}}\Big)^{(1-\epsilon)(1+\frac{1}{p_k})} = |I_k(l)|^{1-\epsilon}. \tag{57}$$

This way we can define $\widehat{X}_{p_k}$ for each $k \in \mathbb{N}$ and a mass distribution μ on $\cap_{k=1}^{\infty} \widehat{X}_{p_k}$ satisfying $\mu(\cap_{k=1}^{\infty} \widehat{X}_{p_k}) = 1$. Suppose $x \in \cap_{k=1}^{\infty} \widehat{X}_{p_k}$ and $0 < r < \left(\frac{1}{B_{p_1}}\right)^{1+\frac{1}{p_1}}$. Choose $k \geq 2$ such that

$$\left(\frac{1}{B_{p_k}}\right)^{1+\frac{1}{p_k}} \leq r < \left(\frac{1}{B_{p_{k-1}}}\right)^{1+\frac{1}{p_{k-1}}}. \tag{58}$$

If $r < \frac{1}{2B_{p_k}}$ and $x \in I_k(l)$ then by (50), (55), (57) and (58)

$$\mu(B(x,r)) \leq \mu(I_k(l)) < \left(\frac{1}{B_{p_k}}\right)^{(1-\epsilon)(1+\frac{1}{p_k})} \leq r^{1-\epsilon}. \tag{59}$$

If

$$\frac{1}{2B_{p_k}} < r < \left(\frac{1}{B_{p_{k-1}}}\right)^{1+\frac{1}{p_{k-1}}}$$

then we can use (50) and (53) by choosing l' such that $x \in I_{k-1}(l')$ to obtain

$$\mu(B(x,r)) \leq (rB_{p_k} + 2)\mu(I_k(l)) < 5rB_{p_k}\mu(I_k(l)) < \tag{60}$$

(using (55), (56) and (58))

$$10r \cdot \left(\frac{1}{B_{p_{k-1}}}\right)^{-\epsilon(1+\frac{1}{p_{k-1}})} < 10r^{1-\epsilon}.$$

This and (59) imply that we can apply the Mass Distribution Principle (Theorem 5). Indeed, if $|U| < \left(\frac{1}{B_{p_1}}\right)^{1+\frac{1}{p_1}}$ then either $U \cap \bigcap_{k=1}^{\infty} \widehat{X}_{p_k} = \emptyset$ and then $\mu(U) = 0$, or there exists $x \in \cap_{k=1}^{\infty} \widehat{X}_{p_k}$ such that taking $r = |U|$ we have $\mu(U) \leq \mu(B(x,r)) < 10r^{1-\epsilon}$.

□

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References

[1] Z. Buczolich and R. D. Mauldin "Divergent Square Averages", *Ann. of Math.* **171**, (2010), 1479-1530.

[2] J-P. Conze, “Convergence des moyennes ergodiques pour des sous suites”, *Bull. Soc. Math. France*, **35** (1973), 7-15.

[3] J-P. Conze, “(Question) on the a.e. divergence of L^1 functions along the squares”, *Ergodic Theory, Contemp. Math.*, **485**, Amer. Math. Soc., Providence, RI, (2009), 160.

[4] K. Falconer, *Fractal Geometry,* John Wiley & Sons, (1990).

[5] D. Ornstein and B. Weiss, “Ergodic Theory of Amenable Group Actions. I: The Rohlin Lemma,” *Bulletin of the Amer. Math. Soc.* **2** No. 1, (1980), 161-164.

[6] Patrick LaVictoire, “Universally L^1-Bad Arithmetic Sequences,” to appear in *J. Anal. Math.* Preprint: `http://arxiv.org/abs/0905.3865`.

[7] J. Rosenblatt and M. Wierdl “Pointwise ergodic theorems via harmonic analysis”, *Ergodic theory and its connections with harmonic analysis (Alexandria, 1993)*, 3–151, London Math. Soc. Lecture Note Ser., 205, Cambridge Univ. Press, Cambridge, 1995.

[8] S. Sawyer, “Maximal inequalities of weak type”, *Ann. of Math.* **84** (1966) no. 4, 157-174.